Path Following for Underactuated Marine Vessels

D.J.W. Belleter * C. Paliotta * M. Maggiore **
K.Y. Pettersen *

* *Centre for Autonomous Marine Operations and Systems (NTNU AMOS) and the Department of Engineering Cybernetics, Norwegian University of Science and Technology, NO7491 Trondheim, Norway*
{dennis.belleter,claudio.paliotta,kristin.y.pettersen}@itk.ntnu.no
** *Department of Electrical and Computer Engineering, University of Toronto, 10 King's College rd., Toronto, ON, M5S3G4, Canada,*
maggiore@ece.utoronto.ca

Abstract: This paper investigates the problem of making an underactuated marine vessel follow an arbitrary differentiable Jordan curve. A solution is proposed which relies on a hierarchical control methodology involving the simultaneous stabilization of two nested sets, and results in a smooth, static, and time-invariant feedback. The methodology in question effectively reduces the control problem to one of path following for a kinematic point-mass. It is shown that as long as the curvature of the path is smaller than a quantity dependent on the mass and damping parameters of the ship, path following is achieved with uniformly bounded sway speed.

1. INTRODUCTION

This paper presents a control methodology for underactuated marine vessels with two control inputs (thrust and torque) and three degrees-of-freedom (position and rotation). The control specification is path following: make the ship approach a path and follow it with nonzero speed without requiring any time parametrization. While in the trajectory tracking problem one would seek to make the ship follow a moving reference point, in path following one wants to stabilize a suitable controlled-invariant subset of the state space (see Nielsen et al. (2010)), and no exogenous signal drives the control loop.

The path following and trajectory tracking problems have been the subject of significant research in the context of marine vessels. We mention some of the relevant references. Straight-line/waypoint path following for underactuated vessels is considered in Fredriksen and Pettersen (2006), Børhaug et al. (2008), Oh and Sun (2010), and Aguiar and Pascoal (2007). Path following for curved paths is considered in Do and Pan (2006) where the path is parametrized by a path-variable that propagates along the path with a velocity dependent on the desired vessel velocity. The papers Aguiar and Hespanha (2007) and Skjetne et al. (2005) investigate the trajectory tracking problem. Path following of curved paths for underactuated vessels using a Serret-Frenet path frame is considered in Li et al. (2009); Moe et al. (2014) and Lapierre and Soetanto (2007).

The papers listed above consider path following of straight-line paths or path-following/trajectory-tracking of curved paths that are parametrized by time or a path variable. To the best of our knowledge, in the context of marine vessels, the problem of finding a smooth, static, and time invariant feedback solving the path following problem for general unparametrized paths remains open. In this paper, we make an initial step towards its solution. Our approach leverages the hierarchical control methodology presented in El-Hawwary and Maggiore (2013), a methodology which has been used in Roza and Maggiore (2014) to derive almost global position controllers for underactuated flying vehicles. The idea is to first design a path following control law for a kinematic point-mass. Then from this feedback extract a desired heading angle, and view it as a reference for a torque controller. Carrying out these two separate design steps corresponds to the simultaneous stabilization of two nested subsets of the state space, and the a reduction theorem from El-Hawwary and Maggiore (2013) is used to show overall stability. In particular, we show that if the curvature of the path is not too large in relation to a constant that depends on the ship's parameters, then the sideways velocity is uniformly bounded.

The challenge in solving the path following problem for marine vessels is that, due to the presence of sideways motion, in order to stay on a curved path the ship cannot head tangent to it, and its angle of attack relative to the path's tangent depends on the sway speed.

2. PRELIMINARIES AND NOTATION

In this paper we adopt the following notation. We denote by $\mathbb{S}^1$ the set of real numbers modulo 2π, with the differentiable manifold structure making it diffeomorphic to the unit circle. If $\psi \in \mathbb{S}^1$, R_ψ is the rotation matrix

[1] M. Maggiore was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC). This research was performed while M. Maggiore was on sabbatical leave at the Laboratoire des Signaux et Systèmes, Gif sur Yvette, France.
[2] D.J.W. Belleter, C. Paliotta, and K.Y. Pettersen were supported by the Research Council of Norway through its Centres of Excellence funding scheme, project No. 223254 AMOS.

$$R_\psi = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}.$$

If $f(x, y)$ is a differentiable function of two scalar variables, we denote by $\partial_x f$, $\partial_y f$ the partial derivatives with respect to x and y, respectively. Similarly, we define $\partial^2_{xy} f := \partial_x \partial_y f$, and similarly for the other second-order partial derivatives. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable vector function and $p \in \mathbb{R}^n$, df_p is the $m \times n$ Jacobian matrix of f at p. If Γ is a closed subset of a metric space (M, d) and $x \in M$, then we denote by $\|x\|_M$ the point-to-set distance of x to M, $\|x\|_M = \inf_{y \in M} d(x - y)$.

The following stability definitions are taken from El-Hawwary and Maggiore (2013). Let $\Sigma : \dot{\chi} = f(\chi)$ be a smooth dynamical system with state space a Riemannian manifold $\mathcal{X}$ with associated metric d. Let $\phi(t, \chi_0)$ denote the local phase flow generated by Σ, and let $B_\delta(x)$ denote the ball of radius δ centred at $x \in M$.

Consider a closed set $\Gamma \subset \mathcal{X}$ which is positively invariant for Σ, i.e., for all $\chi_0 \in \Gamma$, $\phi(t, \chi_0) \in \Gamma$ for all $t > 0$ for which $\phi(t, \chi_0)$ is defined. Then we have the following stability definitions taken from El-Hawwary and Maggiore (2013).

Definition 1. The set Γ is *stable* for Σ if for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\phi(t, \chi_0) \in B_\varepsilon(\Gamma)$, for all $t > 0$ for which $\phi(t, \chi_0)$ is defined. The set Γ is *attractive* for Σ if there exists a neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0$. The *domain of attraction of* Γ is the set $\{\chi_0 \in \mathcal{X} : \lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0\}$. The set Γ is *globally attractive* for Σ if it is attractive with domain of attraction $\mathcal{X}$. The set Γ is *locally asymptotically stable (LAS)* for Σ if it is stable and attractive. The set Γ is *globally asymptotically stable* for Σ if it is stable and globally attractive. If $\Gamma_1 \subset \Gamma_2$ are two closed positively invariant sets, then Γ_1 is *asymptotically stable relative to* Γ_2 if Γ_1 is asymptotically stable for the restriction of Σ to Γ_2. System Σ is *locally uniformly bounded (LUB) near* Γ if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}_+, B_\lambda(x)) \subset B_m(x)$. $\triangle$

The following result is key in the development of this paper.

Theorem 1. (El-Hawwary and Maggiore (2013)). Let Γ_1, Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed sets that are positively invariant for Σ and suppose that Γ_1 is not compact. If

(i) Γ_1 is asymptotically stable relative to Γ_2,
(ii) Γ_2 is asymptotically stable, and
(iii) Σ is LUB near Γ_1,

then Γ_1 is asymptotically stable for Σ.

3. THE PROBLEM

Consider the 3-degrees-of-freedom vessel depicted in Figure 1, which may describe an autonomous surface vessel (ASV) or an autonomous underwater vehicle (AUV) moving in the horizontal plane. We denote by $p \in \mathbb{R}^2$ the position of the vessel on the plane and $\psi \in \mathbb{S}^1$ its heading (or yaw) angle. The yaw rate $\dot{\psi}$ is denoted by r.

We attach at the point p of the vessel a body frame aligned with the main axes of the vessel, as depicted in the figure, with the standard convention that the z-axis points into the plane (towards the sea bottom). We represent the velocity vector $\dot{p}$ in body frame coordinates as (u, v), where u, the longitudinal component of the velocity vector, is called the surge speed, while v, the lateral component, is called the sway speed. Finally, the control inputs of the vessel are the surge trust T_u and the rudder angle T_r. In terms of these variables, the model derived in Fossen (2011) is

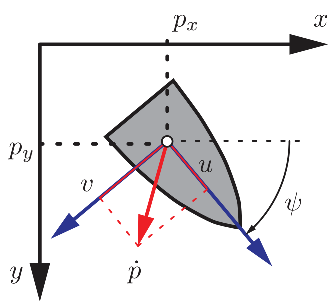

Fig. 1. Illustration of the ship's kinematic variables.

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \begin{bmatrix} R_\psi & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\nu} \\ \boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}\boldsymbol{\nu} &= \boldsymbol{B}\boldsymbol{f} \end{aligned} \tag{1}$$

with $\boldsymbol{\eta} \triangleq [p, \psi]^\top$, $\boldsymbol{\nu} \triangleq [u, v, r]^\top$, and $\boldsymbol{f} \triangleq [T_u, T_r]^\top$. The matrices $\boldsymbol{M}$, $\boldsymbol{D}$, and $\boldsymbol{B}$ are given by

$$\boldsymbol{M} \triangleq \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{23} & m_{33} \end{bmatrix}, \quad \boldsymbol{D} \triangleq \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{bmatrix}, \boldsymbol{B} \triangleq \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \\ 0 & b_{32} \end{bmatrix}$$

with $\boldsymbol{M} = \boldsymbol{M}^\top > 0$ the symmetric positive definite inertia matrix including added mass, $\boldsymbol{D} > 0$ is the hydrodynamic damping matrix, and $\boldsymbol{B}$ is the actuator configuration matrix. The matrix $\boldsymbol{C}(\boldsymbol{\nu})$ is the matrix of Coriolis and centripetal forces and can be obtained from $\boldsymbol{M}$ (see Fossen (2011)). We place the origin of the body frame at a point on the center-line of the vessel with distance ϵ from the centre of mass. Following Fredriksen and Pettersen (2006), assuming that the vessel is starboard symmetric, there exists ϵ such that the resulting dynamics have mass and damping matrices satisfying this relation: $\boldsymbol{M}^{-1}\boldsymbol{B}\boldsymbol{f} = [\tau_u, 0, \tau_r]^\top$. Thus, with this choice of origin of the body frame, the sway dynamics become decoupled from the rudder control input, making it easier to analyze the stability properties of the sway dynamics. Using this convention, the model of the marine vessel (1) can be represented as

$$\begin{aligned} \dot{p} &= R_\psi \begin{bmatrix} u \\ v \end{bmatrix} \\ \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} F_u(v, r) - \frac{d_{11}}{m_{11}} u + \tau_u \\ X(u) r + Y(u) v \end{bmatrix} \\ \dot{\psi} &= r \\ \dot{r} &= F_r(u, v, r) + \tau_r. \end{aligned} \tag{2}$$

The functions $X(u)$ and $Y(u)$ are linear. Their expressions are given in Appendix A together with those of F_u and F_r. Denoting by $\chi := (p, u, v, \psi, r)$ the state of the vessel, the state space is $\mathcal{X} := \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}$.

Assumption 1. We assume that $Y(u) < 0$ $\forall u \in [0, V_{\max}]$.

This is a realistic assumption, since $Y(\bar{u}) \geq 0$ would imply that the sway dynamics are undamped or unstable when the yaw rate r is zero.

Assumption 2. The ocean current is zero.

This assumption is made to simplify the exposition of the ideas. The results of this paper can be adapted to handle unknown constant current.

Consider a planar Jordan[3] curve γ expressed in implicit form as $\gamma = \{p : h(p) = 0\}$, where h is a C^1 function whose gradient never vanishes on γ. We assume that $h : \mathbb{R}^2 \to \mathbb{R}$ is a proper function, i.e., all its sublevel sets $\{p : h(p) \leq c\}$, $c \in \mathbb{R}$, are compact. Since γ is assumed to be compact, there is no loss of generality in this assumption.

Path Following Problem (PFP). Design a smooth time-invariant feedback such that, for suitable initial conditions, the position vector $p(t) \to \{p : h(p) = 0\}$, and the speed $\|\dot{p}(t)\|$ satisfies $0 < \|\dot{p}(t)\| \leq \sup_t \|\dot{p}(t)\| < \infty$. In other words, we want to make the position of the ship converge to the path, travel along it without stopping, while guaranteeing that its speed is bounded.

Geometric objects. Associated with the implicit representation $h(p) = 0$ of γ there are three geometric objects: the unit tangent and normal vectors, and the signed curvature. The unit normal vector at p is

$$N(p) := dh_p^\top / \|dh_p\|.$$

The unit tangent vector at p is the counterclockwise rotation of $N(p)$ by $\pi/2$,

$$T(p) := R_{\pi/2} N(p).$$

Finally, the signed curvature $\kappa(p)$ is defined as

$$\kappa(p) = -\frac{(\partial_y h)^2\, \partial^2_{xx} h - 2\partial^2_{xy} h\, \partial_x h\, \partial_y h + \partial^2_{yy} h\, (\partial_x h)^2}{\left((\partial_x h)^2 + (\partial_y h)^2\right)^{(3/2)}}. \tag{3}$$

The quantities $N(p), T(p), \kappa(p)$ are defined not just on γ, but at all points p such that $dh_p \neq [0 \ \ 0]$. If $p_0 \notin \gamma$, then $N(p_0), T(p_0), \kappa(p_0)$ are the normal vector, tangent vector, and curvature at p_0 of the curve $\{p : h(p) = p_0\}$.

4. HIERARCHICAL CONTROL APPROACH

The idea of the proposed solution is hierarchical in nature.

(1) We regulate the surge speed u to a desired constant $\bar{u} > 0$.
(2) We consider the kinematic point-mass system
$$\dot{p} = \mu,$$
and we solve the PFP with the constraint that $\|\mu\| = (\bar{u}^2 + v^2)^{(1/2)}$. The result of this design is a function $\mu(p, v)$.
(3) Having found $\mu(p, v)$, we find the desired heading angle $\psi_d(p, v)$ such that
$$R_{\psi_d} \begin{bmatrix} \bar{u} \\ v \end{bmatrix} = \mu.$$
This equation has a solution because, by construction, $\|\mu\| = (\bar{u}^2 + v^2)^{(1/2)}$. Intuitively, when $\psi = \psi_d$ and $u = \bar{u}$, the marine vessel behaves like a kinematic point-mass subject to a path following control law.
(4) Having found $\psi_d(p, v)$, we define the output function $e = \psi - \psi_d$ and we show that, under certain conditions on $\bar{u}$ (possibly any $\bar{u} > 0$), the system with input τ_r and output e has relative degree 2. We define a controller $\tau_r(\chi)$ that stabilizes the set where $e = \dot{e} = 0$.
(5) We show that, if the curvature of the path is not too large, then the sway speed v remains bounded. We use Theorem 1 to prove that the hierarchical approach described above does indeed solve the PFP if the curvature of the path is not too large.

[3] A curve is said to be Jordan if it is closed and has no self-intersections.

5. CONTROL DESIGN

In this section we carry out the design steps 1-4 outlined above. The stability analysis of step 5 is carried out in the next section.

Step 1: regulation of surge speed. This step is trivial, we choose the feedback linearizing control law

$$\tau_u = -F_u(v, r) + \frac{d_{11}}{m_{11}} u - K_u(u - \bar{u}),\ K_u > 0. \tag{4}$$

Step 2: solution of the PFP for a kinematic point-mass. Consider the kinematic point-mass system

$$\dot{p} = \mu, \tag{5}$$

where the velocity vector $\mu \in \mathbb{R}^2$ is the control input. We are to design μ such that $\|\mu\| = (\bar{u}^2 + v^2)^{(1/2)}$ and the set $\{h(p)\}$ is asymptotically stable. To this end, consider the output $z = h(p)$. The derivative is

$$\dot{z} = dh_p \mu = \|dh_p\| N(p)^\top \mu. \tag{6}$$

Define

$$\mu(p, v) := -\Big[\bar{u}\sigma(h(p))\Big] N(p) + w(p, v) T(p). \tag{7}$$

This control input is composed of two terms. The first term is orthogonal to all level sets of h (in particular, to γ) and is responsible for making $z \to 0$, as we shall see in a moment. The second term is tangent to the level sets of h and it will be designed to guarantee that $\|\mu\| = (\bar{u}^2 + v^2)^{(1/2)}$. The function $\sigma : \mathbb{R} \to (-a, a)$, $a \in (0, 1)$, is a saturation function, chosen to be smooth, monotonically increasing, zero in zero, and such that $\lim_{|z|\to\infty} |\sigma(z)| = a$. The positive scalar a is a design parameter.

Since $\{T(p), N(p)\}$ is an orthonormal frame, substitution of (7) into (6) gives

$$\dot{z} = -\|dh_p\| \bar{u} \sigma(z).$$

Since, by assumption, $\|dh_p\| \neq 0$ on γ, by continuity of h we have that $\|dh_p\| \neq 0$ in a neighborhood of γ. Therefore, for any $\bar{u} > 0$, the set $\{p : h(p) = 0\}$ is asymptotically stable.

Next we design $w(p, v)$ such that $\|\mu(p, v)\| = (\bar{u}^2 + v^2)^{(1/2)}$. Referring to the identity (7), since $\{T(p), N(p)\}$ form an orthonormal frame, we have

$$\|\mu\|^2 = \bar{u}^2 \sigma^2(h(p)) + w^2(p, v).$$

Setting

$$w(p, v) := \left(\bar{u}^2(1 - \sigma^2(h(p))) + v^2\right)^{(1/2)}, \tag{8}$$

we have $\|\mu(p, v)\| = (\bar{u}^2 + v^2)^{(1/2)}$, as required. Note that the above expression of $w(p, v)$ is well-defined and smooth because, by construction, $|\sigma| < a \leq 1$.

In conclusion, we have the following result.

Lemma 1. The feedback $\mu(p,v)$ defined in (7) and (8) makes the set $\{p \in \mathbb{R}^2 : h(p) = 0\}$ asymptotically stable for the kinematic point-mass system (5).

Step 3: definition of ψ_d. We need to find a smooth function $\psi_d(p,v)$ such that

$$R_{\psi_d} \begin{bmatrix} \bar{u} \\ v \end{bmatrix} = \mu(p,v).$$

The vector on the left-hand side of the identity above has norm $(\bar{u}^2+v^2)^{(1/2)}$ and, by construction, so does the vector on the right-hand side. Thus ψ_d is just the phase of the vector μ,

$$\psi_d(p,v) := \operatorname{atan2}(\mu_2(p,v), \mu_1(p,v)), \tag{9}$$

where atan2 is the four-quadrant arctangent function such that $\operatorname{atan2}(\sin(\theta), \cos(\theta)) = \theta \bmod 2\pi$.

Step 4: regulation of ψ to ψ_d. We define the output function $e = \psi - \psi_d$. Then

$$\dot{e} = g(p,u,v)r + f(p,u,v,\psi), \tag{10}$$

where

$$g(p,u,v) = 1 - \big(\partial_v \psi_d(p,v)\big) X(u),$$
$$f(p,u,v,\psi) = -\big(\partial_p \psi_d(p,v)\big) R_\psi \begin{bmatrix} u \\ v \end{bmatrix} - \partial_v \psi_d Y(u) v.$$

Taking one more time derivative along (2) we get

$$\ddot{e} = g(p,u,v)\big(F_r(v,r) + \tau_r\big) + \dot{g}(\chi) r + \dot{f}(\chi).$$

Lemma 2. The following identity holds:

$$\partial_v \psi_d = -\frac{\bar{u}}{\bar{u}^2 + v^2}\left[1 + \frac{\sigma(h(p))v}{w(p,v)}\right], \tag{11}$$

where $w(p,v)$ is given in (8). Suppose that

$$1 - \frac{\bar{u}|X(\bar{u})|}{\bar{u}^2 + v^2} > 0 \tag{12}$$

for all $v \in \mathbb{R}$. Then, the parameter $a \in (0,1]$ in the saturation σ can be chosen small enough that system (2) with input τ_r and output $e = \psi - \psi_d(p,v)$ has relative degree 2 at any point $\chi = (p,u,v,\psi,r)$ such that $u = \bar{u}$.

The proof of the lemma is omitted due to space limitations.

Remark 1. Condition (12) is met for all $\bar{u}$, for the ship parameters in Fredriksen and Pettersen (2004) that are used in our simulations.

Assuming that (12) holds, we define the smooth feedback linearizing control law

$$\begin{aligned} \tau_r = {} & -F_r(v,r) + \frac{1}{g(p,u,v)}\Big(- \dot{f}(\chi) - \dot{g}(\chi) r \\ & - K_p \sin(\psi - \psi_d(p,v)) - K_d(r - \dot{\psi}_d(\chi))\Big), \end{aligned} \tag{13}$$

where dot on a function denotes the time derivative of the function along the vector field (2) with τ_u as in (4). With the feedback above, we obtain $\ddot{e} + K_p \sin(e) + K_d \dot{e} = 0$. This is the equation of a pendulum with friction. Thus the equilibrium $(e, \dot{e}) = (0,0)$ is almost globally asymptotically stable. This implies that the set $\{\chi \in \mathcal{X} : \psi = \psi_d(p,v), r = \dot{\psi}_d(\chi)\}$ is stable. Moreover, this set is also asymptotically stable if the original system (2) with the chosen feedbacks τ_u and τ_r has no finite escape times. The absence of finite escape times will be proved in the next section.

Summary of feedback design. We have designed the following feedback control law

$$\begin{aligned} \tau_u = {} & -F_u(v,r) + \frac{d_{11}}{m_{11}} u - K_u(u - \bar{u}), \\ \tau_r = {} & -F_r(v,r) + \frac{1}{g(p,u,v)}\Big(- \dot{f}(\chi) - \dot{g}(\chi) r \\ & - K_p \sin(\psi - \psi_d(p,v)) - K_d(r - \dot{\psi}_d(\chi))\Big), \end{aligned} \tag{14}$$

where $\bar{u}, K_u, K_p, K_d > 0$ are design parameters and

$$\begin{aligned} \psi_d(p,v) &= \operatorname{atan2}(\mu_2(p,v), \mu_1(p,v)), \\ \mu(p,v) &= -\Big[\bar{u}\sigma(h(p))\Big] N(p) \\ &\quad + \big(\bar{u}^2(1 - \sigma^2(h(p))) + v^2\big)^{(1/2)} T(p). \end{aligned}$$

Finally, $\sigma(z)$ is any smooth, monotonically increasing function such that $\sigma(0) = 0$ and $\lim_{|z|\to\infty} |\sigma(z)| = a$, where $a \in (0,1]$ is sufficiently small as in Lemma 2. For instance, $\sigma(z) = a \tanh(Kz)$, $K > 0$, has the desired properties.

As we discussed, in the absence of finite escape times the feedback above asymptotically stabilizes the set $\Gamma_2 := \{\chi \in \mathcal{X} : u = \bar{u}, \psi = \psi_d(p,v), r - \dot{\psi}_d(p,u,v,r) = 0\}$. In Theorem 2 below we show that it solves the PFP.

6. STABILITY ANALYSIS

As we shall see in a moment, the control design procedure developed in the previous section amounts to the simultaneous stabilization of the two nested closed sets $\Gamma_1 \subset \Gamma_2$

$$\begin{aligned} \Gamma_2 &= \{\chi \in \mathcal{X} : u = \bar{u}, \psi = \psi_d(p,v), r = \dot{\psi}_d(\chi)\}, \\ \Gamma_1 &= \{\chi \in \Gamma_2 : h(p) = 0\}. \end{aligned}$$

On Γ_2, the ship behaves like a kinematic point-mass subject to a path following control law. On Γ_1, the ship is on the path with a desired surge speed $\bar{u}$. Showing that the feedback (14) solves the PFP amounts to showing that Γ_1 is asymptotically stable. To prove this property, we will use Theorem 1.

To begin, we observe that, by design, Γ_2 is stable, and asymptotically stable if solutions starting in a neighborhood of Γ_2 have no finite escape times. Assume for a moment that this is the case. On Γ_2, we have

$$\dot{p} = R_{\psi_d} \begin{bmatrix} \bar{u} \\ v \end{bmatrix}.$$

By the construction in step 2,

$$R_{\psi_d} \begin{bmatrix} \bar{u} \\ v \end{bmatrix} = \mu(p,v),$$

and thus $\dot{p} = \mu(p,v)$. By Lemma 1, the set $\{h(p) = 0\}$ is asymptotically stable for the above dynamics. In the absence of finite escape times, this implies that Γ_1 is asymptotically stable relative to Γ_2. Therefore, in order to prove asymptotic stability of Γ_1, we will prove that the closed-loop system has no finite escape times near Γ_2 and, in addition, property (iii) of Theorem 1 holds. This is done in the next lemma.

Lemma 3. Consider system (2) with the feedbacks defined in (14), and suppose Assumptions 1 and 2 hold. Suppose further that the desired surge speed $\bar{u} \in [0, V_{\max}]$ is such that $1+\bar{u}X(\bar{u})/(\bar{u}^2+v^2) \neq 0$. Then for any initial condition in a neighborhood of Γ_2, the solution is defined for all

$t \geq 0$. Moreover, if the curvature κ of γ satisfies the bound $\max_{p\in\gamma} |\kappa(p)| < \frac{|Y(\bar{u})|}{|X(\bar{u})|}$, then the closed-loop system is LUB near Γ_1.

The proof of the lemma is omitted due to space limitations. The idea is roughly this. Using the fact that, by design, Γ_2 is stable, one can show that the path following error $z = h(p)$ is uniformly bounded. Using this fact, one can show that Γ_1 is stable. In a neighborhood of Γ_1, the v-subsystem describing the sway dynamics of the ship can be viewed as a perturbation of the "nominal" dynamics. The assumption $1 + \bar{u}X(\bar{u})/(\bar{u}^2 + v^2) \neq 0$ guarantees that the equilibrium $v = 0$ is exponentially stable for the nominal dynamics, and that v remains uniformly bounded in the presence of the perturbation, so that the system is LUB near Γ_1.

Using the result of Lemma 3 and of Theorem 1, we get the following result.

Theorem 2. Consider system (2) with the feedbacks defined in (14), suppose that Assumptions 1 and 2 hold, and assume that the desired surge speed $\bar{u} \in [0, V_{\max}]$ is chosen such that condition (12) holds. If the curvature κ of γ satisfies the bound $\max_{p\in\gamma} |\kappa(p)| < \frac{|Y(\bar{u})|}{|X(\bar{u})|}$, then Γ_1 and Γ_2 are asymptotically stable, implying that feedback (14) solves the PFP.

Remark 2. It is interesting to note that in (Moe et al., 2014, Theorem 1), the authors present a stability result for a path following control law with a similar, but more restrictive, curvature bound, $\max |\kappa| < (1/3)|Y(\bar{u})/X(\bar{u})|$ compared to that in Theorem 2.

7. SIMULATION RESULTS

In this section two case studies are presented to verify the proposed path following strategy. For this purpose we consider a supply vessel described by the model (2) with the function descriptions and model parameters given in Appendix A. In the first case study we consider the case of following of following a straight-line path. Note that in the proof we assume the curves are Jordan, which the straight-line is not since it is not closed. However, the straight-line is a common test case and serves as a good proof of concept of the control strategy. The second case study considers following of a Cassini oval. Both the simulations presented in the following are based on the model parameters given in Fredriksen and Pettersen (2004).

7.1 Case 1: Straight-Line Path

In this case study the goal is to follow a straight-line path aligned with the inertial x-axis. Hence, $h(p) \triangleq -p_y$ and the implicit representation of the path is given by $\gamma = \{p : -p_y = 0\}$. This assures that the unit normal vector, $N(p)$, points in the negative y-direction and the unit tangent vector, $T(p)$, points in the positive x-direction. The desired velocity is chosen as $\bar{u} = 2$ [m/s] and the saturation function is set to $\sigma(h(p)) = 2/\pi \tan^{-1}(h(p))$. The initial conditions are given by $\chi_0 := ([0, 100], 0, 0, \pi/2, 0)$ and the controller gains from (14) are given by $K_u = 0.5$, $K_p = 0.4$, and $K_d = 2$. The trajectory of the ship in the x-y plane can be seen in Figure 2, where the ship icons superimposed on the path give the orientation of the ship at those points.

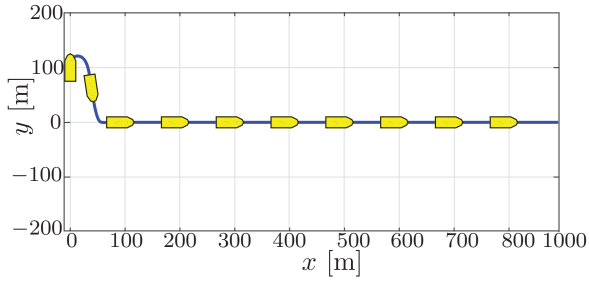

Fig. 2. Path of the ship (the ship is not to scale).

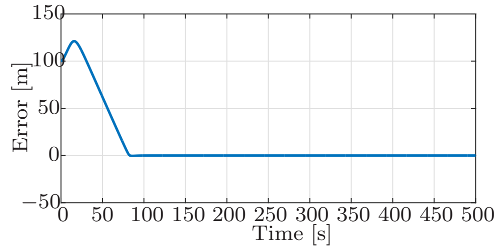

Fig. 3. Path following error of the ship.

From Figure 2 it can be seen that the trajectory converges to the x-axis and that the ship travels in the direction of the unit tangent vector $T(p)$. The positional error of the ship w.r.t. the path, i.e. p_y, can be seen in Figure 3, from which it can clearly be seen that the error converges to zero.

7.2 Case 2: Cassini Oval

In this case study the goal is to follow a Cassini oval. This implies that $h(p) \triangleq (p_x^2 + p_y^2)^2 - 2a^2(p_x^2 - p_y^2) + a^4 - b^4$ and that the path is implicitly described by

$$\gamma = \{p : (p_x^2 + p_y^2)^2 - 2a^2(p_x^2 - p_y^2) + a^4 - b^4 = 0\}.$$

where in this case study $a = 22.5$ [m] and $b = 24.9$ [m]. This results in a curve for which the maximum curvature $\max_{p\in\gamma} |\kappa(p)| = 0.0785$ and with a desired velocity $\bar{u} = 2$ [m/s] the ratio $|Y(\bar{u})|/|X(\bar{u})| = 0.2483$. Note that this curve satisfies the curvature condition of Theorem 2 showing that this is not a very restrictive condition, since it allows a ship with a length of approximately 83 meters to follow a curve whose diameter (the maximum distance between any two of its points) is approximately 70 metres. The saturation function is set to $\sigma(h(p)) = 2/\pi \tan^{-1}(\alpha h(p))$, where α is a parameter that can be used to tune the slope of the saturation function. In this case the magnitude of $h(p)$ is large, therefore α needs to be small to make the saturation effective close to the path and we choose $\alpha = 10^{-4}$. The initial conditions are given by $\chi_0 := ([15, 45], 0, 0, -2/3\pi, 0)$ and the controller gains from (14) are given by $K_u = 1$, $K_p = 30$, and $K_d = 5$. The trajectory of the ship and the desired oval can be seen in Figure 4. From Figure 4 we can clearly see convergence to the desired oval and from the superimposed ships it can be seen that the heading of the vessel is not tangent to the oval. Its velocity vector, on the other hand, is tangent to the path. From the plot of the sway velocity in Figure 5 it can be seen that this motion induces quite large sway velocities relative to the desired surge velocity $\bar{u} = 2$ [m/s]. The value of $h(p)$ is plotted in Figure 6 which shows that $h(p)$ is driven to zero as the ship converges to the path,

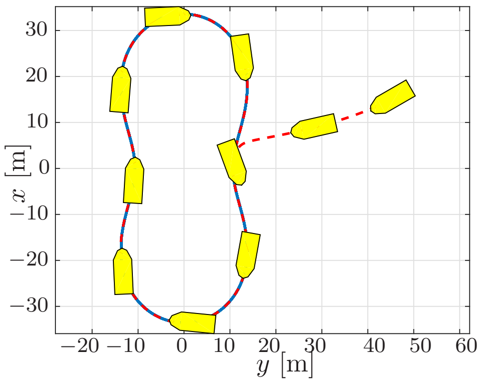

Fig. 4. Ship's path and the Cassini oval (ship not in scale).

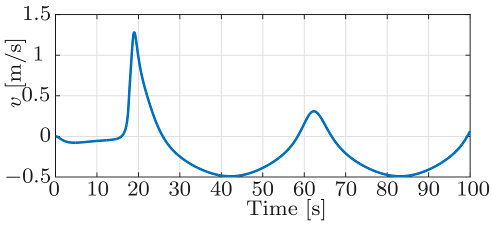

Fig. 5. Sway velocity of the ship.

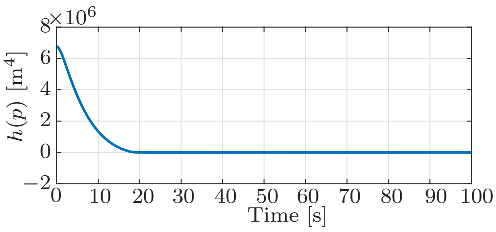

Fig. 6. Magnitude of $h(p)$ as the ship converges to the path.

showing that the ship is able to track the specified Cassini oval in accordance with the theoretical analysis.

8. CONCLUSIONS AND FUTURE WORK

In this paper we presented a methodology to design path following controllers for a class of underactuated marine vessels. This methodology allows one to migrate a path following controller designed for a point-mass to one that is guaranteed to work for the underactuated vessel. As we mentioned in the introduction, the proposed solution is an initial step. For simplicity, we assumed the curve to be Jordan and the ocean current to be absent. We will remove these assumptions in future work.

Appendix A. FUNCTIONS USED IN THE MODEL

The functions F_u, $X(u)$, $Y(u)$, and F_r are given by:

$$\begin{aligned}
F_u &\triangleq \tfrac{1}{m_{11}}(m_{22}v + m_{23}r)r,\\
X(u) &\triangleq \tfrac{m_{23}^2 - m_{11}m_{33}}{m_{22}m_{33} - m_{23}^2}u + \tfrac{d_{33}m_{23} - d_{23}m_{33}}{m_{22}m_{33} - m_{23}^2},\\
Y(u) &\triangleq \tfrac{(m_{22} - m_{11})m_{23}}{m_{22}m_{33} - m_{23}^2}u - \tfrac{d_{22}m_{33} - d_{32}m_{23}}{m_{22}m_{33} - m_{23}^2},\\
F_r(u,v,r) &\triangleq \tfrac{m_{23}d_{22} - m_{22}(d_{32} + (m_{22} - m_{11})u)}{m_{22}m_{33} - m_{23}^2}v\\
&\quad + \tfrac{m_{23}(d_{23} + m_{11}u) - m_{22}(d_{33} + m_{23}u)}{m_{22}m_{33} - m_{23}^2}r.
\end{aligned}$$

REFERENCES

Aguiar, A.P. and Hespanha, J.P. (2007). Trajectory-tracking and path-following of underactuated autonomous vehicles with parametric modeling uncertainty. *IEEE Transactions on Automatic Control*, 52(8), 1362–1379.

Aguiar, A.P. and Pascoal, A.M. (2007). Dynamic positioning and way-point tracking of underactuated AUVs in the presence of ocean currents. *International Journal of Control*, 80(7), 1092–1108.

Børhaug, E., Pavlov, A., and Pettersen, K.Y. (2008). Integral LOS control for path following of underactuated marine surface vessels in the presence of ocean currents. In *Proc. of the 47th IEEE Conference on Decision and Control*, 4984–4991.

Do, K.D. and Pan, J. (2006). Global robust adaptive path following of underactuated ships. *Automatica*, 42(10), 1713–1722.

El-Hawwary, M.I. and Maggiore, M. (2013). Reduction theorems for stability of closed sets with application to backstepping control design. *Automatica*, 49(1), 214–222.

Fossen, T.I. (2011). *Handbook of Marine Craft Hydrodynamics and Motion Control*. Wiley.

Fredriksen, E. and Pettersen, K.Y. (2004). Global κ-exponential way-point manoeuvering of ships. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 5, 5360–5367. IEEE.

Fredriksen, E. and Pettersen, K.Y. (2006). Global κ-exponential way-point maneuvering of ships: Theory and experiments. *Automatica*, 42(4), 677–687.

Lapierre, L. and Soetanto, D. (2007). Nonlinear path-following control of an AUV. *Ocean engineering*, 34(11), 1734–1744.

Li, Z., Sun, J., and Oh, S. (2009). Design, analysis and experimental validation of a robust nonlinear path following controller for marine surface vessels. *Automatica*, 45(7), 1649–1658.

Moe, S., Caharija, W., Pettersen, K.Y., and Schjølberg, I. (2014). Path following of underactuated marine surface vessels in the presence of unknown ocean currents. In *American Control Conference (ACC), 2014*, 3856–3861. IEEE.

Nielsen, C., Fulford, C., and Maggiore, M. (2010). Path following using transverse feedback linearization: Application to a maglev positioning system. *Automatica*, 46(3), 585–590.

Oh, S.R. and Sun, J. (2010). Path following of underactuated marine surface vessels using line-of-sight based model predictive control. *Ocean Engineering*, 37(2), 289–295.

Roza, A. and Maggiore, M. (2014). A class of position controllers for underactuated VTOL vehicles. *IEEE Transactions on Automatic Control*, 59(9), 2580–2585.

Skjetne, R., Fossen, T.I., and Kokotović, P.V. (2005). Adaptive maneuvering, with experiments, for a model ship in a marine control laboratory. *Automatica*, 41(2), 289–298.