Case Studies on Passivity-Based Stabilization of Closed Sets

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Abstract

We present a novel control design procedure for the passivity-based stabilization of closed sets which leverages recent theoretical advances. The procedure involves using part of the control freedom in order to enforce a detectability property, while the remaining part is used for passivity-based stabilization. The procedure is illustrated in four case studies of path following coordination for one or two kinematic unicycles, and variations of these problems. Among other things, we present a smooth global path following controller making the unicycle converge to an arbitrary closed and strictly convex curve, and a coordinated path following controller for two unicycles.

1 Introduction

One of the far-reaching stabilization approaches for nonlinear control systems is based on the notion of passivity. This approach involves selecting a candidate storage function whose minimum represents the stabilization objective, defining an output function as the derivative of the storage in the direction of the input vector field, and checking whether the system with this output enjoys a property of detectability. If it does, then stabilization of the minimum of the storage is accomplished by a simple static output feedback controller with arbitrarily small gain. At times, the selection of the storage function may derive from trial and error, but more often such function arises from physical considerations or structural properties of the system. This approach has proven to be successful in many applications and has even been used as a modeling paradigm for the important class of port-Hamiltonian systems (see, for instance, [1, 2, 3] and [4]). One of the reasons for the success of passivity-based control approaches is that they allow one to think of stability in terms of energy, and to view the stabilization problem in terms of interconnection of subsystems exchanging energy. This particular point of view emerges elegantly in the aforementioned work on port-controlled Hamiltonian systems by Ortega, van der Schaft, and others, and has its foundations in the pioneering work of [5, 6]. Subsequently to Willems' work, many authors focused on the *equilibrium stabilization* problem for passive systems. Important contributions in this direction were

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made by [7, 8, 9, 10], and by [11]. This latter work, in particular, established a clear relationship between stabilizability by passivity-based feedback and a detectability property referred to as *zero-state detectability*.

Most of the nonlinear systems literature, including the papers cited above, focuses on the equilibrium stabilization problem, assuming that the storage function is positive definite. In certain applications, researchers use positive semi-definite storage functions and design passivity-based controllers to stabilize their zero level sets. Such results are common in the multi-agent systems literature whenever controllers are designed based on so-called artificial potentials. Two representative works in this area are those by [12], and [13]. On the theoretical side, the equilibrium theory of Byrnes-Isidori-Willems finds a straightforward extension to the stabilization of sets in the work of [14] where, once again, the goal set is the zero level set of a positive semi-definite storage function and, additionally, it is assumed to be compact. Imposing that the goal set coincides with a level set of the storage is too restrictive, because it inextricably links the control objective to the storage function and limits the flexibility of passivity-based control design. Our recent work, [15], overcomes this problem by allowing the goal set to be a *subset* of the zero level set of the storage function, without imposing that it be compact. The end result in [15], reviewed in Section 2.1 of this paper, is a set of necessary and sufficient conditions for a passivity-based feedback to stabilize a given goal set, expressed in terms of a new notion of Γ-detectability (where Γ denotes the goal set) that generalizes the zero-state detectability property of Byrnes-Isidori-Willems. In the setting of [15], the property of Γ-detectability corresponds to asymptotic stability of Γ when the system dynamics are restricted to a special invariant subset of the state space.

In this paper we leverage the theory in [15] and present a control design procedure for passivity-based stabilization of closed sets. The idea behind the procedure is to use part of the control freedom to enforce detectability, while the remaining part is used for passivity-based stabilization. Whenever feasible, this methodology has the advantage of simplifying the control design, because stabilizing the goal set Γ amounts to designing a stabilizer for a system of smaller dimension, so the dimensionality of the problem is effectively reduced. The control design procedure is presented in Section 2.2. To illustrate the procedure, in Sections 3 to 6 we present four case studies concerning the path following problem for one kinematic unicycle, the coordinated path following problem for two unicycles, and variations of these two problems. Our examples have independent interest, but their primary objective is to elucidate different aspects of the theory in [15], and demonstrate the design flexibility gained by eliminating the requirement that the goal set coincides with the zero level set of the storage function. Of course, ours is not the only possible approach to solving these case studies.

In this paper we use the following notation, we denote by $\mathbb{R}^+$ the positive real line $[0, +\infty)$, and by S^1 the set $\mathbb{R} \bmod 2\pi$, where two scalars x and $x + 2\pi$ are identified. If A and B are two matrices or vectors, $\mathrm{col}(A, B)$ denotes the matrix $[A^\top \;\; B^\top]^\top$, and $\mathrm{blockdiag}(A, B)$ denotes the block-diagonal matrix with blocks A and B. By $\phi(t, x_0)$ we denote the solution of $\dot{x} = f(x)$ with initial condition x_0. Given an interval I of the real line and a set $S \in \mathcal{X}$, denote by $\phi(I, S)$ the set $\phi(I, S) := \{\phi(t, x_0) : t \in I, x_0 \in S\}$. We use $\| \cdot \|_S$ to denote the point-to-set distance to a set $S \subset \mathcal{X}$, $B_\alpha(x)$ an open ball of radius α centered at x, and $B_\alpha(S)$ the set of points with distance $< \alpha$ to S. Denote by $\mathcal{N}(S)$ a generic open neighbourhood of S. We use the standard notation $L_f V$ to denote the Lie derivative of a C^1 function V along a vector field f on $\mathcal{X}$.

2 Passivity-based Set Stabilization

In this section we review recent results on the stabilization of closed sets by means of passivity-based feedback. Consider the control-affine system

$$
\begin{aligned}
\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i := f(x) + g(x)u \\
y &= h(x)
\end{aligned}
\tag{1}
$$

with state space $\mathcal{X} \subset \mathbb{R}^n$, set of input values $\mathcal{U} \subset \mathbb{R}^m$ and set of output values $\mathcal{Y} \subset \mathbb{R}^m$. We assume that $\mathcal{X}$ is either an open subset of $\mathbb{R}^n$ or a smooth submanifold therein. Further, f and g_i, $i = 1, \ldots, m$, are smooth vector fields on $\mathcal{X}$, and h is a smooth mapping.

Suppose that system (1) is passive with C^1 nonnegative storage function $V : \mathcal{X} \to \mathbb{R}$. In other words, for all piecewise-continuous functions $u : [0, \infty) \to \mathcal{U}$, for all $x_0 \in \mathcal{X}$, and for all t in the maximal interval of existence of the solution $x(t)$, one has the dissipation inequality

$$
V(x(t)) - V(x_0) \leq \int_0^t u(\tau)^\top y(\tau) d\tau,
$$

where $y(t) = h(x(t))$. For smooth control-affine systems, the above passivity property can be equivalently stated as follows ([7])

$$
(\forall x \in \mathcal{X}) \ L_f V(x) \leq 0 \text{ and } L_g V(x) = h(x)^\top, \tag{2}
$$

where $L_g V = [L_{g_1} V \ \cdots \ L_{g_m} V]$.

2.1 Preliminary definitions and set stabilization theorem

Definition 2.1. A smooth function $u = -\varphi(x)$, where $\varphi(x)$ is such that $\varphi(x) = 0$ whenever $h(x) = 0$, and $h(x)^\top \varphi(x) > 0$ whenever $h(x) \neq 0$, is called a *passivity-based feedback (PBF) with respect to the output* $h(x)$.

The simplest example of PBF is the function $\varphi(x) = -h(x)$. Before presenting the stabilization result, we review the basic notions of set stability that are needed in the sequel. The next few definitions concern a smooth system $\Sigma : \dot{x} = f(x), \ \ x \in \mathcal{X}$, and a closed set $\Gamma \subset \mathcal{X}$.

Definition 2.2. The set Γ is

1. *stable* for Σ if, for all $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma)) \subset B_\varepsilon(\Gamma)$.

2. an *attractor* for Σ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all $x_0 \in \mathcal{N}(\Gamma)$, $\lim_{t\to\infty} \|\phi(t, x_0)\|_\Gamma = 0$. It is a *global attractor* if it is a attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$. It is an *almost global* attractor if it is an attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$ minus a set of measure zero.

3. *[globally, almost globally] asymptotically stable* for Σ if it is stable and attractive [globally attractive, almost globally attractive] for Σ.

All stability notions in Definition 2.2 can be relativized to a subset of the state space as follows.

Definition 2.3. Let $\mathcal{O} \subset \mathcal{X}$ be such that $\mathcal{O} \cap \Gamma \neq \emptyset$. We say that Γ is *stable relative to* $\mathcal{O}$ for Σ if, for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma) \cap \mathcal{O}) \subset B_\varepsilon(\Gamma)$. Similarly, one modifies all other notions in Definition 2.2 by restricting initial conditions to lie in $\mathcal{O}$.

Next, we define a notion of boundedness near a set, uniform in some ball around each point of the set.

Definition 2.4. Σ is *locally uniformly bounded (LUB) near* Γ if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$.

We now return to the control system (1) and define a notion of detectability that is closely related to stabilizability by passivity-based feedback. Let $\mathcal{O}$ denote the *maximal* open-loop invariant set contained in $h^{-1}(0)$, that is, the set with the property that if $\hat{\mathcal{O}}$ is any other open-loop invariant set contained in $h^{-1}(0)$, then $\hat{\mathcal{O}} \subset \mathcal{O}$. If (1) is linear time-invariant, then $\mathcal{O}$ is the unobservable subspace.

Definition 2.5. System (1) is *locally* Γ*-detectable* if Γ is asymptotically stable relative to $\mathcal{O}$ for the open-loop system, and Γ*-detectable* if Γ is globally asymptotically stable relative to $\mathcal{O}$ for the open-loop system.

In the LTI setting, when Γ is the origin, Γ-detectability coincides with the classical notion of detectability. The next stabilization result was presented in [15].

Theorem 2.6. Consider system (1) with a PBF. If Γ is compact, then

- Γ is asymptotically stable for the closed-loop system if, and only if, system (1) is locally Γ-detectable,
- if all trajectories of the closed-loop system are bounded, then Γ is globally asymptotically stable for the closed-loop system if, and only if, system (1) is Γ-detectable.

If Γ is unbounded and the closed-loop system is LUB near Γ, then

- Γ is asymptotically stable for the closed-loop system if, and only if, system (1) is locally Γ-detectable.
- if all trajectories of the closed-loop system are bounded, then Γ is globally asymptotically stable for the closed-loop system if, and only if, system (1) is Γ-detectable.

In the case when V is positive definite and $\Gamma = V^{-1}(0) = \{0\}$ is an equilibrium, an analogous result to Theorem 2.6 was proved by [11]. This results was later extended by [14] to handle the case when V isn't positive definite, but still $\Gamma = V^{-1}(0)$ and Γ is a compact set. Theorem 2.6 encompasses both results and extends them in two directions, by not requiring Γ to be compact, and by allowing $\Gamma \subsetneq V^{-1}(0)$. One of the aims of this paper is to illustrate the greater flexibility in control design that these two extensions provide.

2.2 Passivity-based control design

The aim of this paper is to outline a procedure for set stabilization by means of passivity-based feedback based on Theorem 2.6, and to illustrate its various aspects through case studies.

Set stabilization procedure: Let Γ be a closed goal set that is controlled invariant (i.e., there exists a smooth feedback rendering it invariant) for (1).

1. *Candidate storage function and feedback transformation.*
   (a) Find a candidate C^1 storage function $V : \mathcal{X} \to \mathbb{R}^+$ such that $\Gamma \subset V^{-1}(0)$ and $L_f V(x) \leq 0$ for all $x \in \mathcal{X}$.
   (b) Find, if possible, a locally Lipschitz matrix-valued function $\beta_1(x) : \mathcal{X} \to \mathbb{R}^{m\times k}$, for some $k \in \{1, \ldots, m-1\}$, such that $\beta_1(x)$ has full rank k and $L_g V(x)\beta_1(x) = 0_{1\times k}$ for all $x \in \mathcal{X}$.
   (c) Let $\beta_2(x) : \mathcal{X} \to \mathbb{R}^{m\times m-k}$ be any locally Lipschitz function such that $[\beta_1(x) \quad \beta_2(x)]$ is nonsingular for all $x \in \mathcal{X}$, and define the feedback transformation
   $$u = \beta_1(x)\bar{u} + \beta_2(x)\tilde{u}, \tag{3}$$
   where $\bar{u} \in \mathbb{R}^k$ and $\tilde{u} \in \mathbb{R}^{m-k}$ are new control inputs. Define an output function $h(x) := L_{g\beta_2} V(x)^\top$.

2. Γ-*detectability enforcement.* Find, if possible, a feedback $\bar{u}(x)$ such that Γ is (globally) asymptotically stable relative to $\mathcal{O}$ for the system $\dot{x} = [f(x) + g(x)\beta_1(x)\bar{u}(x)]|_{\mathcal{O}}$, where $\mathcal{O}$ is the maximal subset of $h^{-1}(0)$ invariant under the vector field $f + g\beta_1\bar{u}$.

3. *Passivity-based stabilization.* Pick any PBF $\tilde{u}(x)$, and let $u(x) = \beta_1(x)\bar{u}(x) + \beta_2(x)\tilde{u}(x)$, where $\bar{u}(x)$ is the feedback chosen in step 2.

Remark 2.7. (a) The inputs $\bar{u}$ and $\tilde{u}$ after feedback transformation (3) represent control directions tangential and transversal to $V^{-1}(0)$, so that when $\tilde{u} = 0$ and $\bar{u}(x)$ is any smooth feedback, the set $V^{-1}(0)$ is invariant. Moreover, since for any smooth $\bar{u}(x)$ it holds that $L_{f+g\beta_1\bar{u}}V(x) = L_f V(x) + L_g V(x)\beta_1(x)\bar{u}(x) = L_f V(x) \leq 0$, the system with input $\tilde{u}$ and output $h(x) = L_{g\beta_2}V(x)^\top$ is passive. The idea then is to use $\bar{u}(x)$ to enforce Γ-detectability (step 2), while $\tilde{u}$ is chosen to be any passivity-based feedback (step 3).

(b) In step 1a, it may be possible to ensure that $L_f V \leq 0$ through a preliminary feedback $u_p(x)$, so that $L_{f+gu_p}V \leq 0$. In this case, we would let $u = u_p(x) + \hat{u}$, define $\hat{f} := f + gu_p$, and continue the procedure with $\hat{f}$ and $\hat{u}$ in place of f and u. The preliminary controller $u_p(x)$ has the role of passifying feedback.

(c) Suppose that the set Γ is expressed as the level set of C^1 functions, $\Gamma = \{x \in \mathcal{X} : \psi_1(x) = 0, \ldots, \psi_l(x) = 0\}$. In this case, the functions $\psi_i(x)$ can be used to produce guesses for the storage V by setting, for instance, $V = \psi_i^2(x)$ or $V = \psi_i^2(x) + \psi_j^2(x)$, and so on. Since we only require Γ to be a subset of $V^{-1}(0)$, there is some freedom in which of the functions to use, and how to combine them. The storage functions in the four case studies presented in this paper are chosen using this method.

(d) A feature of the set stabilization procedure is that, whenever it is feasible, it allows one to *reduce* the control design to the design of a controller $\bar{u}(x)$ that asymptotically stabilizes Γ for the system $\dot{x} = [f(x) + g(x)\beta_1(x)\bar{u}(x)]\big|_{\mathcal{O}}$, with state space $\mathcal{O}$. Typically, $\mathcal{O}$ is a submanifold of the state space, and hence the restriction of $f + g\beta_1\bar{u}$ to $\mathcal{O}$ is a system of dimension smaller than the original system (1).

(e) As shown in Proposition 2.10 below, the outcome of the control design procedure is independent of the choice of $\beta_2(x)$ in step 1c.

Proposition 2.8. The feedback $u(x)$ designed according to the procedure above has the following properties:

(a) If Γ is compact, then $u(x)$ asymptotically stabilizes it.

(b) If Γ is closed and unbounded, then $u(x)$ asymptotically stabilizes it provided that the closed-loop system is LUB near Γ.

(c) In both cases above, if all trajectories of the closed-loop system are bounded, and the Γ-detectability property enforced in step 2 of the procedure is global, then the stabilization of Γ is global as well.

Remark 2.9. When Γ is unbounded, a suitable choice of PBF $\tilde{u}(x)$ may help achieve the LUB property. This fact is illustrated in case studies 3 and 4 below.

Proof. Let $u = \beta_1(x)\bar{u}(x) + \beta_2(x)\tilde{u}$, where $\bar{u}(x)$ is as in step 2 of the procedure, and consider the system

$$\begin{aligned}\dot{x} &= [f(x) + g(x)\beta_1(x)\bar{u}(x)] + g(x)\beta_2(x)\tilde{u} \\ y &= L_{g\beta_2}V(x)^\top.\end{aligned}$$

Since $L_{f+g\beta_1\bar{u}}V(x) = L_fV(x) + L_gV(x)\beta_1(x)\bar{u}(x) = L_fV(x) \leq 0$, the system above is passive. By the construction in step 2, Γ is [globally] asymptotically stable relative to $\mathcal{O}$, and hence the system above is locally Γ-detectable [Γ-detectable]. Now the proposition follows directly from Theorem 2.6. $\square$

Proposition 2.10. Steps 2 and 3 of the set stabilization procedure are independent of the choice of $\beta_2(x)$.

Proof. We need to show that the set $\mathcal{O}$ is independent of the choice of $\beta_2(x)$. Let $\beta_1(x), \beta_2(x)$ be as in step 1 of the procedure and let $\hat{\beta}_2(x)$ be another locally Lipschitz function $\mathcal{X} \to \mathbb{R}^{m \times m-k}$ such that $[\beta_1(x) \;\; \hat{\beta}_2(x)]$ is nonsingular. Denote $\hat{h}(x) = L_{g\hat{\beta}_2}V(x)^\top$ the corresponding output. Since $[\beta_1 \;\; \beta_2]$ is nonsingular, there exist continuous matrix-valued functions $K_1(x)$ and $K_2(x)$ such that $\hat{\beta}_2(x) = \beta_1(x)K_1(x) + \beta_2(x)K_2(x)$ and therefore

$$\hat{h}(x)^\top = L_{g\hat{\beta}_2}V(x) = L_{g\beta_1}V(x)K_1(x) + L_{g\beta_2}V(x)K_2(x) = L_{g\beta_2}V(x)K_2(x) = h(x)^\top K_2(x).$$

Since the matrix $[\beta_1 \;\; \hat{\beta}_2]$ is nonsingular, the matrix-valued function $K_2 : \mathcal{X} \to \mathbb{R}^{m-k \times m-k}$ must be nonsingular. Therefore, the sets $h^{-1}(0)$ and $\hat{h}^{-1}(0)$ coincide, proving that the set $\mathcal{O}$ is the same for both outputs $h(x)$ and $\hat{h}(x)$. $\square$

The rest of this paper is dedicated to the application of the procedure above to four case studies illustrating different aspects of Theorem 2.6:

CS1. Path following control design for the kinematic unicycle and strictly convex paths. In this case, we will have $\Gamma = V^{-1}(0)$ and Γ compact.

CS2. Stabilizing the kinematic unicycle to the unit circle with a constant heading requirement on the circle. Here, $\Gamma \subsetneq V^{-1}(0)$ and Γ is compact.

CS3. Coordinated path following for two unicycles: make the unicycles follow a unit circle (no specified centre) while keeping a constant distance between each other. Here, $\Gamma \subsetneq V^{-1}(0)$ and Γ is unbounded.

CS4. Coordination of two unicycles: make two unicycles meet at a fixed distance facing each other. Here, $\Gamma \subsetneq V^{-1}(0)$ and Γ is unbounded.

3 Case study 1: path following for the kinematic unicycle

We consider the path following problem for the kinematic unicycle model with state $(x_1, x_2, x_3) \in \mathbb{R}^2 \times S^1$,

$$\begin{aligned}\dot{x}_1 &= u_1 \cos x_3 \\ \dot{x}_2 &= u_1 \sin x_3 \\ \dot{x}_3 &= u_2,\end{aligned} \tag{4}$$

and a smooth regular path $\mathcal{C} \subset \mathbb{R}^2$ which is closed and does not have self-intersections (i.e., it is a Jordan curve). The path following problem for kinematic unicycles and, more generally, for wheeled vehicles was the subject of considerable research in the 1990s. The seminal work by [16] (see also the review paper [17]) proposed a smooth time-varying control law based on the conversion of the path following problem to equilibrium stabilization by using Ferret-Serret frames moving along the path. The idea of using Frenet-Serret frames for path following is also found in [18], where a virtual target is used to make a unicycle converge to the path. Virtual targets are further explored in the recent work [19]. A global discontinuous path following controller for a circle is proposed in the work of [20]. No global solution to the unicycle path following problem has been found by means of a smooth, static, and time-invariant feedback. In this section, we present the first such global solution for the class of *strictly convex* paths, i.e., paths with strictly positive signed curvature. The next lemma provides a useful characterization of strictly convex paths.

Lemma 3.1. If $\mathcal{C}$ is a smooth Jordan curve, then the following statements are equivalent:

i. $\mathcal{C}$ is strictly convex.

ii. There exists a regular parameterization $\sigma : S^1 \to \mathbb{R}^2$ of $\mathcal{C}$ such that, for each $\theta \in S^1$, the angle of the tangent vector $\sigma'(\theta)$ is precisely $\theta \bmod 2\pi$. In other words, $\sigma'(\theta) = \|\sigma'(\theta)\| \operatorname{col}(\cos\theta, \sin\theta)$.

Proof. Let $\tilde{\sigma} : \mathbb{R} \to \mathbb{R}^2$ be a unit speed parameterization of $\mathcal{C}$, and for each t denote by $\varphi(t)$ the angle of the vector $\tilde{\sigma}'(t)$ modulo 2π. If L is the length of $\mathcal{C}$, then $\tilde{\sigma}$ is L-periodic, and we change the domain of $\tilde{\sigma}$ from $\mathbb{R}$ to $\mathbb{R} \bmod L$, so that $\tilde{\sigma}$ maps $\mathbb{R} \bmod L$ diffeomorphically onto $\mathcal{C}$.

(i) $\Rightarrow$ (ii). The curvature of $\mathcal{C}$ at a point $\tilde{\sigma}(t)$ is $\varphi'(t)$, and it is a smooth function. Since $\varphi'(t) > 0$ for all $t \in \mathbb{R}$, the function $t \mapsto \varphi(t)$ is invertible, its inverse $\varphi^{-1} : \theta \in S^1 \to t \in \mathbb{R} \bmod L$ is smooth, and the derivative of $\varphi^{-1}(\cdot)$ is positive. The function $\sigma(\theta) := \tilde{\sigma} \circ \varphi^{-1}(\theta)$ has the required properties: its derivative $\sigma'(\theta) = \tilde{\sigma}'(\varphi^{-1}(\theta))(\varphi^{-1})'(\theta)$ is never zero, and so it is a regular parametrization. Moreover, the angle of $\sigma'(\theta)$ is the same as that of $\tilde{\sigma}'(\varphi^{-1}(\theta))$, which is precisely θ.

(ii) $\Rightarrow$ (i). Let $\sigma : S^1 \to \mathbb{R}^2$ be a regular parameterization of $\mathcal{C}$ such that $\sigma'(\theta) = \|\sigma'(\theta)\| \operatorname{col}(\cos(\theta), \sin(\theta))$. The signed curvature $k(\theta)$ is given by the formula $k(\theta) = [\sigma_1'(\theta)\sigma_2''(\theta) - \sigma_2'(\theta)\sigma_1''(\theta)]/\|\sigma'(\theta)\|^3 = 1/\|\sigma'(\theta)\|$, which is everywhere positive. $\square$

Example 3.2. Suppose that $\mathcal{C}$ is a circle of radius r centred at the origin, and consider the regular parameterization $\sigma(\theta) = r \operatorname{col}(\sin\theta, -\cos\theta)$. The tangent vector at $\sigma(\theta)$ is $\sigma'(\theta) = r\operatorname{col}(\cos\theta, \sin\theta)$, whose angle is θ. Next, suppose that $\mathcal{C}$ is an ellipse with major semi-axis a and minor semi-axis b, centred at the origin. The regular parameterization

$$\sigma(\theta) = \begin{bmatrix} \frac{a^2 \sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \\ \frac{-b^2\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \end{bmatrix}$$

satisfies $\sigma'(\theta) = \|\sigma'(\theta)\| \operatorname{col}(\cos\theta, \sin\theta)$, where $\|\sigma'(\theta)\| = a^2b^2/(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}$.

We now return to the path following problem for the unicycle. Suppose that $\mathcal{C}$ is a strictly convex curve with parametrization $\sigma(\theta) : S^1 \to \mathbb{R}^2$, as in Lemma 3.1. We will design a global path following controller making the unicycle follow the curve in the counter-clockwise direction. In order to make the unicycle follow $\mathcal{C}$ in the clockwise direction, it suffices to replace θ by $-\theta$ in the definition of σ.

If $(x_1(t), x_2(t), x_3(t))$ is a solution of (4), then $x_3(t)$ is the angle of the tangent vector to the curve $(x_1(t), x_2(t))$. This fact, and the property, due to strict convexity, that the angle of $\sigma'(\theta)$ is θ, together imply that solving the path following problem is equivalent to stabilizing the controlled invariant set

$$\Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^2 \times S^1 : x_1 = \sigma_1(x_3),\ x_2 = \sigma_2(x_3)\}. \tag{5}$$

Remark 3.3. Note that the set $\bar{\Gamma} = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathcal{C}\}$ is not controlled invariant unless $u_1(t) \equiv 0$. For, if $(x_1, x_2) \in \mathcal{C}$ and the unicycle's heading is not tangent to $\mathcal{C}$, then the unicycle will leave $\mathcal{C}$. The set Γ in (5) is a connected component of the largest controlled invariant subset of $\bar{\Gamma}$ subject to the requirement that u_1 is bounded away from zero. The other connected component corresponds to motion around $\mathcal{C}$ in the clockwise direction.

Step 1: Candidate storage function. We make the obvious choice

$$V(x) = \frac{1}{2}\left[(x_1 - \sigma_1(x_3))^2 + (x_2 - \sigma_2(x_3))^2\right].$$

Note that $\Gamma = V^{-1}(0)$ and Γ is a compact set because $x_3 \in S^1$, which is compact. For the unicycle (4), $f = \operatorname{col}(0,0,0)$ and $g = [g_1 \quad g_2]$, with $g_1 = \operatorname{col}(\cos(x_3), \sin(x_3), 0)$, $g_2 = \operatorname{col}(0,0,1)$. Since $L_f V = 0$, V satisfies the requirements of step 1a of the procedure. Next, we find a feedback transformation of the form (3). We have

$$L_g V = [(x_1 - \sigma_1)\cos x_3 + (x_2 - \sigma_2)\sin x_3 \quad -(x_1 - \sigma_1)\sigma_1' - (x_2 - \sigma_2)\sigma_2'],$$

and since, by strict convexity, $\sigma'(x_3) = \|\sigma'(x_3)\| \operatorname{col}(\cos x_3, \sin x_3)$, setting $\beta_1(x) = \operatorname{col}(1, 1/\|\sigma'(x_3)\|)$ we have $L_g V(x)\beta_1(x) = 0$. Next, we need to pick a vector β_2 that is linearly independent of β_1. We choose $\beta_2 = \operatorname{col}(0, 1)$. The feedback transformation

$$u = \beta_1(x)\bar{u} + \beta_2(x)\tilde{u} = \begin{bmatrix} 1 \\ 1/\|\sigma'(x_3)\| \end{bmatrix} \bar{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}$$

guarantees that, for any smooth $\bar{u}(x)$, the system with input $\tilde{u}$ and output $y = h(x) := L_{g\beta_2} V(x)^\top$,

$$h(x) := -[x_1 - \sigma_1(x_3)]\sigma_1'(x_3) - [x_2 - \sigma_2(x_3)]\sigma_2'(x_3) \tag{6}$$

is passive.

Step 2: Γ*-detectability enforcement.*

Lemma 3.4. Let $\bar{u}(x)$ be any smooth positive feedback bounded away from 0, i.e., $\inf_x \bar{u}(x) > \varepsilon > 0$, for some $\varepsilon > 0$. Then, the maximal subset $\mathcal{O}$ of $h^{-1}(0)$ invariant under $f + g\beta_1\bar{u}$ is Γ.

Proof. If $u = \beta_1(x)\bar{u}(x)$, we have

$$\begin{aligned} \dot{x}_1 &= \bar{u}(x)\cos x_3 \\ \dot{x}_2 &= \bar{u}(x)\sin x_3 \\ \dot{x}_3 &= \frac{\bar{u}(x)}{\|\sigma'(x_3)\|}. \end{aligned}$$

Using the fact that $\sigma'(\theta) = \|\sigma'(\theta)\| \operatorname{col}(\cos\theta, \sin\theta)$, we have

$$\frac{d}{dt}\begin{bmatrix} x_1 - \sigma_1(x_3(t)) \\ x_2 - \sigma_2(x_3(t)) \end{bmatrix} = 0,$$

and so the vector $\operatorname{col}(x_1(t), x_2(t)) - \sigma(x_3(t))$ is constant. Therefore, if $\inf_x \bar{u}(x) > \varepsilon > 0$, the curve $t \mapsto \operatorname{col}(x_1(t), x_2(t))$ coincides with $\mathcal{C}$ modulo a translation. Suppose that, for suitable initial conditions, the output signal $y(t)$ is identically zero. Then, the vectors $\operatorname{col}(x_1(t) - \sigma_1(x_3(t)), x_2(t) - \sigma_2(x_3(t)))$ and $\sigma'(x_3(t))$ are orthogonal for all $t \geq 0$. Therefore, either $\operatorname{col}(x_1(t) - \sigma_1(x_3(t)), x_2(t) - \sigma_2(x_3(t)))$ is zero (i.e., $x(t) \in \Gamma$), or $\sigma'(x_3(t))$ has a constant angle. However, the angle of $\sigma'(x_3(t))$ is $x_3(t)$, whose derivative is positive. Thus, $x(t) \in \Gamma$, proving that $\mathcal{O} = \Gamma$. □

Letting $u = \beta_1(x)\bar{u} + \beta_2(x)\tilde{u}$, Lemma 3.4 guarantees that the system with input $\tilde{u}$ and output $h(x)$ is Γ-detectable.

Step 3: Passivity-based stabilization. The next result is a direct consequence of Theorem 2.6.

Proposition 3.5. For any smooth $\bar{u}(x) : \mathbb{R}^2 \times S^1 \to \mathbb{R}$ bounded away from zero, i.e., $\bar{u}(x) > \epsilon > 0$, and any PBF $\tilde{u}(x) = -\varphi(x)$ with respect to the output $h(x)$ in (6), the feedback

$$\begin{aligned} u_1 &= \bar{u}(x) \\ u_2 &= \frac{\bar{u}(x)}{\|\sigma'(x_3)\|} + \varphi(x) \end{aligned} \tag{7}$$

globally asymptotically stabilizes the set Γ in (5), and thus solves the path following problem for $\mathcal{C}$ globally.

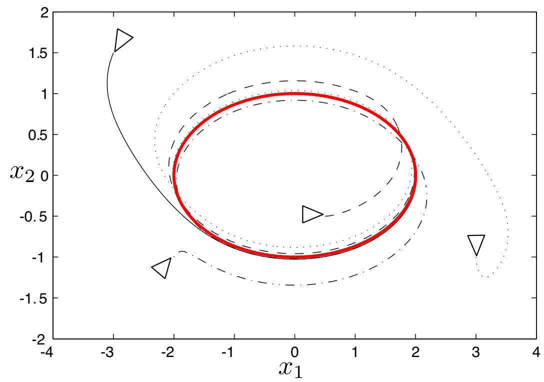

Figure 1: Simulation results for the global path following controller in (7), where $\mathcal{C}$ is an ellipse with major semi-axis length 2 and minor semi-axis length 1.

Example 3.6. If $\mathcal{C}$ is a circle of radius r centred at the origin, then a global solution to the path following problem in the counter-clockwise direction is given by the feedback

$$\begin{aligned} u_1 &= v \\ u_2 &= \frac{v}{r} + r(x_1 \cos x_3 + x_2 \sin x_3). \end{aligned}$$

If $\mathcal{C}$ is an ellipse centered at the origin with major semi-axis a and minor semi-axis b, then a global solution to the path following problem in the counter-clockwise direction is given by the feedback

$$\begin{aligned} u_1 &= v \\ u_2 &= v\frac{\mu(x)^{3/2}}{a^2 b^2} + a^2 b^2 \left[\frac{(b^2 - a^2)\sin x_3 \cos x_3}{\mu(x)^2} + \frac{x_1 \cos x_3 + x_2 \sin x_3}{\mu(x)^{3/2}} \right], \end{aligned}$$

where $\mu(x) = a^2 \sin^2 x_3 + b^2 \cos^2 x_3$. Simulation results for this controller, with $a = 2$ and $b = 1$, are displayed in Figure 1.

Remark 3.7. An important advantage of the feedback (7) is that it can be made to be compatible with any input saturation constraint. For, if the controller is subject to saturation constraints $|u_1| \leq U_1$, $|u_2| \leq U_2$, one can choose $\bar{u}(\cdot) > 0$ small enough that $\bar{u}(\cdot) \leq U_1$ and $\bar{u}/\|\sigma'(x_3)\| < U_2$. Then, choose $\varphi(y)$ to be any odd function of $h(x)$ such that $\bar{u}/\|\sigma'(x_3)\| + \sup_{\mathbb{R}} |\varphi(\cdot)| \leq U_2$.

4 Case study 2: stabilizing the unicycle to a circle with heading angle requirement

Consider again the kinematic unicycle model in (4), and the problem of stabilizing the unicycle to a unit circle centered at the origin, with a constant desired heading on the circle. This problem can be stated equivalently as the stabilization of the set

$$\Gamma = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, x_3 = a \bmod 2\pi\}, \tag{8}$$

where a is the desired reference heading.

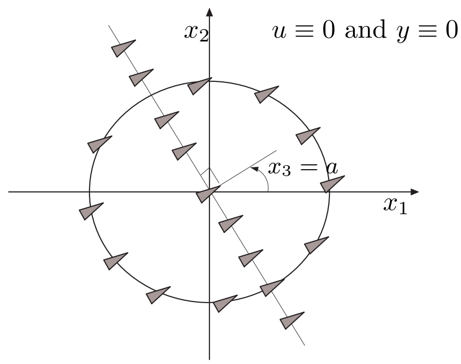

Figure 2: Failure of Γ-detectability in case study 2 when $\Gamma = V^{-1}(0)$.

Step 1: Candidate storage function. If one chooses a storage function V such that $V^{-1}(0) = \Gamma$, then a passivity-based feedback does not stabilize Γ. In order to illustrate this fact, consider the storage function $V = (x_1^2 + x_2^2 - 1)^2/2 + (x_3 - a)^2/2$. The unique value of u rendering $V^{-1}(0)$ invariant is $u = 0$, so the feedback transformation (3) becomes trivial, $u = \tilde{u}$. Since $f = 0$ for the kinematic unicycle, the system is passive with any storage function $V(x_1, x_2, x_3)$, and output

$$L_g V^\top = \begin{bmatrix} 2(x_1^2 + x_2^2 - 1)(x_1 \cos x_3 + x_2 \sin x_3) \\ x_3 - a \end{bmatrix}.$$

We now show that the system with input u and the output above is *not* Γ-detectable and hence, since Γ-detectability is a necessary condition for passivity-based stabilization, no PBF can stabilize Γ with the above choice of V. In order to check Γ-detectability, we need to find $\mathcal{O}$. Suppose that $u(t) \equiv 0$ and $L_g V(t) \equiv 0$. Then, the unicycle dynamics are stationary and $\mathcal{O} = \{x : (x_1^2 + x_2^2 - 1)(x_1 \cos a + x_2 \sin a) = 0,\ x_3 = a\}$, and all points on $\mathcal{O}$ are equilibria. Figure 2 illustrates the set of configurations of the unicycle on $\mathcal{O}$. It is clear that $\mathcal{O}$ contains and is not equal to Γ in (8). Therefore, Γ is not asymptotically stable relative to $\mathcal{O}$, and the system is not Γ-detectable. More generally, if we choose for the system a storage function $V(e_1, e_2)$, where $e_1 = (x_1^2 + x_2^2 - 1)/2$, $e_2 = x_3 - a$, and $(e_1, e_2) \mapsto V(e_1, e_2)$ is positive definite, then

$$L_{g_1} V = \frac{\partial V}{\partial e_1}(x_1 \cos x_3 + x_2 \sin x_3),$$

gives the same obstruction to Γ-detectability.

The above suggests that if one wants to stabilize Γ in (8) using a passivity-based approach, one should *not* attempt to find a storage V with the property that $V^{-1}(0) = \Gamma$. Guided by this principle, we choose a simple storage V such that $\Gamma \subsetneq V^{-1}(0)$, namely

$$V(x) = \frac{1}{4}(x_1^2 + x_2^2 - 1)^2.$$

Next, we define a feedback transformation according to step 2 of the procedure. Since $L_g V = (x_1^2 + x_2^2 - 1)[x_1 \cos x_3 + x_2 \sin x_3 \;\; 0]$, we choose $\beta_1 = \mathrm{col}(0, 1)$, so $L_g V(x)\beta_1 = 0$, and $\beta_2 = \mathrm{col}(1, 0)$, so the matrix $[\beta_1 \; \beta_2]$ is nonsingular. The feedback transformation

$$u = \beta_1(x)\bar{u} + \beta_2(x)\tilde{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{u} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tilde{u}.$$

guarantees that, for any feedback $\bar{u}(x)$, the system with input $\tilde{u}$ and output $y = h(x) := L_{g\beta_2}V(x)^\top$ below is passive,

$$h(x) := (x_1^2 + x_2^2 - 1)(x_1 \cos x_3 + x_2 \sin x_3). \tag{9}$$

Step 2: Γ-detectability enforcement.

Lemma 4.1. Let $\bar{u}(\cdot)$ be any feedback such that, for any solution $x(t)$ of $\dot{x} = f + g\beta_1\bar{u}$, $\bar{u}(x(t)) \equiv 0$ implies $V(x(t)) \equiv 0$. Then, the maximal subset of $h^{-1}(0)$ invariant under the vector field $f + g\beta_1\bar{u}$ is

$$\mathcal{O} = V^{-1}(0) \cup \{x : x_1 = x_2 = 0\}.$$

Proof. We have $f + g\beta_1\bar{u} = \mathrm{col}(0, 0, \bar{u})$, and so $x_1(t)$ and $x_2(t)$ are constant. If $h(x(t)) \equiv 0$, then either $x(t) \in V^{-1}(0)$, or $x_1(t)\cos x_3(t) + x_2(t)\sin x_3(t) \equiv 0$. If $x(t) \notin V^{-1}(0)$, then the latter identity can only be satisfied if $x_1(t) \equiv x_2(t) \equiv 0$, because otherwise we would have $x_3(t)$ =constant, implying that $\dot{x}_3(t) \equiv \bar{u}(x(t)) \equiv 0$ and this, by assumption, can only hold on $V^{-1}(0)$. □

Under the assumption of the above lemma, $\mathcal{O}$ is the union of two disconnected components, $V^{-1}(0)$ and $\{x : x_1 = x_2 = 0\}$. On $V^{-1}(0)$, $f + g\beta_1\bar{u} = \mathrm{col}(0, 0, \bar{u})$. To enforce Γ-detectability, choose $\bar{u} = -\varphi_1(x_3 - a) - \varphi_2(x_1^2 + x_2^2 - 1)\sin t$, where $\varphi_1(\cdot)$ is 2π-periodic and such that $\varphi_1(y)\sin y > 0$ for all $y \neq 0, \pi \bmod 2\pi$, and φ_2 is positive definite. If $\bar{u}(x(t)) \equiv 0$, then $x_3(t)$ is constant. Thus, $\varphi_1(x_3(t) - a)$ is constant and so $-\varphi_1(x_3(t) - a) - \varphi_2(x_1^2(t) + x_2^2(t) - 1)\sin t$ can only be zero if $x_1^2(t) + x_2^2(t) \equiv 1$. Therefore, this choice of $\bar{u}$ satisfies the assumption of Lemma 4.1. Moreover, on $V^{-1}(0)$ we have $\dot{x}_3 = -\varphi_1(x_3 - a)$. By the choice of φ_1, $x_3 = a \bmod 2\pi$ is almost globally asymptotically stable for this differential equation, with domain of attraction $x_3 \neq a + \pi \bmod 2\pi$. Thus, Γ is almost globally asymptotically stable relative to $V^{-1}(0)$, and hence almost globally asymptotically stable relative to $\mathcal{O}$ (because the set $\{x_1 = x_2 = 0\}$ has measure zero).

Step 3: Passivity-based stabilization.

Proposition 4.2. Let $\varphi_1(y)$ be a locally Lipschitz and 2π-periodic function such that $\varphi_1(y)\sin y > 0$ for all $y \neq 0, \pi \bmod 2\pi$, and let $\varphi_2 : \mathbb{R} \to \mathbb{R}^+$ be positive definite. Then, for any PBF $\varphi(x)$ with respect to the output $h(x)$ in (9), the feedback

$$\begin{aligned} u_1 &= -\varphi(x) \\ u_2 &= -\varphi_1(x_3 - a) + \varphi_2(x_1^2 + x_2^2 - 1)\sin t \end{aligned} \tag{10}$$

almost globally stabilizes the set

$$V^{-1}(0) = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1\}$$

with domain of attraction $\mathcal{D} = (\mathbb{R}^2 \times S^1) \backslash \{(x_1, x_2, x_3) : x_1 = x_2 = 0\}$, and asymptotically stabilizes the set

$$\Gamma = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, x_3 = a \bmod 2\pi\}.$$

Simulation results for the controller (10) solving case study 2 are found in Figure 3, in which we have chosen $\varphi_1(\cdot) = \sin(\cdot)$, $\varphi_2 = \|\cdot\|$, and $\varphi(x) = \arctan(h(x))$.

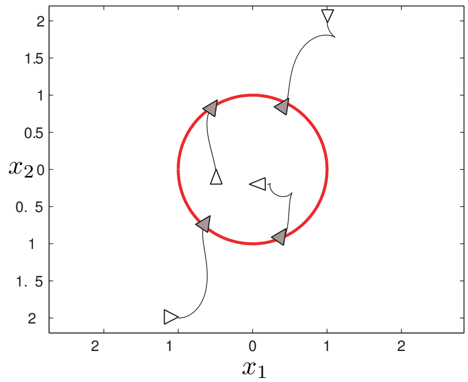

Figure 3: Simulation results for the controller in (10).

Proof. In order to handle the presence of the term $\sin t$ in the control input, consider the augmented system

$$\begin{aligned}
\dot{x}_1 &= \tilde{u}\cos x_3 \\
\dot{x}_2 &= \tilde{u}\sin x_3 \\
\dot{x}_3 &= -\varphi_1(x_3 - a) + \varphi_2(x_1^2 + x_2^2 - 1)\sin\theta \\
\dot{\theta} &= 1,
\end{aligned}$$

with $(x_1, x_2, x_3, \theta) \in \mathbb{R}^2 \times S^1 \times S^1$. For notational simplicity, we will still denote by Γ, $\mathcal{O}$, $V^{-1}(0)$, and $h^{-1}(0)$ the lift of these sets to the augmented state space. Thus, for instance, we will denote by Γ the set $\{(x_1, x_2, x_3, \theta) : (x_1, x_2, x_3) \in \Gamma\}$. We have shown in step 2 of the procedure that the system above with input $\tilde{u}$ and output $h(x)$ in (9) is passive and locally Γ-detectable. Let $\tilde{u} = -\varphi(x)$ be any PBF with respect to the output $h(x)$. By Theorem 2.6, since Γ is compact, Γ is asymptotically stable for the closed-loop system. Moreover, since V is proper, all trajectories of the closed-loop system are bounded. On $\{x_1 = x_2 = 0\}$, V has a local maximum. Therefore, for any initial condition in $\mathcal{D}$, the corresponding solution of the closed-loop system remains in $\mathcal{D}$ and converges to the maximal invariant subset of $\{\dot{V} = 0\} = h^{-1}(0)$, i.e., it converges to $\mathcal{O} \cap \mathcal{D} = V^{-1}(0)$. This fact, together with the properness of V, implies that $V^{-1}(0)$ is almost globally asymptotically stable with domain of attraction $\mathcal{D}$. □

5 Case study 3: coordinated path following for two unicycles

Consider two kinematic unicycles

$$\begin{aligned}
\dot{x}_1 &= u_1^x \cos x_3 \qquad & \dot{z}_1 &= u_1^z \cos z_3 \\
\dot{x}_2 &= u_1^x \sin x_3 \qquad & \dot{z}_2 &= u_1^z \sin z_3 \\
\dot{x}_3 &= u_2^x \qquad & \dot{z}_3 &= u_2^z,
\end{aligned} \tag{11}$$

and let $\chi = \mathrm{col}(x, z)$. For this system, we have $f = 0$ and

$$g = \mathrm{blockdiag}\left\{ \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos z_3 & 0 \\ \sin z_3 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

In this section we design a controller making the two unicycles follow a common circle of radius r (unspecified centre) in the counter-clockwise direction, while maintaining a constant distance d from each other, where $0 < d < 2r$. Our design can be easily modified to follow a circle in the clockwise direction. This problem is sometimes referred to as coordinated path following, and falls within the class of cooperative control problems, although the case of two vehicles considered here is very special. In [21], a Lie group setting is used to control formations of vehicles moving along parallel straight lines or circles. A controller is provided for two vehicles, and one is conjectured for the general case of n vehicles. The authors in [22] study closed related problems in the same framework and provide controllers for various types of straight line and circular coordination problems based on the construction of potential functions whose minimum corresponds to the desired coordinated motion. In [23], the authors propose an approximate solution to the circular coordinated path following problem which aims at decoupling the path following task from that of inter-vehicle coordination. The solution relies on Jacobian linearization and gain scheduling.

Define the functions $c_x(x)$ and $c_z(z)$ as

$$\begin{aligned} c_x(x) &= \mathrm{col}(x_1 - r\sin x_3, x_2 + r\cos x_3) \\ c_z(z) &= \mathrm{col}(z_1 - r\sin z_3, z_2 + r\cos z_3). \end{aligned} \tag{12}$$

For any x, the point $c_x(x)$ lies at distance r from (x_1, x_2), and the vector $\mathrm{col}(x_1, x_2) - c_x(x)$ is orthogonal to the normalized velocity vector $(\cos x_3, \sin x_3)$ of the unicycle. Therefore, the point $c_x(x)$ is the centre of the circle that the unicycle would follow if its controls were chosen as $u_1^x = v$ and $u_2^x = v/r$. The same observation holds for $c_z(z)$. The control specification for the coordinated path following problem is to stabilize the set $\Gamma = \{\chi : c_x(x) = c_z(z), \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2} = d\}$, and guarantee that the linear velocities u_1^x, u_1^z are bounded away from zero. When the unicycles lie on a common circle of radius r, i.e., when $c_x(x) = c_z(z)$, the distance between them is $2r\sin(|x_3 - z_3|/2)$. Therefore, the set Γ can be equivalently expressed as

$$\Gamma = \{\chi : c_x(x) = c_z(z),\ |x_3 - z_3| = 2\sin^{-1}(d/2r) \bmod 2\pi\}. \tag{13}$$

The goal set Γ is unbounded because we are not putting any restriction on the location of the centres $c_x(x)$ and $c_z(z)$.

Remark 5.1. Note that the representation of Γ in relative coordinates $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (x_1 - z_1, x_2 - z_2, x_3 - z_3)$ is compact. However, the stabilization of Γ in relative coordinates *does not imply* stabilization of Γ in original coordinates because the map $(x, z) \mapsto \tilde{x}$ is not a diffeomorphism. For instance, it may happen that the centres of rotation c_x and c_z drift off to infinity with a finite escape time while the relative positions of the unicycles remain bounded.

Step 1: Candidate storage function. Consider the candidate storage function

$$V = \frac{1}{2}\|c_x(x) - c_z(z)\|^2,$$

and note that $\Gamma \subset V^{-1}(0)$. We define the feedback transformation

$$\begin{bmatrix} u_1^x \\ u_2^x \\ u_1^z \\ u_2^z \end{bmatrix} = \beta_1 \bar{u} + \beta_2 \tilde{u} = \begin{bmatrix} 1 & 0 \\ 1/r & 0 \\ 0 & 1 \\ 0 & 1/r \end{bmatrix} \begin{bmatrix} \bar{u}^x \\ \bar{u}^z \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}^x \\ \tilde{u}^z \end{bmatrix}, \tag{14}$$

which has the property that $L_g V(x)\beta_1 = 0$. For any feedback $\bar{u}(\chi) = \mathrm{col}(\bar{u}^x(\chi), \bar{u}^z(\chi))$, when $\tilde{u} = 0$ the unicycles travel along circles of radius r and the centres of rotation remain constant, implying that V is constant along trajectories or, what is the same, $L_{f+g\beta_1\bar{u}}V = 0$. Therefore, for any feedback $\bar{u}(\chi)$, the system with input $\tilde{u}$ and output $y = h(\chi) := L_{g\beta_2}V(\chi)^\top$ below is passive,

$$y = h(\chi) = \begin{bmatrix} \partial V/\partial x_3 \\ \partial V/\partial z_3 \end{bmatrix} = \begin{bmatrix} -r(c_x(x) - c_z(z))^\top \begin{bmatrix} \cos x_3 \\ \sin x_3 \end{bmatrix} \\ r(c_x(x) - c_z(z))^\top \begin{bmatrix} \cos z_3 \\ \sin z_3 \end{bmatrix} \end{bmatrix}. \tag{15}$$

Step 2: Γ*-detectability enforcement.*

Lemma 5.2. Let $\bar{u}(\chi)$ be any feedback which is bounded away from zero component-wise, i.e., for some $\varepsilon > 0$, $\inf_\chi \bar{u}^x(\chi) \geq \varepsilon > 0$ and $\inf_\chi \bar{u}^z(\chi) \geq \varepsilon > 0$. Then, the maximal subset of $h^{-1}(0)$ invariant under the vector field $f + g\beta_1\bar{u}$ is $V^{-1}(0)$, i.e., $\mathcal{O} = V^{-1}(0)$.

Proof. As observed earlier, if $\tilde{u} = 0$ and $\inf_\chi \bar{u} > \varepsilon > 0$ component-wise, then each unicycle moves along a circle of radius r, and so the vector $c_x(x(t)) - c_z(z(t))$ is constant. Suppose that, for some solution $\chi(t)$ of the open-loop system, $h(\chi(t)) \equiv 0$. Then, either $\chi(t) \in V^{-1}(0)$, or the constant vector $c_x(x(t)) - c_z(z(t))$ is perpendicular to the vectors $\mathrm{col}(\cos x_3(t), \sin x_3(t))$ and $\mathrm{col}(\cos z_3(t), \sin z_3(t))$ for all $t \in \mathbb{R}$, implying that the linear velocities of the unicycles have constant angle. However, the unicycles move along two circles with nonzero linear velocities vectors whose angles are not constant. $\square$

We mentioned earlier that the functions $c_x(x)$, $c_z(z)$ remain constant along solutions of (11) with feedback transformation (14) and $\tilde{u} = 0$. Therefore, when $\tilde{u} = 0$, the dynamics of the unicycles are entirely described by those of their angular velocities x_3 and z_3. In other words, the restriction of the vector field $f + g\beta_1\bar{u}$ to $\mathcal{O}$ is

$$\dot{x}_3 = \frac{1}{r}\bar{u}^x, \quad \dot{z}_3 = \frac{1}{r}\bar{u}^z. \tag{16}$$

We rewrite the set Γ as $\Gamma = \{\chi \in V^{-1}(0) : |x_3 - z_3| = 2\sin^{-1}(d/2r) \bmod 2\pi\}$. Let $\alpha := 2\sin^{-1}(d/2r)$. Since $d \in (0, 2r)$, we have that $0 < \alpha < \pi$. Referring to the restriction of the system dynamics on $\mathcal{O}$ in (16), in order to enforce Γ-detectability we need to design $\bar{u}$ to stabilize the set $\{|x_3 - z_3| = \alpha\}$. In designing the stabilizer, we must take into account the fact that $x_3, z_3 \in S^1$, so the stabilization must be performed modulo 2π. To fulfill the assumption of Lemma 5.2, we also need both $\bar{u}^x$ and $\bar{u}^z$ to be bounded away from zero. There are many ways to fulfill these objectives. We base our design on the candidate Lyapunov function

$$W = \frac{1}{2}\left[\cos(x_3 - z_3) - \cos\alpha\right]^2, \tag{17}$$

whose derivative along (16) is

$$\dot{W} = -\frac{1}{r}\left[\cos(x_3 - z_3) - \cos\alpha\right]\sin(x_3 - z_3)(\bar{u}^x - \bar{u}^z).$$

The feedback

$$\begin{aligned} \bar{u}^x &= v + \varphi_1\left((\cos(x_3 - z_3) - \cos\alpha)\sin(x_3 - z_3)\right) \\ \bar{u}^z &= v - \varphi_1\left((\cos(x_3 - z_3) - \cos\alpha)\sin(x_3 - z_3)\right) \end{aligned} \tag{18}$$

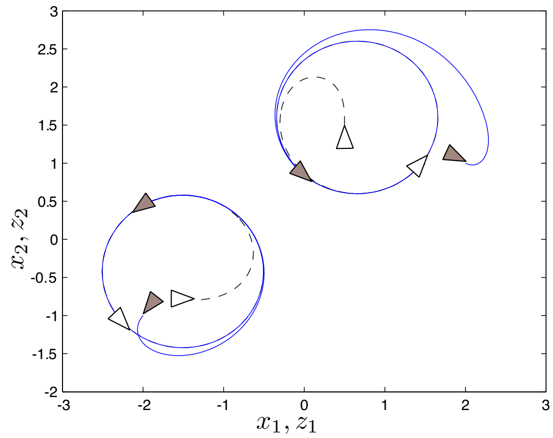

Figure 4: Simulation results for the coordinated path following controller in (19) for two different initial conditions.

where $v > 0$ is a design constant, and $\varphi_1(\cdot)$ is an odd function such that $\sup_{\mathbb{R}} |\varphi_1(\cdot)| < v$, is bounded away from zero component-wise. Moreover, it almost globally stabilizes the set $|x_3 - z_3| = \alpha$ with domain of attraction $\{(x_3, z_3) : \sin(x_3 - z_3) \neq 0\}$, and thus enforces local Γ-detectability for system (11) after feedback transformation (14), with input $\tilde{u}$ and output $y = h(\chi)$ in (15).

Step 3: Passivity-based stabilization.

Proposition 5.3. Let v be a positive scalar, and $\varphi_1 : \mathbb{R} \to \mathbb{R}$ be a smooth odd function which is strictly increasing and such that $\sup_{\mathbb{R}} |\varphi_1(\cdot)| < v$. Then, there exists $K^\star > 0$ such that for all $K \in (0, K^\star)$ the feedback

$$\begin{aligned} u_1^x &= v + \varphi_1\left((\cos(x_3 - z_3) - \cos\alpha)\sin(x_3 - z_3)\right) \\ u_2^x &= \frac{u_1^x}{r} - Kh_1(\chi) \\ u_1^z &= v - \varphi_1\left((\cos(x_3 - z_3) - \cos\alpha)\sin(x_3 - z_3)\right) \\ u_2^z &= \frac{u_1^z}{r} - Kh_2(\chi) \end{aligned} \tag{19}$$

where $\alpha = 2\sin^{-1}(d/2r)$ and $h(x)$ is defined in (15), renders $V^{-1}(0)$ globally exponentially stable and Γ asymptotically stable for the closed-loop system, thus solving the coordinated path following problem. Moreover, for all initial conditions the centres of rotation c_x and c_z of both unicycles converge exponentially to a common constant value.

Simulation results for the controller in (19), with $d = 3r/2$, $v = 1$, $K = 1$, and $\varphi_1(y) = \frac{1.6v}{\pi}\arctan(y)$, are found in Figure 4.

Proof. Since u_1^x, u_1^z are bounded away from zero, if we show that Γ is asymptotically stable for the closed-loop system then we can conclude that the coordinated path following problem is solved. Referring to the feedback transformation (14), the feedback (19) corresponds to choosing $\bar{u}$ as in (18) and the PBF $\tilde{u} = -Kh(\chi)$. Therefore, in light of Proposition 2.8, to show that Γ is asymptotically stable we only need to show that the closed-loop system is LUB near Γ. Since x_3 and z_3 belong to S^1, a compact set, we need to prove the LUB property for the displacements x_1, x_2, z_1, z_2 or, equivalently, for the centres of rotation $c_x(x)$, $c_z(z)$. First off, note that the trajectories of the closed-loop system are defined for all $t \geq 0$ because $|\dot{x}_1|, |\dot{x}_2| \leq |u_1^x|$, $|\dot{z}_1|, |\dot{z}_2| \leq |u_1^z|$, and both u_1^x and u_1^z are bounded by v. Next, the time derivatives of the centres of rotation along solutions of the closed-loop system are

$$\dot{c}_x = \begin{bmatrix} r\cos x_3 \\ r\sin x_3 \end{bmatrix} Kh_1(\chi) = -Kr^2 \begin{bmatrix} \cos^2 x_3 & \sin x_3 \cos x_3 \\ \sin x_3 \cos x_3 & \sin^2 x_3 \end{bmatrix} (c_x(x) - c_z(z))$$
$$\dot{c}_z = \begin{bmatrix} r\cos z_3 \\ r\sin z_3 \end{bmatrix} Kh_2(\chi) = Kr^2 \begin{bmatrix} \cos^2 z_3 & \sin z_3 \cos z_3 \\ \sin z_3 \cos z_3 & \sin^2 z_3 \end{bmatrix} (c_x(x) - c_z(z)).$$

Letting $S(\cdot) = [\cos(\cdot) \quad \sin(\cdot)]^\top [\cos(\cdot) \quad \sin(\cdot)]$, we have

$$\begin{bmatrix} \dot{c}_x \\ \dot{c}_z \end{bmatrix} = -Kr^2 \begin{bmatrix} S(x_3)(c_x - c_z) \\ -S(z_3)(c_x - c_z) \end{bmatrix}. \tag{20}$$

Letting $e = c_x - c_z$, we have $\dot{e} = -Kr^2(S(x_3) + S(z_3))e$, which can be viewed as a linear time-varying system. We study its stability using averaging theory. The averaged system is $\dot{e}_{\text{avg}} = -Kr^2(\bar{S}_1 + \bar{S}_2)e$, where $\bar{S}_1 = \lim_{T\to\infty}(1/T)\int_0^T S(x_3(\tau))d\tau$ and $\bar{S}_2 = \lim_{T\to\infty}(1/T)\int_0^T S(z_3(\tau))d\tau$. By Cauchy-Schwarz's inequality we have

$$\left(\int_0^T \sin x_3(\tau)\cos x_3(\tau) d\tau\right)^2 \leq \int_0^T \sin^2(x_3(\tau))d\tau \int_0^T \cos^2(x_3(\tau))d\tau,$$

with equality holding only if $x_3(t)$ is constant. The same inequality holds for $z_3(t)$. Therefore, assuming for the time being that $x_3(t), z_3(t)$ are not constant, the matrices $\bar{S}_1$ and $\bar{S}_2$ are positive definite and therefore the origin $e_{\text{avg}} = 0$ of the averaged system is globally exponentially stable. By the averaging theorem in [24, Theorem 10.5], there exists $K^\star > 0$ such that for all $K \in (0, K^\star)$ the equilibrium $e = 0$ is globally exponentially stable as well, proving the global exponential stability of $V^{-1}(0)$. Referring to equation (20), the exponential convergence of $c_x(x(t)) - c_z(z(t))$ to zero implies that $c_x(x(t)), c_z(z(t))$ are bounded and have a constant limit as $t \to \infty$. Moreover, in light of (20), the bound on c_x, c_z can be expressed as $\|c_x(x(t))\|, \|c_z(z(t))\| \leq M\|c_x(x(0)) - c_z(z(0))\|$ for some $M > 0$. This bound is uniform over Γ, proving the LUB property. We are left to show that $x_3(t), z_3(t)$ are not constant. Suppose they were, then $u_2^x(t), u_2^z(t) \equiv 0$. By design, $u_1^x(t), u_1^z(t) > 0$ and $K > 0$, so referring to (19), $h_1(\chi(t)), h_2(\chi(t)) > 0$ for all t. By (15), this can only happen if $x_3 \neq z_3 \bmod 2\pi$. Therefore, the unicycles travel along nonparallel straight lines with positive speed, implying that their centres of rotation diverge from each other, $\|c_x(x(t)) - c_z(z(t))\| \to \infty$. In other words, $V(\chi(t)) \to \infty$, contradicting the fact that the storage function is nonincreasing along solutions of the closed-loop system. □

Remark 5.4. a. In [22] the authors address the stabilization of n-unicycles to a circular formation using an all-to-all communication model. Their results can be used to solve this case study by appropriate choice of a phase potential function.

b. In [25] we address the extension of this case study to the circular formation control problem for n-unicycles with the extra requirement of stabilizing arbitrary phase formations on the circle.

We address different situations with different information flow graphs the most general of which is one with arbitrary static directed graph. These results provide a more general solution to the problem than that in [22] which relies on all-to-all communication, and those in [26] which address only symmetrical formations using dynamic feedbacks.

c. As in case studies 1 and 2, a candidate storage function V such that $\Gamma = V^{-1}(0)$ is not a feasible starting point to solve this case study. If, for instance, we set

$$V = \frac{1}{2}\|c_x(x) - c_z(z)\|^2 + \frac{1}{2}\left[(x_1 - z_1)^2 + (x_2 - z_2)^2 - d^2\right]^2,$$

and we let

$$\begin{aligned} u_1^x &= v,\ u_2^x = v/r + \tilde{u}^x \\ u_1^z &= v,\ u_2^z = v/r + \tilde{u}^z \end{aligned} \tag{21}$$

so as to make $V^{-1}(0)$ invariant when $\tilde{u}^x, \tilde{u}^z = 0$, then the system with input $(\tilde{u}^x, \tilde{u}^z)$ and output $L_{\tilde{g}}V^\top$, with $\tilde{g} = [g_2\ g_4]$, would not be passive because V fails to be nonincreasing along trajectories of the open-loop system. The reason is that, when $\tilde{u}^x, \tilde{u}^z = 0$, the unicycles move along circles of radius r, and so $c_x(x)$, $c_z(z)$ remain constant along solutions, but the distance between the unicycles is not constant when the centres of rotation do not coincide, and it fails to be nonincreasing.

6 Case study 4: coordination of two unicycles

Consider again the two kinematic unicycles in (11), but this time consider the problem of making the unicycles meet at a distance $\Delta > 0$ facing each other. Solving this problem corresponds to stabilizing the set

$$\Gamma = \left\{\chi \in \mathcal{X} : \sqrt{d_1(\chi)^2 + d_2(\chi)^2} = \Delta, z_3 = \theta(\chi),\ x_3 = \theta(\chi) + \pi\right\}. \tag{22}$$

where $d_1(\chi) = x_1 - z_1$, $d_2(\chi) = x_2 - z_2$, and $\theta(\chi) = \arg(d_1(\chi) + i\, d_2(\chi))$, with $\theta \in S^1$.

Step 1: Candidate storage function. Once again, choosing a candidate storage function V with the property that $\Gamma = V^{-1}(0)$ does not lead to a solution of the problem, because such a choice would lead to an obstruction to Γ-detectability similar to the one described in case study 2. For, consider a storage function $V(e_1, e_2)$ where $e_1 = (x_1 - z_1)^2 + (x_2 - z_2)^2 - \Delta^2$, $e_2 = x_3 - z_3 + \pi$ and $(e_1, e_2) \mapsto V(e_1, e_2)$ is positive definite. Then,

$$L_gV = \begin{bmatrix} \frac{\partial V}{\partial e_1}(d_1 \cos x_3 + d_2 \sin x_3) & \frac{\partial V}{\partial e_2} & -\frac{\partial V}{\partial e_1}(d_1 \cos z_3 + d_2 \sin x_3) & -\frac{\partial V}{\partial e_2} \end{bmatrix}.$$

From this it is clear that any configuration with $z_3 - x_3 = \pi$ and (d_1, d_2) perpendicular to x_3 is an equilibrium that belongs to the set $\mathcal{O}$, and thus the system with output $L_gV^\top$ is not Γ-detectable.

Instead, we choose

$$V(\chi) = \frac{1}{4}\left[d_1(\chi)^2 + d_2(\chi)^2 - \Delta^2\right]^2, \tag{23}$$

which has the property that $\Gamma \subsetneq V^{-1}(0)$. We choose the feedback transformation

$$\begin{bmatrix} u_1^x \\ u_2^x \\ u_1^z \\ u_2^z \end{bmatrix} = \beta_1\bar{u} + \beta_2\tilde{u} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{u}^x \\ \bar{u}^z \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{u}^x \\ \tilde{u}^z \end{bmatrix}. \tag{24}$$

For any feedback $\bar{u}(\chi)$, when $\tilde{u} = 0$, we have that the unicycles rotate without translating, and therefore the distance between them, $(d_1(\chi), d_2(\chi))$, remains constant. In other words, $L_g V \beta_1 = 0$. Therefore, for any feedback $\bar{u}(\chi)$, the system with input $\tilde{u}$ and output $y = h(\chi) := L_{g\beta_2} V(\chi)^\top$ below is passive,

$$y = h(\chi) = (d_1^2 + d_2^2 - \Delta^2) \begin{bmatrix} d_1 \cos x_3 + d_2 \sin x_3 \\ -(d_1 \cos z_3 + d_2 \sin z_3) \end{bmatrix}. \tag{25}$$

Step 2: Γ*-detectability enforcement.*

Lemma 6.1. Let $\bar{u}(\cdot)$ be any feedback which does not vanish on the set $\{\chi : (d_1(\chi), d_2(\chi)) \neq 0, d_1(\chi)\cos x_3 + d_2(\chi)\sin x_3 = 0, d_1(\chi)\cos z_3 + d_2(\chi)\sin z_3 = 0\}$. Then, the maximal subset $\mathcal{O}$ of $h^{-1}(0)$ invariant under the vector field $f + g\beta_1\bar{u}$ is $\mathcal{O} = V^{-1}(0) \cup \{\chi : d_1(\chi) = d_2(\chi) = 0\}$.

Proof. The solutions of $\dot{\chi} = f + g\beta_1\bar{u}$ correspond to the two unicycles rotating and not translating. Therefore, d_1, d_2 are constant along solutions. The set $\{\chi : d_1(\chi) = d_2(\chi) = 0\}$, being invariant under the vector field $f + g\beta_1\bar{u}$ and contained in $h^{-1}(0)$, is contained in $\mathcal{O}$. Now suppose that $(d_1(\chi(0)), d_2(\chi(0))) \neq 0$, so that $(d_1(\chi(t)), d_2(\chi(t))) \neq 0$ for all $t \in \mathbb{R}$, and that $h(\chi(t)) \equiv 0$. Then, $dh(\chi(t))/dt \equiv 0$ so either $d_1^2(\chi(t)) + d_2^2(\chi(t)) \equiv \Delta^2$ (i.e., $\chi(t) \in V^{-1}(0)$), or

$$\begin{aligned} &d_1 \cos x_3(t) + d_2 \sin x_3(t) \equiv 0 \\ &d_1 \cos z_3(t) + d_2 \sin z_3(t) \equiv 0 \\ &(-d_1 \sin x_3(t) + d_2 \cos x_3(t))\bar{u}^x \equiv 0 \\ &(-d_1 \sin z_3(t) + d_2 \cos z_3(t))\bar{u}^z \equiv 0. \end{aligned}$$

By assumption, $\bar{u}^x$, $\bar{u}^z$ are not zero on the set where the first two equations are satisfied. Therefore, the equations can only be satisfied if $d_1(\chi(t)) = d_2(\chi(t)) \equiv 0$, which is not the case. $\square$

The sets $V^{-1}(0)$ and $\{d_1(\chi) = d_2(\chi) = 0\}$ are disjoint and, for any $\bar{u}$, they are invariant under $f + g\beta_1\bar{u}$. In order to enforce Γ-detectability, we need to design $\bar{u}$ such that Γ is asymptotically stable relative to $V^{-1}(0)$ and $\bar{u} \neq 0$ on the set $\{\chi : (d_1(\chi), d_2(\chi)) \neq 0, d_1(\chi)\cos x_3 + d_2(\chi)\sin x_3 = 0, d_1(\chi)\cos z_3 + d_2(\chi)\sin z_3 = 0\}$. The restriction of $f + g\beta_1\bar{u}$ to $V^{-1}(0)$ is

$$\begin{aligned} &\dot{x}_1 = 0,\ \dot{x}_2 = 0, \dot{x}_3 = \bar{u}^x \\ &\dot{z}_1 = 0,\ \dot{z}_2 = 0, \dot{z}_3 = \bar{u}^z. \end{aligned}$$

The function $\theta(\chi)$ is constant along solutions of the above differential equation, so stabilizing Γ corresponds to stabilizing the equilibria $x_3 = \theta(\chi) + \pi$, $z_3 = \theta(\chi)$ modulo 2π. There are many ways to achieve this goal. We choose

$$\begin{aligned} \bar{u}^x &= -K_1\sqrt{d_1^2 + d_2^2}\sin(x_3 - \theta(\chi) - \pi) \\ &= K_1\left[d_1 \sin x_3 - d_2 \cos x_3\right] \\ \bar{u}^z &= -K_1\sqrt{d_1^2 + d_2^2}\sin(z_3 - \theta(\chi)) \\ &= K_1\left[-d_1 \sin z_3 + d_2 \cos z_3\right], \end{aligned}$$

with $K_1 > 0$, which almost globally stabilizes Γ relative to $V^{-1}(0)$, with domain of attraction $\{\chi \in V^{-1}(0) : x_3 \neq \theta(\chi),\ z_3 \neq \theta(\chi) + \pi\}$. Our choice of $\bar{u}$ is not zero on the set $\{\chi : (d_1(\chi), d_2(\chi)) \neq$

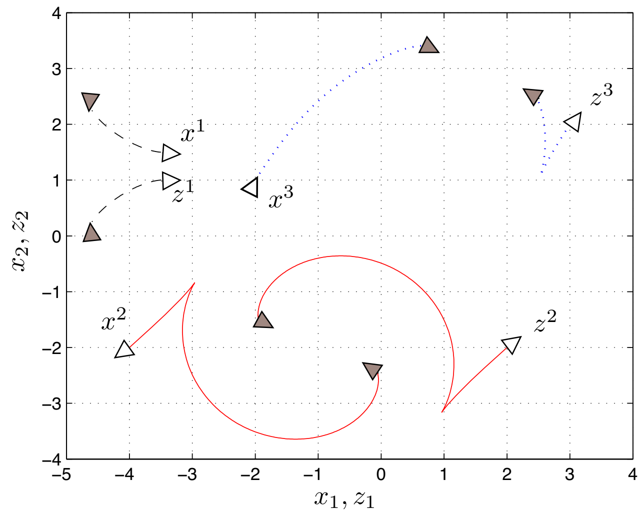

Figure 5: Simulation results for the coordination controller in (26).

$0, d_1(\chi)\cos x_3 + d_2(\chi)\sin x_3 = 0, d_1(\chi)\cos z_3 + d_2(\chi)\sin z_3 = 0\}$ and therefore, by Lemma 6.1, the feedback above almost globally stabilizes Γ relative to $\mathcal{O}$ with domain of attraction $\{\chi \in \mathcal{O} : d_1(\chi) \neq 0,\ d_2(\chi) \neq 0,\ x_3 \neq \theta(\chi),\ z_3 \neq \theta(\chi) + \pi\}$, and thus ensures local Γ-detectability for system (11) with feedback transformation (24), input $\tilde{u}$ and output y defined in (25).

Step 4: Passivity-based stabilization.

Proposition 6.2. For any positive scalars K_1, K_2, the feedback

$$\begin{aligned} u_1^x &= -K_2 h_1(\chi) \\ u_2^x &= K_1 \left[d_1(\chi)\sin x_3 - d_2(\chi)\cos x_3\right] \\ u_1^z &= -K_2 h_2(\chi) \\ u_2^z &= K_1 \left[-d_1(\chi)\sin z_3 + d_2(\chi)\cos z_3\right], \end{aligned} \tag{26}$$

where $h(\chi)$ is defined in (25), renders Γ asymptotically stable for the closed-loop system and solves the coordination problem.

Simulation results for the controller in (26), with $K_1 = K_2 = 1$, are found in Figure 5.

Proof. In order to prove that Γ is asymptotically stable, by Theorem 2.6 it suffices to show that the closed-loop system is LUB near Γ. Solutions of the closed-loop system are defined for all $t \geq 0$ because x_3 and z_3 are variables in S^1, a compact set, and $u_1^x(\chi(t))$, $u_1^z(\chi(t))$ are uniformly bounded,

$$|u_1^x|, |u_1^z| \leq 2K_2\sqrt{V(2\sqrt{V} + \Delta^2)}, \tag{27}$$

and V is nonincreasing along solutions. The derivative of the storage function along closed-loop solutions is

$$\dot{V} = -4V\Big[(d_1\cos x_3 + d_2\sin x_3)^2 + (d_1\cos z_3 + d_2\sin z_3)^2\Big]. \tag{28}$$

We will show that for all initial conditions in some neighborhood $\mathcal{N}(\Gamma)$, the term in square brackets in (28) is bounded away from zero. This fact then implies the LUB property. For, the claim implies that on $\mathcal{N}(\Gamma)$, V converges to zero exponentially and thus, by (25), $h(\chi(t))$ tends to zero exponentially. Therefore, u_1^x, u_1^z tend to zero exponentially and so x_1, x_2, z_1, z_2 are bounded. Moreover, their bound is uniform on Γ, proving the LUB property.

It is easy to see that $(d_1 \cos x_3 + d_2 \sin x_3)^2 = (d_1^2 + d_2^2)\cos^2(x_3 - \theta)$. Since V is nonincreasing, if $d_1(\chi_0)^2 + d_2(\chi_0)^2 \neq 0$ and $V(\chi_0) < (1/4)\Delta^4$, then the solution $\chi(t)$ is such that, for all $t \geq 0$, $d_1^2(\chi(t)) + d_2^2(\chi(t)) \geq \Delta^2 - 2\sqrt{V(\chi_0)} > 0$. Therefore, for the purpose of showing that the term in square brackets in (28) is bounded away from zero, it is enough to show that there exists a neighborhood $\mathcal{N}(\Gamma) \subset \{V < (1/4)\Delta^4\}$ such that all closed-loop solutions originating in $\mathcal{N}(\Gamma)$ yield, for all $t \geq 0$, $\cos^2(x_3(t) - \theta(t)) \geq 1/2$. Let

$$W(\chi) = \frac{1}{2}\left[x_3 - \theta(\chi) - \pi\right]^2.$$

The time derivative of W along closed-loop solutions is

$$\begin{aligned}\dot{W} &= -(x_3 - \theta - \pi)\sin(x_3 - \theta - \pi)\sqrt{d_1^2 + d_2^2}\Big(1 + -\frac{u_1^x}{d_1^2 + d_2^2}\Big) + (x_3 - \theta - \pi)\sin(z_3 - \theta)\frac{u_1^z}{\sqrt{d_1^2 + d_2^2}} \\ &\leq -\sqrt{2W}\sin(\sqrt{2W})\sqrt{d_1^2 + d_2^2}\Big(1 - \frac{|u_1^x|}{d_1^2 + d_2^2}\Big) + \sqrt{2W}\frac{|u_1^z|}{\sqrt{d_1^2 + d_2^2}}.\end{aligned}$$

Note that, when $u_1^x = u_1^z = 0$, if $W(\chi_0) < \pi^2/2$, then the solution asymptotically converges to $\{W = 0\}$. Moreover, given any c, with $0 < c < \pi^2/2$, there exists $U > 0$ such that, for $|u_1^x|, |u_1^z| < U$ the set $\{\chi \in \mathcal{X} : W(\chi) \leq c\}$ is positively invariant. Pick $c = 1/2(\pi/4)^2$, and let U be as above. Given any $V_0 > 0$, by the inequalities in (27) and the fact that V is nonincreasing along solutions of the closed-loop system, for any initial condition $\chi_0 \in \{\chi \in \mathcal{X} : V(\chi) \leq V_0\}$, we have

$$|u_1^x(t)|, |u_1^z(t)| \leq 2K_2\sqrt{V_0\big(2\sqrt{V_0} + \Delta^2\big)}.$$

Let V_0 be small enough that $2K_2\sqrt{V_0\big(2\sqrt{V_0} + \Delta^2\big)} < U$ and $V_0 < (1/4)\Delta^4$. Consider the set

$$\mathcal{N}(\Gamma) = \{\chi : V < V_0\} \cap \{\chi : W < 1/2(\pi/4)^2\}.$$

On Γ, $V = 0$ and $W = 0$, so $\mathcal{N}(\Gamma)$ is a neighborhood of Γ. By construction, the set $\mathcal{N}(\Gamma)$ is positively invariant. In particular, for all $\chi_0 \in \mathcal{N}(\Gamma)$ and all $t \geq 0$, $|x_3(t) - \theta(t) - \pi| < \pi/4$, and hence

$$\cos^2(x_3(t) - \theta(t) - \pi) > 1/2,$$

as required. $\square$

Conclusions

We presented four case studies illustrating a novel set stabilization procedure for closed sets based on recent theoretical advances in [15]. In case studies 2-4 the stabilization of the goal set is only proved to be local, although the detectability property is almost global. We conjecture[1] that, in fact, the controllers in case studies 2-4 stabilize the respective goal sets almost globally.

[1]Such a result would follow by extending those in [15] to the case when Γ-detectability is satisfied except on a set of Lebesgue measure zero. This would require one to show that the set of initial conditions that approach this set is of measure zero as well.

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