Control of a Bicycle Using Virtual Holonomic Constraints

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***Abstract*— The problem of making a bicycle trace a strictly convex Jordan curve with bounded roll angle and bounded speed is investigated. The problem is solved by enforcing a virtual holonomic constraint which specifies the roll angle of the bicycle as a function of its position along the curve. It is shown that virtual holonomic constraints can be generated as periodic solutions of a scalar periodic differential equation. Finally, it is shown that if the mean curvature of the path is sufficiently small the virtual holonomic constraint can be asymptotically stabilised and the speed of the bicycle is asymptotically periodic.**

I. Introduction

This paper investigates the problem of maneuvering a bicycle along a closed Jordan curve $\mathcal{C}$ in the horizontal plane in such a way that the bicycle does not fall over and its velocity is bounded. The simplified bicycle model we use in this paper, developed by Neil Getz [1], [2], views the bicycle as a point mass with a side slip velocity constraint, and models its roll dynamics as those of an inverted pendulum, see Figure 1. The model neglects, among other things, the wheels dynamics and the associated gyroscopic effect. The dynamics of Getz's bicycle when the contact point of the rear wheel is made to follow the curve $\mathcal{C}$ are Euler-Lagrange.

In [3], Hauser-Saccon-Frezza investigate the maneuvering problem for Getz's bicycle using a dynamic inversion approach to determine bounded roll trajectories. They constrain the bicycle on the curve and, given a desired velocity signal $v(t)$, they find a trajectory with the property that the velocity of the bicycle is $v(t)$ and its roll angle φ is in the interval $(-\pi/2, \pi/2)$, i.e, the bicycle doesn't fall over. In [4], Hauser-Saccon develop an algorithm to compute the minimum-time speed profile for a point-mass motorcycle compatible with the constraint that the lateral and longitudinal accelerations do not make the tires slip, and apply their algorithm to Getz's bicycle model.

The problem of maneuvering Getz's bicycle along a closed curve is equivalent to moving the pivot point of an inverted pendulum around the curve without making the pendulum fall over. On the other hand, the seemingly different problem of maneuvering Hauser's PVTOL aircraft [5] along a closed curve in the vertical plane can be viewed as the problem of moving the pivot of an inverted pendulum around the curve without making the pendulum fall over. The two problems are, therefore, closely related, the main difference being the fact that in the former case the pendulum lies on a plane which is orthogonal to the plane of the curve, while in the latter case it lies on the same plane. In [6], the path following problem for the PVTOL was solved by enforcing a virtual holonomic constraint which specifies the roll angle of the PVTOL as a function of its position on the curve. In this paper we follow a similar approach for the bicycle model and impose a virtual holonomic constraint relating the bicycle's roll angle to its position along the curve, but rather than finding one feasible virtual constraint, we show how to generate a class of feasible virtual constraints as periodic solutions of a scalar periodic differential equation which we call the *virtual constraint generator*. We show that if the curvature of the path is sufficiently small compared to the height of the bicycle's centre of mass, then on the constraint manifold the velocity of the bicycle converges exponentially to a periodic profile. In other words, the bicycle traverses the entire curve with bounded speed and its speed profile is asymptotically periodic. Finally, we design a controller which exponentially stabilises the virtual constraint manifold and recovers the asymptotic properties of the bicycle on the constraint manifold just described.

The idea of virtual holonomic constraint is a useful paradigm for the control of oscillations and goes well beyond the example investigated in this paper. To the best of our knowledge, this notion originated with the work of Grizzle and collaborators on biped locomotion (e.g., [7] and [8]). The recent work in [9], [10], [11] investigated virtual holonomic constraints for Euler-Lagrange systems. There, an integral of motion for the dynamics on the virtual constraint was given explicitly, and a methodology to stabilise desired limit cycles on the virtual constraint manifold was given based on a time-varying linearization of the system on the limit cycle. In [12], this ideas were applied to the stabilisation of oscillations in the Furuta pendulum. In [13], we gave conditions for a virtual holonomic constraint to be feasible, and we exposed the role of the virtual constraint generator in producing feasible virtual constraints that can always be locally exponentially stabilised. We also presented sufficient conditions in order that the reduced system describing the motion on the virtual constraint manifold be Euler-Lagrange. The bicycle model in this paper does not meet the sufficient conditions of [13]; in fact, we will show that the reduced system is not Euler-Lagrange. Getz's bicycle model is therefore an example showing that the reduced motion of an Euler-Lagrange system subjected to a virtual holonomic constraint may not be Euler-Lagrange. See Remark 3.4 for more details.

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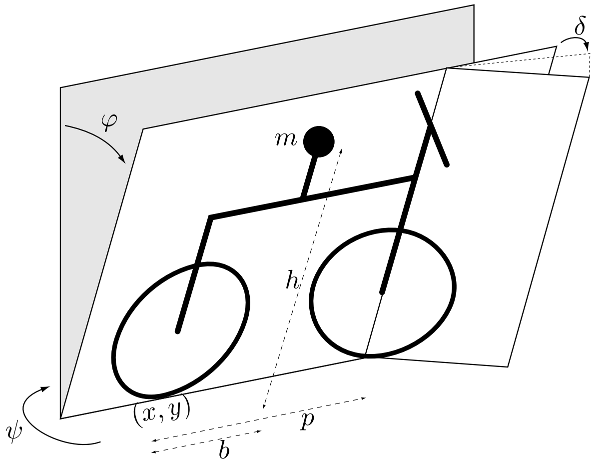

Fig. 1. Getz's bicycle model.

We adopt the notational conventions in [3] to describe the simplified bicycle model depicted in Figure 1:

- (x, y) - coordinates of the point of contact of the rear wheel
- φ - roll angle
- ψ - yaw angle
- δ - projected steering angle on the (x, y) plane
- b - distance between the projection of the centre of mass and the point of contact of the rear wheel
- p - wheel base
- h - pendulum length
- v - forward linear velocity of the bicycle
- f - reaction force of the ground on the rear wheel.

We will denote $\bar{\kappa} = \tan\delta/p = \dot{\psi}/v$. For a given velocity signal $v(t)$ and steering angle signal $\delta(t)$, $\bar{\kappa}(t)$ represents the curvature of the path $(x(t), y(t))$ traced by the point of contact of the rear wheel. The model of the bicycle in Figure 1 was presented in [2] and is given by

$$M \begin{bmatrix} \ddot{\varphi} \\ \dot{v} \end{bmatrix} = F + B \begin{bmatrix} \dot{\bar{\kappa}} \\ f \end{bmatrix}, \tag{1}$$

where, letting $s_\varphi = \sin\varphi$ and $c_\varphi = \cos\varphi$,

$$\begin{aligned} M &= \begin{bmatrix} h^2 & bhc_\varphi\bar{\kappa} \\ bhc_\varphi\bar{\kappa} & 1 + (b^2 + h^2 s_\varphi^2)\bar{\kappa}^2 - 2h\bar{\kappa}s_\varphi \end{bmatrix}, \\ F &= \begin{bmatrix} ghs_\varphi - (1 - h\bar{\kappa}s_\varphi)hc_\varphi\bar{\kappa}v^2 \\ (1 - h\bar{\kappa}s_\varphi)2hc_\varphi\bar{\kappa}v\dot{\varphi} + bh\bar{\kappa}s_\varphi\dot{\varphi}^2 \end{bmatrix}, \\ B &= \begin{bmatrix} -bhc_\varphi v & 0 \\ -(b^2\bar{\kappa} - hs_\varphi(1 - h\bar{\kappa}s_\varphi))v & 1/m \end{bmatrix}. \end{aligned}$$

Consider a C^3 closed Jordan curve $\mathcal{C}$ in the (x, y) plane with regular parametrization $\sigma(s) : \mathbb{R} \bmod T \to \mathbb{R}^2$ (T is the period of the function σ), not necessarily unit speed. Let $\kappa(s)$ denote the signed curvature of $\mathcal{C}$. Throughout this paper, we assume the following.

Assumption 1: The curve $\mathcal{C}$ is strictly convex, i.e., $\kappa(s) > 0$ for all $s \in \mathbb{R} \bmod T$.

In this paper we investigate the dynamics of the bicycle when the point (x, y) is constrained to move along the curve $\mathcal{C}$. In order to derive the constrained dynamics, suppose that $(x(0), y(0)) \in \mathcal{C}$, i.e., $(x(0), y(0)) = \sigma(s_0)$, for some $s_0 \in \mathbb{R} \bmod T$. A point $\sigma(s(t))$ moving on $\mathcal{C}$ has linear velocity

$$v(t) = \|\sigma'(s(t))\|\dot{s}(t) \tag{2}$$

and acceleration

$$\dot{v}(t) = \|\sigma'(s(t))\|\ddot{s}(t) + \frac{\dot{s}^2(t)}{\|\sigma'(s(t))\|}\sigma'(s(t))^\top\sigma''(s(t)). \tag{3}$$

Therefore, for an arbitrary velocity signal $v(t)$, $(x(t), y(t))$ traverses $\mathcal{C}$ with velocity $v(t)$ if and only if $(x(0), y(0)) \in \mathcal{C}$, $(\dot{x}(0), \dot{y}(0))$ is tangent to $\mathcal{C}$, and the steering angle $\delta(t)$ is chosen to be $\delta(t) = \arctan[p\,\kappa(s(t))]$, where $s(t) = \left(\int_0^t [v(\tau)/\|\sigma'(s(\tau))\|]d\tau\right) \bmod T$, so that

$$\bar{\kappa}(t) = \kappa(s(t)). \tag{4}$$

The motion of the bicycle on the curve $\mathcal{C}$ is thus found by substituting (2), (3), (4) in (1):

$$\tilde{M}\begin{bmatrix} \ddot{\varphi} \\ \|\sigma'\|\ddot{s} + \frac{(\sigma')^\top\sigma''}{\|\sigma'\|}\dot{s}^2 \end{bmatrix} = \tilde{F} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} f, \tag{5}$$

where $\tilde{M} = M|_{\bar{k}=k(s)}$ and

$$\tilde{F} = \left(F + B\begin{bmatrix} 1 \\ 0 \end{bmatrix} k'(s)\dot{s}\right)\Bigg|_{\bar{k}=k(s),\, v=\|\sigma'(s)\|\dot{s}}.$$

The motion of the bicycle on $\mathcal{C}$ in (5) is an Euler-Lagrange system with configuration variables (φ, s) and Lagrangian $L = T - V$, with

$$T = \frac{1}{2}[\dot{\varphi} \quad \|\sigma'(s)\|\dot{s}]\tilde{M}\begin{bmatrix} \dot{\varphi} \\ \|\sigma'(s)\|\dot{s} \end{bmatrix}, \quad V = gh\cos\varphi.$$

Since the control force f enters nonsingularly in the $\ddot{s}$ equation, we can define a feedback transformation for f in (5) such that $\ddot{s} = u$, where u is the new control input. With this transformation, the motion of the bicycle on $\mathcal{C}$ reads as

$$\begin{aligned} h\ddot{\varphi} &= gs_\varphi - \Big[(1 - h\kappa(s)s_\varphi)\kappa(s)\|\sigma'(s)\| + b\kappa'(s) \\ &\quad + \frac{b\kappa(s)}{\|\sigma'(s)\|^2}\sigma'(s)^\top\sigma''(s)\Big]c_\varphi\|\sigma'(s)\|\dot{s}^2 - b\kappa(s)c_\varphi\|\sigma'(s)\|u \\ \ddot{s} &= u. \end{aligned} \tag{6}$$

Letting $q = (\varphi, s)$, the state space is $\mathcal{X} = \{(q, \dot{q}) \in S^1 \times (\mathbb{R} \bmod T) \times \mathbb{R}^2\}$. In our simulations we will use $h = 1\,\mathrm{m}$ and $b = 1.5\,\mathrm{m}$.

Remark 1.1: If $\sigma(s)$ is a unit speed parametrization of $\mathcal{C}$, then the first differential equation in (6) reduces to

$$h\ddot{\varphi} = gs_\varphi - \Big[(1 - h\kappa(s)s_\varphi)\kappa(s) + b\kappa'(s)\Big]c_\varphi\dot{s}^2 - b\kappa(s)c_\varphi u.$$

The objective of this paper is to solve the following

Maneuvering Problem. Find a feedback $u(q, \dot{q})$ for system (6) such that there exists a set of initial conditions Ω with the property that, for all $(q(0), \dot{q}(0)) \in \Omega$, the bicycle does not overturn, i.e., $|\varphi(t)| < \pi/2$ for all $t \geq 0$, and traverses the entire curve $\mathcal{C}$ in one direction, i.e., there exists

$\bar{t} > 0$ such that $|\dot{s}(t)| > 0$ for all $t \geq \bar{t}$. Moreover, the speed $\dot{s}(t)$ of the bicycle on $\mathcal{C}$ should remain bounded.

Our solution of this problem relies on the notion of virtual holonomic constraint.

Definition 1.2: A function $\varphi = \Phi(s)$, $\Phi : \mathbb{R} \bmod T \to S^1$ is a *virtual holonomic constraint* for system (6) if the *constraint manifold*

$$\Gamma = \{(q, \dot{q}) \in \mathcal{X} : \varphi = \Phi(s),\ \dot{\varphi} = \Phi'(s)\dot{s}\}$$

is controlled invariant, i.e., there exits a smooth feedback $u(q, \dot{q})$ rendering it invariant.

The constraint manifold Γ is the collection of all those phase curves of (6) such that $\varphi(t) = \Phi(s(t))$ for all t for which the solution is defined. It is a two-dimensional submanifold of $\mathcal{X}$ parametrized by $(s, \dot{s})$, and therefore diffeomorphic to the cylinder $(\mathbb{R} \bmod T) \times \mathbb{R}$. Our approach to solving the maneuvering problem is to look for virtual holonomic constraints $\varphi = \Phi(s)$ such that $|\Phi(s)| < \pi/2$ for all $s \in \mathbb{R} \bmod T$. The advantage of this approach, as opposed to searching for individual bounded roll trajectories, is that each virtual constraint Φ provides a *family* of bounded roll trajectories corresponding to arbitrary choices of $(s(0), \dot{s}(0)) \in (\mathbb{R} \bmod T) \times \mathbb{R}$.

II. The virtual constraint generator

In this section we show that virtual holonomic constraints for (6) can be generated as solutions of a first-order differential equation, which we call the *virtual constraint generator.* This idea was first presented in our previous work [13]. We begin with a sufficient condition for a function Φ to be a feasible virtual holonomic constraint for (6).

Lemma 2.1: A C^1 function $\varphi = \Phi(s)$, $\mathbb{R} \bmod T \to S^1$ is a virtual holonomic constraint for system (6) if

$$\Phi'(s) + h^{-1} b \kappa(s) \|\sigma'(s)\| \cos \Phi(s) \neq 0 \qquad (7)$$

for all $s \in \mathbb{R} \bmod T$.

Proof: Viewing the function $\varphi - \Phi(s)$ as an output of system (6), condition (7) is simply the requirement that said output has a well-defined uniform relative degree 2 on $\{\varphi - \Phi(s) = 0\}$. The associated zero dynamics manifold is precisely Γ, and it is controlled invariant. ■

The foregoing lemma inspires the following observation. Instead of guessing a virtual constraint and checking whether it is feasible, as in the lemma, one could use (7) to generate feasible virtual holonomic constraints. More precisely, let $r = h^{-1}b$ and consider the scalar differential equation

$$\frac{d\Phi}{ds} = -r\kappa(s)\|\sigma'(s)\| \cos \Phi + \delta(s). \qquad (8)$$

Since s is a cyclic variable in $\mathbb{R} \bmod T$, the above is a T-periodic differential equation. If, for some $\delta(s) \neq 0$, (8) has a T-periodic solution $\Phi(s)$, then in light of Lemma 2.1, $\varphi = \Phi(s)$ is a virtual holonomic constraint. Therefore, one can think of differential equation (8) as a *virtual holonomic constraint generator*, for which one is to pick $\delta(s)$ such that a periodic solution exists. Such problem of existence of periodic solutions is addressed in the next proposition, whose proof is omitted due to space limitations.

Proposition 2.2: Set $\delta(s) = \epsilon\mu(s)$, where $\mu : \mathbb{R} \bmod T \to \mathbb{R}$ is a T-periodic and locally Lipschitz function such that $\mu(s) > 0$ for all $s \in \mathbb{R} \bmod T$, and let

$$K^+ = \max_{s \in \mathbb{R} \bmod T} \left(\frac{\mu(s)}{\kappa(s)\|\sigma'(s)\|} \right),$$
$$K^- = \min_{s \in \mathbb{R} \bmod T} \left(\frac{\mu(s)}{\kappa(s)\|\sigma'(s)\|} \right).$$

Then, for any $\Phi_0 \in (0, \pi/2) \bmod 2\pi$ and $s_0 \in \mathbb{R} \bmod T$,

(i) there exists a unique

$$\epsilon \in [\epsilon^-, \epsilon^+] = [(r \cos \Phi_0)/K^+, (r \cos \Phi_0)/K^-],$$

such that the solution of (8) with $\delta(s) = \epsilon\mu(s)$ and initial condition $\Phi(s_0) = \Phi_0$ is T-periodic.

(ii) If $\mu(s)$ is chosen so that $K^+/K^- < (\cos \Phi_0)^{-1}$, then the image of the T-periodic solution $\Phi(s)$ in part (i) is contained in the interval

$$(\Phi^-, \Phi^+) = \left(\cos^{-1}\Big(\frac{K^+}{K^-} \cos \Phi_0\Big), \cos^{-1}\Big(\frac{K^-}{K^+} \cos \Phi_0\Big) \right),$$

which is a subset of $(0, \pi/2)$.

Remark 2.3: By choosing $\mu(s) = \kappa(s)\|\sigma'(s)\|$, we have $K^+ = K^- = 1$, $\epsilon^+ = \epsilon^- = r \cos \Phi_0$. In this case, the proposition above implies that, for all $\Phi_0 \in (0, \pi/2)$, setting $\delta(s) = r\kappa(s)\|\sigma'(s)\| \cos \Phi_0$, the virtual constraint generator has a T-periodic solution $\Phi(s)$ whose image is in the interval $(0, \pi/2) \bmod 2\pi$. As a matter of fact, one can readily verify that the solution in question is constant, $\Phi(s) = \Phi_0$, which corresponds to the situation when the bicycle has a constant roll angle as it travels around $\mathcal{C}$. The proposition provides great flexibility in finding virtual holonomic constraints with the property that the roll angle is confined within the interval $(0, \pi/2) \bmod 2\pi$. All such constraints are compatible with the maneuvering problem.

Example 2.4: Suppose $\mathcal{C}$ is an ellipse with major semiaxis A, minor semiaxis B, and 2π-periodic parametrization $\sigma(s) = (A \cos s, B \sin s)$, with $A = 15$, $B = 5$. The curvature is $\kappa(s) = AB/(A^2 \sin^2 s + B^2 \cos^2 s)^{3/2}$. For the initial condition of the virtual constraint generator, we pick $\Phi(0) = \pi/3$. Following Proposition 2.2, we need to choose a 2π-periodic function $\mu(s) > 0$, set $\delta(s) = \epsilon\mu(s)$, and find the unique value of $\epsilon > 0$ guaranteeing that the solution with initial condition $\Phi(0) = \pi/3$ is 2π-periodic. There is much freedom in the choice of $\mu(s)$. For instance, picking $\mu(s) = 1$, we numerically find $\epsilon \approx 0.927$. The corresponding virtual holonomic constraint is depicted in Figure 2. The condition, in Proposition 2.2(ii), that $K^+/K^- < (\cos \phi_0)^{-1}$ is very conservative. Indeed, with our choice of μ we have $K^+ = 3$, $K^- = 1/3$ and thus the condition is violated. Yet, the image of the virtual constraint is contained in the interval $(0, \pi/2)$.

III. Motion on the constraint manifold

Having chosen $\delta(s)$ as in Proposition 2.2(ii) and an associated virtual holonomic constraint $\varphi = \Phi(s)$ satisfying (8),

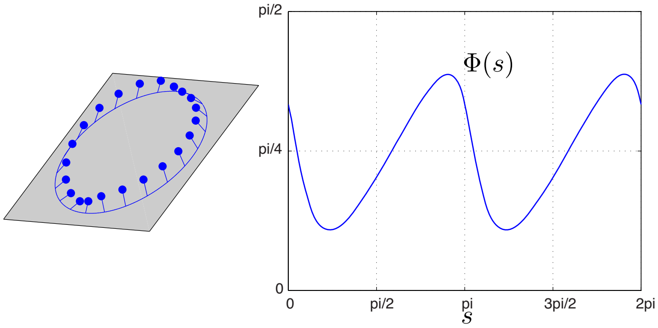

Fig. 2. Virtual holonomic constraint for the ellipse in Example 2.4.

the next step is to analyse the dynamics on the constraint manifold $\Gamma = \{(q,\dot q) \in \mathcal{X} : \varphi = \Phi(s),\ \dot\varphi = \Phi'(s)\dot s\}$. These are the zero dynamics of (6) with output function $\varphi - \Phi(s)$. The feedback making Γ invariant is found by imposing that $\frac{d}{dt}[\Phi'(s)\dot s]\big|_\Gamma = \ddot\varphi\big|_\Gamma$. Expanding both sides of the equation above, using identity (8), and the fact that $\delta(s) \neq 0$, we obtain the feedback making Γ invariant

$$u = \frac{h^{-1} g \sin\Phi}{\delta} - \frac{\dot s^2}{\delta}\Big[\Phi'' + \frac{1}{h}((1 - h\kappa\sin\Phi)\kappa\|\sigma'\| + b\kappa' + b\kappa{\sigma'}^\top\sigma''/\|\sigma'\|^2)\|\sigma'\|\cos\Phi\Big].$$

Substituting this feedback in the s dynamics we get the dynamics on Γ,

$$\ddot s = \Psi_1(s) + \Psi_2(s)\dot s^2, \tag{9}$$

where

$$\begin{aligned}\Psi_1(s) &= \frac{h^{-1} g \sin\Phi(s)}{\delta(s)}\\ \Psi_2(s) &= -\frac{1}{\delta(s)}\Big[\Phi''(s) + \frac{1}{h}((1 - h\kappa(s)\sin\Phi(s))\kappa(s)\|\sigma'(s)\| \\ &+ b\kappa'(s) + b\kappa(s)\sigma'(s)^\top\sigma''(s)/\|\sigma'(s)\|^2)\cos\Phi(s)\|\sigma'(s)\|\Big].\end{aligned} \tag{10}$$

System (9) describes the motion on Γ in the following sense. If the bicycle is initialized on the curve $\mathcal{C}$, with initial roll angle $\varphi(0) = \Phi(s_0)$ for some $s_0 \in \mathbb{R} \bmod T$, and with initial angular velocity $\dot\varphi(0) = \Phi'(s_0)\dot s_0$ for some $\dot s_0 \in \mathbb{R}$, then the bicycle remains on $\mathcal{C}$, its roll angle satisfies $\varphi(t) = \Phi(s(t))$ for all $t \geq 0$, and the position s and velocity $\dot s$ of the bicycle on $\mathcal{C}$ evolve according to (9). In order for the virtual holonomic constraint to be compatible with the maneuvering problem, we need to verify whether or not the bicycle traverses the entire curve $\mathcal{C}$ with bounded speed, i.e., that there exist $\bar t > 0$, $\epsilon > 0$ such that $\dot s(t) > \epsilon > 0$ for all $t \geq \bar t$, and $\limsup_{t\to\infty} \dot s(t) < \infty$. The next result explores general properties of systems of the form (9).

Proposition 3.1: Consider a differential equation of the form (9), where Ψ_1 and Ψ_2 are T-periodic and locally Lipschitz functions such that $\Psi_1(s) > 0$ for all s and $\int_0^T \Psi_2(s)ds < 0$. Then, there exists a real-valued T-periodic function $\nu(s)$, with $\nu(s) > 0$, such that the set $\mathcal{R} = \{(s,\dot s) : \dot s = \nu(s)\}$ is exponentially stable for (9), with domain of attraction containing the set $\mathcal{D} = \{(s,\dot s) : \dot s \geq 0\}$. Moreover, for all initial conditions in $\{(s,\dot s) : \dot s \geq 0\}$, the function $t \mapsto \nu(s(t))$ is periodic and so $\dot s(t)$ is asymptotically periodic.

Remark 3.2: It can be shown that the domain of attraction of the set $\mathcal{R}$ in the foregoing proposition is $\{(s,\dot s) : \dot s > -\nu(s)\}$.

Proof: The set $\{(s,\dot s) : \dot s \geq 0\}$ is positively invariant for (9) because $\ddot s|_{\dot s=0} = \Psi_1(s) > 0$ by assumption. Since the inequality is strict, we have $\dot s(t) > 0$ for all $t > 0$. In the rest of the proof we will restrict initial conditions on $\mathcal{D}$. Letting $z = \dot s^2$, we have $\dot z = 2\dot s(\Psi_1(s) + \Psi_2(s)z)$. Since $\dot s > 0$ for all $t > 0$, we can use s as a time variable:

$$\frac{dz}{ds} = 2\Psi_1(s) + 2\Psi_2(s)z. \tag{11}$$

The above is a scalar linear T-periodic system. As before, if $z(s_0) \geq 0$, then $z(s) > 0$ for all $s > s_0$. Letting $\phi(s) = \exp(2\int_0^s \Psi_2(\tau)d\tau)$, the solution of the linear system with initial condition $z(s_0)$ is

$$z(s) = \phi(s - s_0)z(s_0) + 2\int_{s_0}^{s} \phi(s)\phi^{-1}(\tau)\Psi_1(\tau)d\tau.$$

System (11) has a T-periodic solution if and only if there exists z_0 such that $z_0 = z(s_0) = z(s_0 + T)$, i.e, $z_0 = \phi(T)z_0 + 2\int_{s_0}^{s_0+T} \phi(s_0+T)\phi^{-1}(\tau)\Psi_1(\tau)d\tau$. Using the fact that $\phi(s+u) = \phi(s)\phi(u)$, the condition becomes

$$z_0 = \phi(T)z_0 + 2\int_0^T \phi(T)\phi^{-1}(\tau)\Psi_1(\tau)d\tau. \tag{12}$$

By assumption, $0 < \phi(T) < 1$, so the equation above has a unique solution $z_0 > 0$ and (11) has a unique T-periodic solution $\bar z(s) > 0$. Letting $\tilde z(k) = z(s_0 + kT) - z_0$ and using identity (12), we have $\tilde z(k+1) = \phi(T)\tilde z(k)$. Since $\phi(T) < 1$, the origin of this discrete-time system is globally exponentially stable, proving that the T-periodic solution $\bar z(s)$ is globally exponentially stable for (11). Let $\nu(s) = \sqrt{\bar z(s)}$ and return to system (9). For all initial conditions $(s(0),\dot s(0)) \in \mathcal{D}$, we have $\dot s(t) > 0$ for all $t > 0$, and $\dot s(t) = \sqrt{z(s(t))}$, where $z(s)$ is the solution of (11) with initial condition $z(s(0)) = \dot s(0)^2$. Since $\bar z(s)$ is globally exponentially stable for (11), $\mathcal{R}$ is exponentially stable for (9) with domain of attraction containing $\mathcal{D}$.

It remains to be shown that $t \mapsto \nu(s(t))$ is periodic. Consider the scalar differential equation $\dot s = \sqrt{\bar z(s)}$, whose vector field is T-periodic. Denote by $\varrho(t, s_0)$ its solution with initial condition $s(0) = s_0$ at time t. For all s_0, $\varrho(t, s_0 + T) = \varrho(t, s_0) + T$. Indeed, $\varrho(0, s_0) + T = s_0 + T$ and

$$\frac{d}{dt}[\varrho(t,s_0) + T] = \sqrt{\bar z(\varrho(t,s_0))} = \sqrt{\bar z(\varrho(t,s_0) + T)}.$$

Next, since $\dot s = \sqrt{\bar z(s)} > 0$, there exist unique times $\bar t > 0$ and t_0 such that $\varrho(\bar t, 0) = T$ and $\varrho(t_0, 0) = s_0$, and

$$\begin{aligned}\varrho(t + \bar t, s_0) &= \varrho(t + \bar t + t_0, 0) = \varrho(t + t_0, T)\\ &= \varrho(t + t_0, 0) + T = \varrho(t, s_0) + T.\end{aligned}$$

It then follows that for all t and s_0, $d\varrho(t+\bar{t}, s_0)/dt = d\varrho(t, s_0)/dt$ or, what is the same, for any s_0, $t \mapsto \nu(s(t))$ is periodic. ■

We now show that if the curvature of $\mathcal{C}$ satisfies an integral bound, the bicycle satisfies the hypotheses of Proposition 3.1, and so the motion on the constraint manifold satisfies the requirements of the maneuvering problem.

Proposition 3.3: If the curvature of $\mathcal{C}$ satisfies the inequality

$$\frac{1}{T}\int_0^T \kappa(s) ds < \frac{h}{b^2+h^2} \tag{13}$$

then the functions $\Psi_1(s)$, $\Psi_2(s)$ in (10) satisfy the hypotheses of Proposition 3.1 and therefore there exists at T-periodic function $\nu(s)$, with $\nu(s) > 0$, such that the closed orbit of (9) $\mathcal{R} = \{(s,\dot{s}) : \dot{s} = \nu(s)\}$ is exponentially stable, and $\mathcal{D} = \{(s,\dot{s}) : \dot{s} \geq 0\}$ lies in its domain of attraction. Moreover, for all initial conditions in $\mathcal{D}$, $\dot{s}(t)$ is asymptotically periodic.

Remark 3.4: The existence of the isolated closed orbit $\mathcal{R}$ of (9) which is exponentially stable implies that the reduced motion on the constraint manifold Γ is *not Euler-Lagrange*. Getz's bicycle is therefore an example of an Euler-Lagrange system for which there is a virtual holonomic constraint such that the reduced motion is not Euler-Lagrange.

Proof: By Proposition 2.2(i), the virtual constraint satisfies $\Phi(s) \in (0, \pi/2)$, and so $\sin\Phi(s) > 0$. Since $\delta(s) > 0$ it follows that $\Psi_1(s) > 0$. Using equality (8), we have

$$\begin{aligned}\Phi'' &= \delta' - r\kappa'\|\sigma'\|\cos\Phi - (r\kappa\|\sigma'\|)^2 \sin\Phi\cos\Phi \\ &\quad + r\kappa\|\sigma'\|\delta\sin\Phi - r\kappa{\sigma'}^\top\sigma''/\|\sigma'\|.\end{aligned}$$

Substituting in the expression for Ψ_2 in (10) we get

$$\begin{aligned}\Psi_2 &= -\frac{\delta'}{\delta} - \frac{\kappa\|\sigma'\|}{\delta h}\Big[b\delta\sin\Phi - \|\sigma'\|\cos\Phi \\ &\quad \cdot \left(\kappa\sin\Phi\frac{b^2+h^2}{h} - 1\right)\Big] \\ &\leq -\frac{\delta'}{\delta} + \frac{\kappa\|\sigma'\|^2}{\delta h}\cos\Phi\left(\kappa\sin\Phi\frac{b^2+h^2}{h} - 1\right).\end{aligned}$$

Since $\int_0^T \delta'(s)/\delta(s) = \ln\delta(T) - \ln\delta(0) = 0$, using the bounds in Proposition 2.2(ii), we have

$$\begin{aligned}&\int_0^T \Psi_2(s)ds \leq \int_0^T \frac{\kappa\|\sigma'\|^2}{\delta h}\cos\Phi\left(\kappa\sin\Phi\frac{b^2+h^2}{h} - 1\right)ds \\ &\leq \max_s\left(\frac{\kappa\|\sigma'\|^2}{\delta h}\right)\cos\Phi^- \int_0^T\left(\kappa(s)\frac{b^2+h^2}{h} - 1\right)ds \\ &\leq \max_s\left(\frac{\kappa\|\sigma'\|^2}{\delta h}\right)\cos\Phi^- \left(\frac{b^2+h^2}{h}\int_0^T \kappa(s)ds - T\right) < 0.\end{aligned}$$

■

Remark 3.5: If the parametrization $\sigma(s)$ of $\mathcal{C}$ has unit speed (i.e., $\|\sigma'(s)\| = 1$), then T is the length of $\mathcal{C}$, and the integral $(1/T)\int_0^T \kappa(s)ds$ is equal to (*turning number* of $\mathcal{C}$) $\times 2\pi/T$. The turning number is the number of revolutions that the tangent vector to $\mathcal{C}$ makes as a point is moved once around $\mathcal{C}$. For a Jordan curve, the turning number is ± 1.

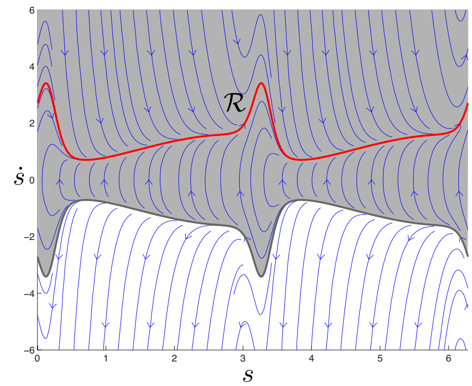

Fig. 3. Phase portrait of the dynamics on Γ and set $\mathcal{R}$ for the ellipse in Example 2.4. The shaded region is the domain of attraction of $\mathcal{R}$.

Example 3.6: We return to ellipse of Example 2.4 and the virtual constraint displayed in Figure 2. For this example, $(1/2\pi)\int_0^{2\pi}\kappa(s)ds \approx 0.142$, and $h/(b^2+h^2) = 0.308$ and thus (13) is satisfied. Indeed, one can numerically check that $\int_0^T \Psi_2(s)ds \approx -27.5 < 0$, and Proposition 3.1 applies. The phase portrait of the dynamics on the constraint manifold is displayed in Figure 3. The figure illustrates the set $\mathcal{R}$, corresponding to the steady-state velocity profile of the bicycle on Γ. The domain of attraction of $\mathcal{R}$, shaded in the figure, is the set $\{(s,\dot{s}) : \dot{s} > -\nu(s)\}$, as pointed out in Remark 3.2.

Example 3.7: Suppose $\mathcal{C}$ is a circle of radius R. The curvature is constant, $\kappa = 1/R$. For any $\Phi_0 \in (0, \pi/2)$, picking $\delta = (r/R)\cos\Phi_0$, as in Remark 2.3, we obtain the constant virtual constraint $\Phi(s) = \Phi_0$. Equation (11) becomes $dz/ds = (gR)/(hr)\tan\Phi_0 - 1/(hr)(1-(h/R)\sin\Phi_0)z$. The above is a linear time-invariant system with constant input which is stable if $R > h\sin\Phi_0$. The periodic solution $\bar{z}(s)$ in this case is simply the equilibrium of the system above, $\bar{z} = gR^2\tan\Phi_0/(R - h\sin\Phi_0)$, and thus the asymptotic velocity of the bicycle on Γ is constant, and reads as $\nu = R\sqrt{g\tan\Phi_0/(R-h\sin\Phi_0)}$. It can be verified that ν is an increasing function of Φ_0. The conclusion is that the bicycle can go around the circle with any constant roll angle in the interval $(0, \pi/2)$. The larger is the roll angle Φ_0, the higher is the asymptotic speed of the bicycle.

IV. Solution of the Maneuvering Problem

Theorem 4.1: Suppose that the mean curvature of $\mathcal{C}$ satisfies inequality (13). If $\Phi(s)$ is a virtual holonomic constraint satisfying (7) and such that $\Phi(s) \in (0, \pi/2)$ for all $s \in \mathbb{R} \bmod T$, then the feedback

$$\begin{aligned}u = \frac{1}{\Delta(q)}\Big(\frac{1}{h}gs_\varphi - \Big(\Phi'' + \frac{1}{h}\Big((1-h\kappa s_\varphi)\kappa\|\sigma'\| + b\kappa' \\ + \frac{b\kappa{\sigma'}^\top\sigma''}{\|\sigma'\|^2}\Big)c_\varphi\|\sigma'\|\Big)\dot{s}^2 + K_1 e + K_2\dot{e}\Big),\end{aligned} \tag{14}$$

where $\Delta(q) = \Phi'(s) + h^{-1}b\kappa(s)c_\varphi\|\sigma'(s)\|$, $e = \varphi - \Phi(s)$, $\dot{e} = \dot{\varphi} - \Phi'(s)\dot{s}$, and K_1, K_2 are positive design parameters, solves the maneuvering problem and has the following properties:

(i) The constraint manifold Γ is invariant and locally exponentially stable for the closed-loop system (6), (14).

(ii) There exists a C^1 and T-periodic function $\nu(s) : \mathbb{R} \bmod T \to \mathbb{R}$, with $\nu(s) > 0$ such that the set $\mathcal{R} = \{(q, \dot{q}) \in \Gamma : \dot{s} = \nu(s)\}$ is asymptotically stable for the closed-loop system and its domain of attraction is a neighbourhood of the set $\{(q, \dot{q}) \in \Gamma : \dot{s} > 0\}$.

(iii) For initial conditions in the domain of attraction of $\mathcal{R}$, the bicycle traverses the entire curve $\mathcal{C}$ and its speed is asymptotically periodic.

Proof: By construction, $\Phi(s)$ satisfies (7) and, as argued in the proof of Lemma 2.1, system (6) with output e has uniform relative degree 2 on Γ. The feedback (14) is a feedback linearizing controller making the origin of the $(e, \dot{e})$ subsystem, and hence Γ, exponentially stable, proving part (i).

As for part (ii), we know from Proposition 3.3 that there exists a T-periodic function $\nu(s)$ such that the set $\mathcal{R}$ is exponentially stable for the restriction of the dynamics on Γ. In order to prove that $\mathcal{R}$ is asymptotically stable for initial conditions outside of Γ, note that $\mathcal{R}$ is a one-dimensional submanifold of Γ diffeomorphic to S^1, and hence compact. Owing to the reduction principle for asymptotic stability of compact sets (see [14], [15]), the asymptotic stability of $\mathcal{R}$ relative to Γ together with the asymptotic stability of Γ implies that $\mathcal{R}$ is asymptotically stable for (6). By Proposition 3.3, its domain of attraction contains the set $\{(q, \dot{q}) \in \Gamma : \dot{s} > 0\}$.

Finally, concerning part (iii), since on $\mathcal{R}$ we have $\dot{s} = \nu(s) > 0$, for all initial conditions in the domain of attraction of $\mathcal{R}$ there exists a time $\bar{t} > 0$ such that $\dot{s}(t) > 0$ for all $t \geq \bar{t}$, and hence the bicycle traverses the entire curve $\mathcal{C}$. Since $\mathcal{R}$ is diffeomorphic to S^1, since it is asymptotically stable, and on it solutions are periodic, $\mathcal{R}$ is a stable limit cycle of the closed-loop system. Therefore, solutions in the domain of attraction of $\mathcal{R}$ are asymptotically periodic. ∎

Example 4.2: We return to example of the ellipse, with the virtual constraint depicted in Figure 2. The simulation results for the closed-loop system with controller (14) and $K_1 = 100$, $K_2 = 10$ are shown in Figures 4, 5 for the initial condition $(\varphi(0), \dot{\varphi}(0), s(0), \dot{s}(0)) = (0, 0, 0, 1)$. Figure 4 illustrates the exponential convergence of $\varphi(t)$ to the constraint $\Phi(s(t))$. Figure 5 displays the projection of the phase curve on the $(s, \dot{s})$ plane and its convergence to the submanifold $\mathcal{R}$.

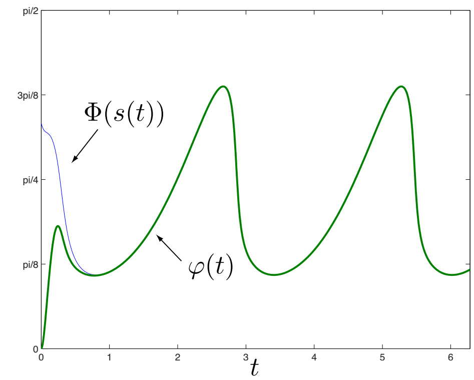

Fig. 4. Simulation of the closed-loop system for the ellipse example. The solution converges to the constraint manifold Γ.

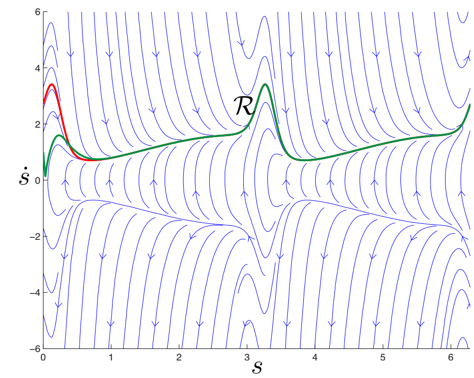

Fig. 5. Simulation of the closed-loop system for the ellipse example. The solution converges to the submanifold $\mathcal{R} \subset \Gamma$.

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