Stabilization of Closed Sets for Passive Systems, Part II: Passivity-Based Control

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Abstract— **In this paper we explore the stabilization of closed invariant sets for passive systems, and present conditions under which a passivity-based feedback makes the set stable, semi-attractive, or semi-asymptotically stable for the closed-loop system. Our results rely on novel reduction principles, presented in Part I of this work. As an application of the theory, we present a coordination problem for two unicycles.**

I. Introduction

In this paper we continue the line of research, initiated in [1] and [2], devoted to the investigation of the set stabilization problem for passive nonlinear systems.

The notion of passivity for state space representations of nonlinear systems, pioneered by Willems in the early 1970's, [3], [4], was instrumental for much research on nonlinear equilibrium stabilization. Key contributions in this area were made in the early 1980's by Hill and Moylan in [5], [6], [7], [8], and later by Byrnes, Isidori, and Willems, in their landmark paper [9]. More recently, in a number of papers [10], [11], [12], Shiriaev and Fradkov addressed the problem of stabilizing compact invariant sets for passive nonlinear systems. Their work is a direct extension of the equilibrium stabilization results by Byrnes, Isidori, and Willems in [9].

The theory presented in this paper generalizes the equilibrium theory of [9], as well as the results in [10], [11], [12]. We investigate the stabilization of a closed set Γ, not necessarily compact, which is open-loop invariant and contained in the zero level set of the storage function. Our results answer this question: *when is it that a passivity-based controller makes Γ stable, attractive, or asymptotically stable for the closed-loop system?* Even in the special case when Γ is an equilibrium, our theory yields novel results, among them necessary and sufficient conditions for the passivity-based asymptotic stabilization of the equilibrium in question without imposing that the storage function be positive definite. The theory in [9], and [10], [11], [12] does not handle this situation.

We show that at the heart of the set stabilization problem under investigation lies the following *reduction problem* for a dynamical system $\Sigma : \dot{x} = f(x)$: *Consider two closed sets Γ and $\mathcal{O}$, with $\Gamma \subset \mathcal{O}$, which are invariant for Σ; suppose that Γ is stable, attractive, or asymptotically stable for the restriction of Σ to $\mathcal{O}$. When is it that Γ is stable, attractive, or asymptotically stable with respect to the whole state space?* This problem is investigated in depth in Part I of this work, and our main stabilization theorems are a direct consequence of the theory therein.

As an application of our passivity-based stabilization results, we present a coordination problem for two kinematic unicycles: design a control law that makes the unicycles converge at a distance $\Delta > 0$ from each other. Moreover, it is desired that the unicycles asymptotically face each other. This problem can be formulated in the set stabilization framework, whereby the underlying set is not compact.

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II. Preliminaries and Problem Statement

In this paper we consider the following control-affine system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y &= h(x) \end{aligned} \tag{1}$$

with state space $\mathcal{X} \subset \mathbb{R}^n$, set of input values $\mathcal{U} \subset \mathbb{R}^m$ and set of output values $\mathcal{Y} \subset \mathbb{R}^m$. We assume that $\mathcal{X}$ is either an open subset of $\mathbb{R}^n$ or a smooth submanifold therein. In both cases, the restriction of a metric $\|\cdot\| : \mathbb{R}^n \to [0, +\infty)$ to $\mathcal{X}$ gives a metric on $\mathcal{X}$. We further assume that f and g_i, $i = 1, \ldots, m$, are smooth vector fields on $\mathcal{X}$, and that h is a smooth mapping.

A. Notation

In addition to the notations introduced in Part I, we use the following. Given either a smooth feedback $u(x)$ or a piecewise-continuous open-loop control $u(t) : \mathbb{R}^+ \to \mathcal{U}$, we denote by $\phi_u(t, x_0)$ the unique solution of (1) with initial condition x_0. This notation stands in contrast to $\phi(t, x_0)$ denoting the solution of the open-loop system $\dot{x} = f(x)$. For an interval I of the real line and a set $S \in \mathcal{X}$, $\phi_u(I, S)$ is defined analogously to $\phi(I, S)$. We use the standard notation $L_f V$ to denote the Lie derivative of a C^1 function V along a vector field f on $\mathcal{X}$, and $dV(x)$ to denote the differential map of V. Finally, $\mathcal{L}^1$ denotes the space of absolutely integrable scalar functions.

B. Passivity

Throughout this paper it is assumed that (1) is passive with smooth nonnegative storage function $V : \mathcal{X} \to \mathbb{R}$, i.e., V is a C^r ($r \geq 1$) nonnegative function such that, for all piecewise-continuous functions $u : [0, \infty) \to \mathcal{U}$, for all $x_0 \in \mathcal{X}$, and for all t in the maximal interval of existence of $\phi_u(\cdot, x_0)$,

$$V(\phi_u(t,x_0)) - V(x_0) \leq \int_0^t u(\tau)^\top y(\tau) d\tau,$$

where $y(t) = h(\phi_u(t,x_0))$. It is well-known (see [5]) that the passivity property above is equivalent to the two conditions

$$(\forall x \in \mathcal{X}) \ L_f V(x) \leq 0 \text{ and } \ L_g V(x) = h(x)^\top, \quad (2)$$

where $L_g V = [L_{g_1} V \ \cdots \ L_{g_m} V]$. Our main objective is to investigate the stabilization of closed sets by means of *passivity-based feedbacks* of the form

$$u = -\varphi(x), \text{ with } \varphi(\cdot)\Big|_{h(x)=0} = 0, \ h(x)^\top \varphi(x)\Big|_{h(x)\neq 0} > 0, \quad (3)$$

where $\varphi : \mathcal{X} \to \mathcal{U}$ is a smooth function. The class of passivity-based feedbacks in (3) includes that of output feedback controllers $u = -\varphi(h(x))$ commonly used in the literature on passive systems.

C. Problem Statement

The main objective of this paper is the stabilization of a closed set Γ using passivity-based feedback for system (1).

Set Stabilization Problem. Given a closed set $\Gamma \subset V^{-1}(0) = \{x \in \mathcal{X} : V(x) = 0\}$ which is positively invariant for the open-loop system in (1), we seek to find conditions under which a passivity-based feedback of the form (3) makes Γ: (a) stable; (b) semi-attractive (or attractive when Γ is compact); and, (c) semi-asymptotically stable (or asymptotically stable when Γ is compact) for the closed-loop system. We also seek to solve the global version of each of the problems above.

The rationale behind passivity-based feedback is the following. Using (2) and the properties of the passivity-based feedback (3), the time derivative of the storage function V along trajectories of the closed-loop system formed by (1) with feedback (3) is given by

$$\begin{aligned}\frac{dV(\phi_u(t,x_0))}{dt} &= L_f V(\phi_u(t,x_0)) \\ &\quad - L_g V(\phi_u(t,x_0))\varphi(\phi_u(t,x_0)) \\ &\leq -h(\phi_u(t,x_0))^\top \varphi(\phi_u(t,x_0)) \leq 0.\end{aligned} \quad (4)$$

Thus, a passivity-based feedback renders the storage function V nonincreasing for the closed-loop system. One expects that if the system enjoys suitable properties, then the storage function should decrease asymptotically to zero and the solutions should approach a subset of $V^{-1}(0)$, hopefully the set Γ. The argument, by now standard, found in the p , shows that the positive limit set of any bounded closed-loop solution, $L_u^+(x_0)$, is contained in $h^{-1}(0)$. Moreover, since a passivity-based feedback vanishes on $h^{-1}(0)$, and since $L_u^+(x_0)$ is an invariant set for the closed-loop system, it follows that $L_u^+(x_0)$ is also an invariant set for the open-loop system. Letting $\mathcal{O}$ denote the *maximal* set contained in $h^{-1}(0)$ which is open-loop invariant, then the positive limit set of any bounded closed-loop solution must be contained in $\mathcal{O}$ (i.e., any bounded closed-loop solution must converge to $\mathcal{O}$). Since $L_f V \leq 0$, $V^{-1}(0)$ is an invariant set for the open-loop system. Moreover, since V is nonnegative, any point $x \in V^{-1}(0)$ is a local minimum of V and hence $dV(x) = 0$. Therefore, $L_g V(x) = h(x)^\top = 0$ on $V^{-1}(0)$, and so $\Gamma \subset V^{-1}(0) \subset h^{-1}(0)$. From this we have that $\mathcal{O}$ is not empty and that

$$\Gamma \subset V^{-1}(0) \subset \mathcal{O} \subset h^{-1}(0). \quad (5)$$

It is then clear that if the trajectories of the closed-loop system in a neighbourhood of Γ are bounded, the least a passivity-based feedback can guarantee is the semi-attractivity of $\mathcal{O}$; but this is not sufficient for our purposes. Notice that, on $\mathcal{O}$, $\varphi(\cdot) = 0$ and so the closed-loop dynamics on $\mathcal{O}$ coincide with the open-loop dynamics. In particular, then, $\mathcal{O}$ is an invariant set for the closed-loop system. In order to ensure the properties of stability, (semi-) attractivity, or (semi-) asymptotic stability of Γ, the open-loop system *must* enjoy similar properties *relative to* $\mathcal{O}$. The *key question*, then, is whether the fulfillment of these properties relative to $\mathcal{O}$ is also sufficient. The answer to this key question is investigated in depth in Part I of this paper.

III. Passivity-based set stabilization

When the storage function V is positive definite, and thus $\Gamma = \{0\}$ is an equilibrium, the most general stabilization result is that by Byrnes, Isidori, and Willems in [9]. This result relied on the notion of zero-state detectability.

When Γ is compact and $\Gamma = V^{-1}(0)$, Shiriaev and co-workers, in a series of papers [10], [11], [13], extended the result in [9] to this case. They used the following notion of V-detectability.

Definition III.1 (V-detectability)**.** System (1) is *locally* V*-detectable* if there exists a constant $c > 0$ such that for all $x_0 \in V^{-1}([0,c])$,

$$h(\phi(t,x_0)) = 0 \text{ for all } t \in \mathbb{R} \implies V(\phi(t,x_0)) \to 0$$

as $t \to \infty$. If $c = \infty$, the system is V*-detectable.*

A. Γ-detectability

In this paper we are interested in the general stabilization problem for a closed set Γ, not necessarily compact, nor equal to $V^{-1}(0)$, but just contained in $V^{-1}(0)$. As discussed in Section II-C, as long as the trajectories of the closed-loop system in a neighbourhood of Γ are bounded, a passivity-based feedback renders the set $\mathcal{O}$ semi-attractive. In order to guarantee stability, semi-attractivity, or semi-asymptotic stability of $\Gamma \subset \mathcal{O}$, the reduction principles in Part I suggest that Γ should be semi-asymptotically stable relative to $\mathcal{O}$ for the open-loop system. This observation motivates the next definition.

Definition III.2 (Γ-detectability)**.** System (1) is *locally* Γ*-detectable* if Γ is semi-asymptotically stable relative to $\mathcal{O}$ for the open-loop system. The system is Γ*-detectable* if Γ is globally semi-asymptotically stable relative to $\mathcal{O}$ for the open-loop system.

Remark. In our previous work [1] and [2], we used a different notion of Γ-detectability which required only semi-attractivity of Γ relative to $\mathcal{O}$. The main theoretical results in both works (Theorem III.1 in [1] and Theorem II.4 in [2]) rely on the assumption that Γ be stable relative to $V^{-1}(0)$. It turns out (see Proposition III.7 in this paper) that the notion of Γ-detectability in Definition III.2 can be replaced by our previous definition plus the assumption of stability of Γ relative to $V^{-1}(0)$.

It can be shown that the notion of Γ-detectability generalizes that of zero-state detectability, [9]. As a matter of fact, when V is positive definite, and thus $\Gamma = \{0\}$, the two notions coincide. Γ-detectability also encompasses the notion of V-detectability. When $\Gamma = V^{-1}(0)$ is a compact set, local Γ-detectability is equivalent to local V-detectability; if V is proper, the global versions coincide. The proof of these relationships are omitted due to space limitations, and will be reported elsewhere. Despite their equivalence when $\Gamma = V^{-1}(0)$ is compact, the two notions of Γ- and V-detectability have a different flavor, in that the latter notion utilizes the storage function $V(x)$ to define a property of the open-loop system, detectability, which is independent of V. On the other hand, the definition of Γ-detectability, being independent of V, is closer in spirit to the original definition of zero-state detectability. Finally, the notion of V-detectability cannot be generalized to the case when Γ is unbounded, even if $\Gamma = V^{-1}(0)$, because in this case $V(\phi(t, x_0)) \to 0$ no longer implies $\phi(t, x_0) \to V^{-1}(0)$.

B. Solution to Set Stabilization Problem

Here, we solve the set stabilization problem in Section II-C by presenting conditions guaranteeing that a passivity-based controller of the form (3) makes Γ stable, attractive, or semi-asymptotically stable for the closed-loop system. All results are straightforward consequences of the reduction principles presented in Part I, and they rely on the next fundamental observation, whose proof is omitted due to space limitations.

Proposition III.3. *Consider the passive system* (1) *with a passivity-based feedback of the form* (3)*. Then, the set $\mathcal{O}$ is locally stable near Γ for the closed-loop system.*

Next, we present conditions under which a passivity-based feedback makes Γ stable for the closed-loop system.

Theorem III.4 (Stability of Γ)**.** *Consider system* (1) *with a passivity-based feedback for the form* (3)*. Then, Γ is stable for the closed-loop system if the following conditions hold:*

(i) System (1) *is locally Γ-detectable,*

(ii) if Γ is unbounded, then the closed-loop system is locally uniformly bounded near Γ.

The statement follows directly from Theorem III.4 and Proposition III.3 in Part I. Next, we present conditions under which a passivity-based feedback makes Γ a semi-attractor for the closed-loop system.

Theorem III.5 (Semi-attractivity of Γ)**.** *Consider system* (1) *with a passivity-based feedback of the form* (3)*. Then, Γ is a global attractor [semi-attractor] for the closed-loop system if the following conditions hold:*

(i) System (1) *is Γ-detectable,*

(ii) All closed-loop solutions in $\mathcal{X}$ [in some neighbourhood of Γ] are bounded.

The local statement of this theorem can be weakened as follows. Instead of condition (i), it is enough to assume local Γ-detectability, and assumption (ii) should then be replaced by assumption (iii) in Theorem III.3 of Part I.

Proof: Recall from Section II-C, that every bounded trajectory of the closed-loop system asymptotically approaches the set $\mathcal{O}$. This follows directly from the proof of Theorem 3.2 in [9]. The global attractivity of Γ, then, is a direct consequence of Theorem III.3 in Part I. As for semi-attractivity, assumption (ii) implies that all solutions in a neighbourhood of Γ are bounded and, therefore, they asymptotically approach $\mathcal{O}$, proving that $\mathcal{O}$ is locally semi-attractive near Γ. Moreover, since by assumption (i) Γ is a global attractor relative to $\mathcal{O}$, assumption (ii) implies assumption (iii) in Theorem III.3 of Part I, once again yielding the desired result. ∎

Now the main result of this paper concerning the passivity-based stabilization of Γ.

Theorem III.6 (Semi-asymptotic stability of Γ)**.** *Consider system* (1) *with a passivity-based feedback of the form* (3)*. If Γ is compact, then Γ is asymptotically stable for the closed-loop system if, and only if, system* (1) *is locally Γ-detectable; if, in addition, all trajectories of the closed-loop system are bounded, then Γ is globally asymptotically stable for the closed-loop system if, and only if, system* (1) *is Γ-detectable. If Γ is unbounded and the closed-loop system is locally uniformly bounded near Γ, then Γ is semi-asymptotically stable for the closed-loop system if, and only if, system* (1) *is locally Γ-detectable; if, in addition, all trajectories of the closed-loop system are bounded, then Γ is globally semi-asymptotically stable for the closed-loop system if, and only if, system* (1) *is Γ-detectable.*

Proof: The sufficiency part of the theorem follows from the following considerations. By Proposition III.3, $\mathcal{O}$ is locally stable near Γ. If Γ is compact, local Γ-detectability, by Theorem III.4, implies that Γ is stable. The stability of Γ and its compactness in turn imply that all closed-loop trajectories in some neighbourhood of Γ are bounded. Since all bounded trajectories asymptotically approach $\mathcal{O}$, $\mathcal{O}$ is locally semi-attractive near Γ. If all trajectories of the closed-loop system are bounded, then $\mathcal{O}$ is globally attractive. Theorem III.2 in Part I yields the required result.

Now suppose that Γ is unbounded. By local uniform boundedness near Γ we have that all closed-loop solutions in some neighbourhood of Γ are bounded and hence $\mathcal{O}$ is locally semi-attractive near Γ. Once again, if all closed-loop trajectories are bounded, then $\mathcal{O}$ is globally attractive. The required result now follows from Theorem III.7 in Part I.

The various necessity statements follow from the following basic observation. Any passivity-based feedback of the form (3) makes $\mathcal{O}$ an invariant set for the closed-loop system (see Section II-C). Therefore, if Γ is [globally] semi-asymptotically stable for the closed-loop system, necessarily Γ is [globally] semi-asymptotically stable relative to $\mathcal{O}$ for the closed-loop system. In other words, (1) is necessarily locally Γ-detectable [Γ-detectable]. ■

We conclude this section with the following result, which gives conditions that are alternatives to the Γ-detectability assumption.

Proposition III.7. *Theorems III.4, III.5, and III.6 still hold if the local Γ-detectability [Γ-detectability] assumption is replaced by the following condition:*

(i') Γ is stable relative to $V^{-1}(0)$ and Γ is [globally] semi-attractive relative to $\mathcal{O}$.

The proof of this proposition relies on essentially identical arguments as those used to prove the reduction principles in Theorems III.3 and III.4 of Part I, and therefore it is omitted. One may find condition (i') in this proposition easier to check than Γ-detectability. This is because verifying whether Γ is stable relative to $V^{-1}(0)$ does not require finding the maximal open-loop invariant subset $\mathcal{O}$ of $h^{-1}(0)$; moreover, checking that Γ is semi-attractive relative to $\mathcal{O}$ amounts to checking the familiar condition

$$h(\phi(t,x_0)) \equiv 0 \implies \phi(t,x_0) \to \Gamma \text{ as } t \to +\infty.$$

Note that, in the framework of [9] and [11], the requirement that Γ be stable relative to $V^{-1}(0)$ is trivially satisfied because in these references it is assumed that $\Gamma = V^{-1}(0)$. We end this section by the following remarks.

Remark. The results in [9] and [11] dealing with the special case of compact Γ where $\Gamma = V^{-1}(0)$ $(= \{0\})$, become corollaries of our main result, Theorem III.6.

Remark. The theory in [9] and [11] does not handle the special case when Γ is compact and $\Gamma \subsetneq V^{-1}(0)$, while our theory does. This case includes the important situation when one wants to stabilize an equilibrium ($\Gamma = \{0\}$) but the storage is only positive semi-definite. As a matter of fact, by Theorem III.6, in order for a passivity-based feedback to asymptotically stabilize the equilibrium, it is necessary and sufficient that the control system be Γ-detectable, in the sense of Definition III.2.

IV. Application

In this section we present an application of the results in Section III for stabilizing a non-compact goal set. The example is a coordination problem for two kinematic unicycles with states $x = (x_1, x_2, x_3)$ and $z = (z_1, z_2, z_3)$, state space $\mathcal{X} = \mathbb{R}^2 \times S^2 \times \mathbb{R}^2 \times S^1$, and dynamics

$$\begin{aligned} \dot{x}_1 &= u_1 \cos x_3 & \dot{z}_1 &= v_1 \cos z_3 \\ \dot{x}_2 &= u_1 \sin x_3 & \dot{z}_2 &= v_1 \sin z_3 \\ \dot{x}_3 &= u_2 & \dot{z}_3 &= v_2 \end{aligned} \tag{6}$$

where $(x_1, x_2), (z_1, z_2) \in \mathbb{R}^2$ are the Cartesian coordinates of the unicycles, $x_3, z_3 \in S^1$ are the headings, and $u = (u_1, u_2, v_1, v_2)$ is the control input, with u_1, v_1 the linear velocities and u_2, v_2 the angular ones. We denote $\chi = \mathrm{col}(x, z)$, and let $d_1(\chi) = x_1 - z_1$, $d_2(\chi) = x_2 - z_2$, and $\theta(\chi) = \arg(d_1(\chi) + i\, d_2(\chi))$, with $\theta \in S^1$. We study the following coordination problem for (6): make the unicycles meet at a distance $\Delta > 0$ facing each other. This problem can be addressed as the stabilization of the unbounded set

$$\begin{aligned} \Gamma = \Big\{ \chi \in \mathcal{X} : & \sqrt{d_1(\chi)^2 + d_2(\chi)^2} = \Delta, \\ & z_3 = \theta(\chi),\ x_3 = \theta(\chi) + \pi \Big\}. \end{aligned} \tag{7}$$

Since system (6) is driftless, given any positive semi-definite C^1 function $V(\chi)$, the system is passive with storage V and output $L_g V^\top$. Consider the storage function

$$V = \frac{1}{4}\left[(x_1 - z_1)^2 + (x_2 - z_2)^2 - \Delta^2\right]^2, \tag{8}$$

and notice that $\Gamma \subsetneq V^{-1}(0)$. It can be shown that other choices of storage V such that $\Gamma = V^{-1}(0)$, or Γ equal to a connected component of $V^{-1}(0)$, give obstructions to Γ-detectability.

For any choice of control inputs (u_2, v_2), system (6), viewed as a system with input (u_1, v_1) and output

$$y = (d_1^2 + d_2^2 - \Delta^2)\begin{bmatrix} d_1 \cos x_3 + d_2 \sin x_3 \\ -(d_1 \cos z_3 + d_2 \sin z_3) \end{bmatrix}, \tag{9}$$

is passive. Our design strategy is this: pick control laws (u_2, v_2) to ensure Γ-detectability and then pick a passivity-based feedback for (u_1, v_1). Consider the following control law, with $k > 0$ a design parameter,

$$\begin{aligned} u_1 &= -k y_1 \\ v_1 &= -k y_2 \\ u_2 &= d_1 \sin x_3 - d_2 \cos x_3 \\ v_2 &= -d_1 \sin z_3 + d_2 \cos z_3. \end{aligned} \tag{10}$$

This control law renders the goal set Γ, given in (7), semi-asymptotically stable. Figure 1 shows simulation results for $k = 1$ and the following initial conditions: $(x^1, z^1) = (-3.5, 1.5, 0, -3.5, 1, 0)$, $(x^2, z^2) = (-4, -2, \frac{-3\pi}{4}, 2, -2, \frac{\pi}{4})$ and $(x^3, z^3) = (-2, 1, \frac{\pi}{3}, 3, 2, \frac{\pi}{3})$. The corresponding evolution of the storage function is depicted in Figure 2, while the heading errors are in Figure 3.

In order to prove that Γ is semi-asymptotically stabilized, in light of Theorem III.6, we need to show that the system

$$\begin{aligned} \dot{x}_1 &= u_1 \cos x_3 & \dot{z}_1 &= v_1 \cos z_3 \\ \dot{x}_2 &= u_1 \sin x_3 & \dot{z}_2 &= v_1 \sin z_3 \\ \dot{x}_3 &= d_1 \sin x_3 - d_2 \cos x_3 & \dot{z}_3 &= -d_1 \sin z_3 + d_2 \cos z_3 \end{aligned} \tag{11}$$

is locally Γ-detectable and that the closed-loop system formed by (6) and (10) is locally uniformly bounded near Γ.

1) Detectability: In order to check local Γ-detectability, we take system (11) with $(u_1, v_1) = (0, 0)$ and find the largest invariant subset contained in $h^{-1}(0)$, with h given in (9). Note that in this setting the unicycles do not translate and so d_1 and d_2 are constant. Suppose that the initial

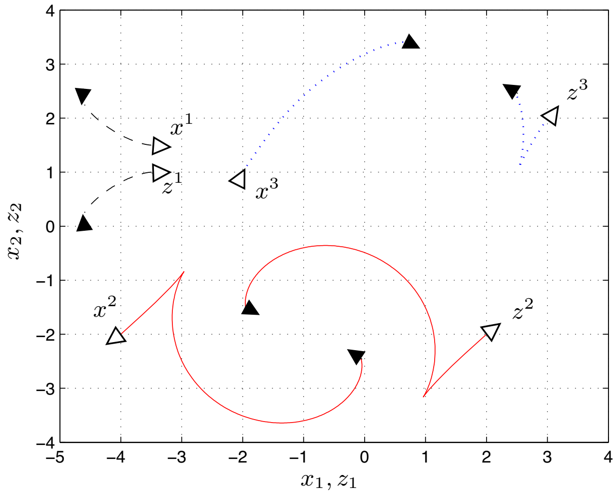

Fig. 1. Motion of the two unicycles on the plane.

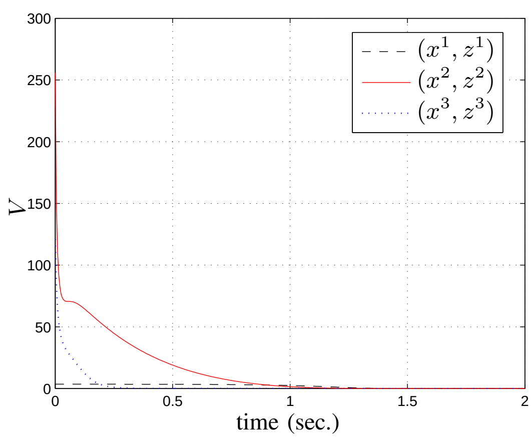

Fig. 2. Storage function V along solutions.

condition $\chi_0 \in h^{-1}(0)$. In order for $\phi(t, \chi_0) \in h^{-1}(0)$ for all $t \in \mathbb{R}$, it must be that the time derivatives of the outputs along solutions are identically zero; this occurs if and only if either $d_1^2(\chi_0) + d_2^2(\chi_0) = \Delta^2$ (i.e., if $V(\chi_0) = 0$), or

$$(d_1 \sin x_{30} - d_2 \cos x_{30}) = 0, \quad (d_1 \sin z_{30} - d_2 \cos z_{30}) = 0.$$

On $h^{-1}(0)$, when $d_1^2 + d_2^2 \neq \Delta^2$, the identities above can only be satisfied if $d_1(\chi_0) = d_2(\chi_0) = 0$. It then follows that

$$\mathcal{O} = V^{-1}(0) \cup \{\chi \in \mathcal{X} : d_1(\chi) = d_2(\chi) = 0\}.$$

Let $\mathcal{N}(\Gamma) = \mathcal{X} \backslash \{\chi : d_1(\chi) = d_2(\chi) = 0\}$. Clearly, $\mathcal{N}(\Gamma)$ is a neighborhood of $V^{-1}(0)$ and therefore also a neighborhood of Γ. Moreover, $\mathcal{N}(\Gamma) \cap \mathcal{O} = V^{-1}(0)$. Consider the restriction to the set $V^{-1}(0)$ of the open-loop dynamics in (11):

$$\begin{aligned} \dot{x}_1 &= 0 & \dot{z}_1 &= 0 \\ \dot{x}_2 &= 0 & \dot{z}_2 &= 0 \\ \dot{x}_3 &= -\Delta \sin(x_3 - \theta(\chi) - \pi) & \dot{z}_3 &= -\Delta \sin(z_3 - \theta(\chi)). \end{aligned}$$

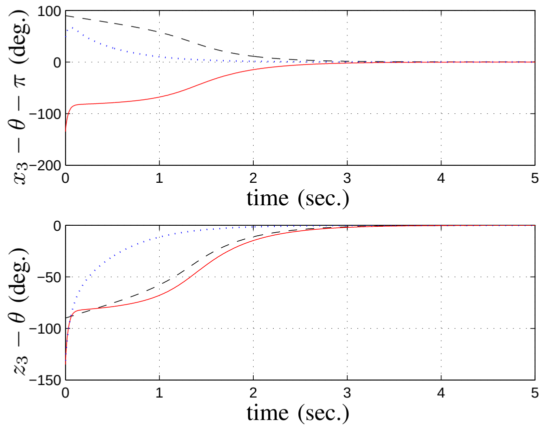

Fig. 3. Heading errors.

The time derivative of $\theta(\chi)$ along the above dynamics is zero, and so $\theta(\chi)$ is a first integral. Recall that $\Gamma = \{\chi \in V^{-1}(0) : x_3 = \theta(\chi) + \pi, \ z_3 = \theta(\chi)\}$. Given any $\chi_0 \in V^{-1}(0)$, the equilibria $x_3 = \theta(\chi_0) + \pi$, $z_3 = \theta(\chi_0)$ of the $\dot{x}_3$ and $\dot{z}_3$ equations are almost globally asymptotically stable with domains of attraction, respectively, $\{x_3 \in S^1 : x_3 \neq \theta(\chi_0)\}$ and $\{z_3 \in S^1 : z_3 \neq \theta(\chi_0) + \pi\}$. This readily implies that Γ is almost globally semi-asymptotically stable relative to $V^{-1}(0) = \mathcal{N}(\Gamma) \cap \mathcal{O}$, with domain of attraction $V^{-1}(0) \backslash (\{\chi : x_3 = \theta(\chi)\} \cup \{\chi : z_3 = \theta(\chi) + \pi\})$, and therefore also that Γ is almost globally semi-asymptotically stable relative to $\mathcal{O}$, with domain of attraction

$$\begin{aligned} \mathcal{O} \backslash \big(\{\chi : d_1(\chi) = d_2(\chi) = 0\} \cup \{\chi : x_3 = \theta(\chi)\} \\ \cup \{\chi : z_3 = \theta(\chi) + \pi\} \big). \end{aligned}$$

Therefore, system (11) is locally Γ-detectable, and actually almost globally so.

2) Local uniform boundedness: Since x_3 and z_3 are variables in S^1, we only need to prove the uniform boundedness property for (x_1, x_2) and (z_1, z_2). Notice, first of all, that the closed-loop system is a complete vector field, that is, all solutions are defined over the entire real line. This follows from the fact that V is nonincreasing along closed-loop solutions, that u_1 and v_1 are bounded as,

$$|u_1|, |v_1| \leq 2\sqrt{V}\sqrt{d_1^2 + d_2^2} \leq 2\sqrt{V}\big(2\sqrt{V} + \Delta^2\big)^{\frac{1}{2}}, \quad (12)$$

and from the fact that x_3 and z_3 are variables in S^1, a compact set. The derivative of the storage function along closed-loop solutions reads as

$$\begin{aligned} \dot{V} = -\,4V\Big[&(d_1 \cos x_3 + d_2 \sin x_3)^2 \\ &+ (d_1 \cos z_3 + d_2 \sin z_3)^2\Big]. \end{aligned} \quad (13)$$

We will show that for all initial conditions in some neighborhood of Γ, the term in square brackets in (13) is bounded away from zero. It is easy to see that

$$(d_1 \cos x_3 + d_2 \sin x_3)^2 = (d_1^2 + d_2^2) \cos^2(x_3 - \theta).$$

Since V is nonincreasing, if $d_1(\chi_0)^2 + d_2(\chi_0)^2 \neq 0$ and $V(\chi_0) < (1/4)\Delta^4$ then

$$d_1^2(\phi_u(t,\chi_0)) + d_2^2(\phi_u(t,\chi_0)) \geq \Delta^2 - 2\sqrt{V(\chi_0)} > 0$$

for all $t \geq 0$. Therefore, for the purpose of showing that the term in square brackets in (13) is bounded away from zero, it is enough to show that there exists a neighborhood $\mathcal{N}(\Gamma) \subset \{V < (1/4)\Delta^4\}$ such that all closed-loop solutions originating in $\mathcal{N}(\Gamma)$ yield, for all $t \geq 0$, $\cos^2(x_3(t) - \theta(t)) \geq 1/2$. Let

$$W(\chi) = \frac{1}{2}\left[x_3 - \theta(\chi) - \pi\right]^2.$$

Taking the derivative of W along the closed-loop vector field (6), (10), and using simple trigonometric identities lead to

$$\begin{aligned}\dot{W} &= (x_3 - \theta - \pi)(\dot{x}_3 - \dot{\theta}) \\ &= -(x_3 - \theta - \pi)\sin(x_3 - \theta - \pi)\sqrt{d_1^2 + d_2^2}\Big(1 + \\ &\quad - \frac{u_1}{d_1^2 + d_2^2}\Big) + (x_3 - \theta - \pi)\sin(z_3 - \theta)\frac{v_1}{\sqrt{d_1^2 + d_2^2}}.\end{aligned}$$

Since, for all $a \in [-\pi, \pi]$, $a \sin a = |a| \sin |a|$, we have

$$\begin{aligned}\dot{W} &\leq -\sqrt{2W}\sin(\sqrt{2W})\sqrt{d_1^2 + d_2^2}\Big(1 - \frac{|u_1|}{d_1^2 + d_2^2}\Big) \\ &\quad + \sqrt{2W}\frac{|v_1|}{\sqrt{d_1^2 + d_2^2}}.\end{aligned}$$

Note that, when $u_1 = v_1 = 0$, if $W(\chi_0) < \pi^2/2$, then the solution asymptotically converges to $\{W = 0\}$. Moreover, given any c, with $0 < c < \pi^2/2$, there exists $U > 0$ such that, for $|u_1|, |v_1| < U$ the set $\{\chi \in \mathcal{X} : W(\chi) \leq c\}$ is positively invariant. Pick $c = 1/2(\pi/4)^2$, and let U be as above.

Given any $V_0 > 0$, by the inequalities in (12) and the fact that V is nonincreasing along solutions of the closed-loop system, for any initial condition $\chi_0 \in \{\chi \in \mathcal{X} : V(\chi) \leq V_0\}$, we have

$$|u_1(t)|, |v_1(t)| \leq 2\sqrt{V_0}\big(2\sqrt{V_0} + \Delta^2\big)^{\frac{1}{2}}.$$

Let V_0 be small enough that $2\sqrt{V_0}\big(2\sqrt{V_0} + \Delta^2\big)^{\frac{1}{2}} < U$ and $V_0 < (1/4)\Delta^4$. Consider the set

$$\mathcal{N}(\Gamma) = \{\chi : V < V_0\} \cap \{\chi : W < 1/2(\pi/4)^2\}.$$

On Γ, $V = 0$ and $W = 0$, so $\mathcal{N}(\Gamma)$ is a neighborhood of Γ. By construction, the set $\mathcal{N}(\Gamma)$ is positively invariant. In particular, for all $\chi_0 \in \mathcal{N}(\Gamma)$ and all $t \geq 0$, $|x_3(t) - \theta(t) - \pi| < \pi/4$, and hence

$$\cos^2(x_3(t) - \theta(t) - \pi) > 1/2,$$

as required. Going back to the derivative of the storage function in (13), we have obtained that for all $\chi_0 \in \mathcal{N}(\Gamma)$, and for all $t \geq 0$, $\dot{V} \leq -2(\Delta^2 - 2\sqrt{V_0})V$, so that, for all $t \geq 0$, $V(\phi_u(t,\chi_0)) \leq V_0 e^{-ct}$, where $c = 2(\Delta^2 - 2\sqrt{V_0}) > 0$. Now notice that

$$|\dot{x}_1| \leq |u_1| \leq 2\sqrt{V}(\Delta^2 + 2\sqrt{V}).$$

The same inequality holds for $|\dot{x}_2|$, $|\dot{z}_1|$, and $|\dot{z}_2|$. Since $t \mapsto V(\phi_u(t,\chi_0))$ is exponentially decreasing, $\dot{x}_1(\cdot)$ belongs to $\mathcal{L}_1$, and hence x_1, (and similarly x_2, z_1, and z_2) is bounded. Moreover, the bound is independent of the initial condition in $\mathcal{N}(\Gamma)$, showing that the closed-loop system is locally uniformly bounded near Γ. This concludes the proof that Γ is semi-asymptotically stabilized by the feedback (10).

Remark. It can be shown that *all* trajectories of the closed-loop system are bounded, and we have seen earlier that system (11) is almost globally Γ-detectable. We conjecture that the controller (10) yields almost global semi-asymptotic stabilization of Γ.

V. Conclusions

New results for the stabilization of closed invariant sets for passive systems have been presented. Using novel reduction principles, presented in Part I, these results provide conditions for set stability, semi-attractivity, and semi-asymptotic stability using passivity-based feedback. The results are applied in a coordination problem for two unicycles.

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