Synchronizing N cart-pendulums using Virtual Holonomic Constraints

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***Abstract*— A solution is presented to the problem of synchronizing a chain of N cart-pendulums using *virtual holonomic constraints*. The approach is based on a master-slave configuration whereby the first cart-pendulum is controlled so as to stabilize a desired oscillation around its unstable equilibrium. Then, each remaining cart-pendulum is controlled so as to fully synchronize it to the previous pendulum.**

I. Introduction

In this paper we consider N cart-pendulum systems whose carts slide on a straight line. The i-th cart-pendulum, depicted in Figure 1, has configuration variable $q_i = (\theta_i, x_i) \in Q_i$, where $Q_i = S^1 \times \mathbb{R}$ is the i-th configuration space, and S^1 is the set of real numbers modulo 2π, diffeomorphic to the unit circle. The i-th pendulum model can be written in the standard form

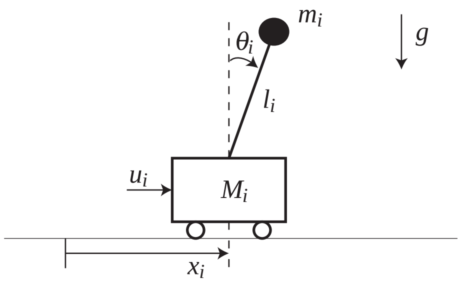

Fig. 1. The i^{th} cart-pendulum system

$$D_i(q_i)\ddot{q}_i + C_i(q_i,\dot{q}_i)\dot{q}_i + \nabla P_i(q_i) = Bu_i, \tag{1}$$

where

$$D_i(q_i) = \begin{bmatrix} m_i l_i^2 & m_i l_i \cos\theta_i \\ m_i l_i \cos\theta_i & M_i + m_i \end{bmatrix},\ \nabla P_i(q_i) = \begin{bmatrix} -m_i g l_i \sin\theta_i \\ 0 \end{bmatrix},$$
$$C_i(q_i,\dot{q}_i) = \begin{bmatrix} 0 & 0 \\ -m_i l_i \sin\theta_i \dot{\theta}_i & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2}$$

We will let $q = (q_1, \ldots, q_N)$ denote the collective configuration variable, and $Q = Q_1 \times \cdots \times Q_N$ denote the collective configuration space.

System (1) is an Euler-Lagrange control system with $2N$ degrees-of-freedom (DOFs) and N actuators. Therefore, the degree of underactuation is N. In this paper we investigate a problem of full synchronization with simultaneous control of oscillations, as follows.

Synchronization of Oscillations Problem (SOP) - Design feedback laws $u_i^\star$, $i = 1, \ldots, N$, meeting the following two specifications:

(i) *Oscillation control for pendulum* 1. Let $\mathscr{C} \subset (\mathbb{R} \times S^1) \times \mathbb{R}^2$ be a closed curve representing a desired oscillation around the unstable equilibrium. For all $(q_1(0), \dot{q}_1(0))$ in a neighborhood of $\mathscr{C}$, the solution $(q_1(t), \dot{q}_1(t))$ asymptotically approaches $\mathscr{C}$.

(ii) *Full synchronization.* For all $i \in \{2, \ldots, N\}$, and for all initial conditions in a suitable set, $\theta_i(t) - \theta_{i-1}(t) \to 0$ and $x_i(t) - x_{i-1}(t) \to d_{i-1}$, where d_{i-1} is a design parameter.

In other words, we wish to fully synchronize the pendulums, with desired separations d_i between the carts, while simultaneously inducing desired oscillatory behaviours on them. Of particular interest is the special case when $\mathscr{C}$ is the unstable equilibrium of the first pendulum. In this case, SOP becomes the simultaneous swing-up and synchronization of the N cart-pendulums. For the problem to be feasible it is required that the lengths l_i of all N pendulums be identical, so we will assume that $l_1 = \cdots = l_N = l$.

SOP has been posed and solved by Shiriaev-Freidovich-Gusev in [1] using virtual holonomic constraints to plan the desired oscillation, and applying the transverse linearization technique to stabilize the oscillation in question. In this paper we present an alternative technique, also based on the notion of virtual holonomic constraint, but relying on different principles.

The solution to SOP presented in this paper relies on a master-slave configuration. Specifically, we design a dynamic feedback $u_1^\star(q_1, \dot{q}_1, s_1, \dot{s}_1)$, where $(s_1, \dot{s}_1)$ is the state of a dynamic compensator, to asymptotically stabilize the desired oscillation $\mathscr{C}$ for pendulum 1. Then, for $i = 2, \ldots, N$, we design a dynamic feedback $u_i^\star(q_i, \dot{q}_i, q_{i-1}, \dot{q}_{i-1}, s_i, \dot{s}_i)$ to fully synchronize pendulum i to pendulum $i-1$. The techniques used to synthesize the dynamic feedbacks $u_i^\star$ rely on recently developed theory in [2], [3] reviewed below.

This paper is organized as follows. In Section II we review the definition of virtual holonomic constraint and basic tools needed in this paper. In Section III we develop a solution to SOP by first focusing on the case $N = 2$, and then generalizing to arbitrary N. The main result of the paper is in Proposition 3.4. In Section IV we present simulation results, and in Section V we draw concluding remarks.

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II. Virtual holonomic constraints

In this section we review some parts of the theory in [2], [3]. Consider a controlled Euler-Lagrange system of the form

$$D(q)\ddot{q}+C(q,\dot{q})\dot{q}+\nabla P(q)=Bu, \tag{3}$$

with n degrees-of-freedom and $n-k$ controls, where k is the degree of underactuation. Let Q denote the configuration space. A **virtual holonomic constraint (VHC)** of order p for system (3) is a relation between the configuration variables of the form $h(q)=0$, where $h:Q\to\mathbb{R}^p$ is a C^1 function such that its Jacobian dh_q has full rank for all $q\in h^{-1}(0)$. A VHC $h(q)=0$ is said to be **feasible** if there exists a feedback $u^\star(q,\dot{q})$ such that the set $\Gamma=\{(q,\dot{q}):h(q)=0,dh_q\dot{q}=0\}$, is invariant for the closed-loop system (3) with $u=u^\star(q,\dot{q})$. We call the set Γ the **constraint manifold**. In other words, a VHC $h(q)=0$ is feasible if, through appropriate feedback, solutions of system (3) can be made to satisfy the constraint $h(q)=0$ whenever their initial condition is chosen such that $h(q(0))=0$ and their initial velocity is tangent to the constraint, $dh_{q(0)}\dot{q}(0)=0$.

If a VHC $h(q)=0$ is feasible, the next step is to enforce it via feedback, i.e., to design a feedback stabilizing the constraint manifold Γ. If we take $h(q)$ to be an output function for system (3), and if the output $y=h(q)$ yields a well-defined vector relative degree $\{2,\dots,2\}$ on Γ, then an input-output feedback linearizing controller will stabilize[1] Γ. A VHC $h(q)=0$ satisfying the vector relative degree condition above is said to be **regular**. Note that regular VHCs are always feasible.

Proposition 2.1: A VHC $h(q)=0$ is regular iff

$$(\forall q\in h^{-1}(0))\ \operatorname{Im}(D^{-1}(q)B(q))\cap\operatorname{Ker}(dh_q)=\{0\},$$

or, equivalently, if the matrix $dh_qD^{-1}(q)B(q)$ has rank p for all $q\in h^{-1}(0)$.

We omit the elementary proof of this proposition, which is adapted from Lemma 2.1 in [2]. The regularity condition above has the following mechanical interpretation: *all* of the acceleration directions that can be imparted via the control input must be *transversal* to the tangent space of the constraint set $\{q:h(q)=0\}$.

Next, we review the notion of reduced dynamics. For this, we will consider the special case when $k=1$, i.e., when the degree of underactuation is one, and consider a VHC of order $n-1$ expressed in explicit form, whereby $n-1$ configuration variables are functions of the remaining configuration variable, $\operatorname{col}(q_1,\dots,q_n)=\phi(q_n)$, so that the associated constraint manifold is $\Gamma=\{(q,\dot{q}):\operatorname{col}(q_1,\dots,q_n)=\phi(q_n),\operatorname{col}(\dot{q}_1,\dots,\dot{q}_{n-1})=\phi'(q_n)\dot{q}_n\}$. If such a VHC is regular, and a feedback $u^\star(q,\dot{q})$ is used to make Γ invariant, the dynamics of the closed-loop system on Γ are called the **reduced dynamics**. These are simply the zero dynamics of system (3) with output $y=\operatorname{col}(q_1,\dots,q_n)-\phi(q_n)$. It can be shown (this fact was discovered in [4]) that the reduced dynamics always take the form

$$\ddot{q}_n=\Psi_1(q_n)+\Psi_2(q_n)\dot{q}_n^2. \tag{4}$$

If q_n is a real variable, (4) is always Euler-Lagrange. If, on the other hand, $q_n\in S^1$, then it was shown in [2] that (4) is Euler-Lagrange provided that ϕ is an odd function. In either case, the total energy of the system is

$$E(q_n,\dot{q}_n)=\frac{1}{2}M(q_n)\dot{q}_n^2+V(q_n), \tag{5}$$

where

$$\begin{aligned}M(q_n)&=\exp\left\{-2\int_0^{q_n}\Psi_2(\tau)d\tau\right\},\\ V(q_n)&=-\int_0^{q_n}\Psi_1(\mu)M(\mu)d\mu.\end{aligned} \tag{6}$$

When $q_n\in S^1$, it was shown in [2] that almost all level sets of $E(q_n,\dot{q}_n)$ are solutions of the reduced system (4) homeomorphic to circles, therefore representing oscillations of the system. If one wishes to stabilize one such oscillation $\{E(q_n,\dot{q}_n)=E_0\}$, one has to break the invariance of the constraint manifold Γ because the reduced system (4) has no control input. In [3], the stabilization of a level set $\{E(q_n,\dot{q}_n)=E_0\}$ was addressed by making the VHC $\operatorname{col}(q_1,\dots,q_n)=\phi(q_n)$ depend on a time-varying parameter s. More precisely, the idea in [3] is to use a modified VHC parametrized by a scalar s, $\operatorname{col}(q_1,\dots,q_n)=\phi^s(q_n)$, with the property that the modified VHC is regular for all s, and that $\phi^0(q_n)=\phi(q_n)$. We call such a constraint a **dynamic VHC**. The time evolution of the scalar s is governed by $\ddot{s}=v$, where v is a new control input to be designed. Instead of stabilizing Γ, an input-output feedback linearizing controller $u^\star(q,\dot{q},s,\dot{s})$ is designed to asymptotically stabilize the set $\bar{\Gamma}=\{\operatorname{col}(q_1,\dots,q_n)=\phi^s(q_n),\operatorname{col}(\dot{q}_1,\dots,\dot{q}_n)=\partial_{q_n}\phi^s(q_n)\dot{q}_n+\partial_s\phi^s(q_n)\dot{s}\}$. Concerning the compensator $\ddot{s}=v$, it is shown in [3] that for appropriate choices of gains K_1,K_2,K_3, the feedback

$$v=K_1(E-E_0)\dot{q}_n+K_2s+K_3\dot{s}, \tag{7}$$

in conjunction with the feedback $u^\star$, asymptotically stabilizes the set $\{E(q_n,\dot{q}_n)=E_0,s=\dot{s}=0\}$ while simultaneously stabilizing $\bar{\Gamma}$.

III. Solution of SOP

In this section we present a solution to SOP adopting a master-slave approach. We begin by considering the case of two pendulums, $N=2$. Viewing the first pendulum as the master, we select a desired oscillation $\mathscr{C}$, and we design a dynamic feedback $u_1^\star$ that asymptotically stabilizes pendulum 1 to $\mathscr{C}$. We do that by looking for regular VHCs of the form $\theta_1=\phi(x_1)$, where $|\phi|<\pi/2$ so that, if the VHC is enforced, the pendulum rod is forced to lie in the upper half-plane. We also impose a second constraint, $\theta_1=\theta_2$, to make the angles of pendulums 1 and 2 synchronize.

[1] Actually, the stabilization of Γ will occur if there exist two class-$\mathscr{K}$ functions α and β such that the function $H(q,\dot{q})=\operatorname{col}(h(q),dh_q\dot{q})$ satisfies the bounds $\alpha(\|(q,\dot{q})\|_\Gamma)\le\|H(q,\dot{q})\|\le\beta(\|(q,\dot{q})\|_\Gamma)$, where $\|\cdot\|_\Gamma$ denotes the point-to-set distance to Γ.

A. Design for the case $N = 2$

Consider the VHC of order 2,

$$\theta_1 = \phi(x_1), \ \theta_2 = \theta_1. \tag{8}$$

Using Proposition 2.1, one can check that VHC (8) is regular if and only if

$$(\forall x_1) \ \cos\phi(x_1) + l\phi'(x_1) \neq 0, \ \cos\phi(x_1) \neq 0. \tag{9}$$

There are various choices of ϕ meeting these two conditions and such that $|\phi| < \pi/2$. For instance, for all $\beta \in \mathbb{R}$ and all $\theta_l \in (0, \pi/2)$, the function

$$\phi(x_1) = -\theta_l \sin\left(\frac{\cos\theta_1}{l\theta_l}(x_1 - \beta)\right) \tag{10}$$

satisfies both properties in (9), making the VHC in (8) regular. Now we need to analyze the reduced dynamics, looking for periodic orbits of interest. The reduced dynamics are obtained by left-multiplying (1) by a left annihilator of B and evaluating the resulting expression on Γ, i.e., letting $\theta_1 = \theta_2 = \phi(x_1)$, $\dot\theta_1 = \dot\theta_2 = \phi'(x_1)\dot x_1$, and $\ddot\theta_1 = \ddot\theta_2 = \phi''(x_1)\dot x_1^2 + \phi'(x_1)\ddot x_1$. By so doing, one obtains the reduced dynamics

$$\ddot x_1 = \Psi_1(x_1) + \Psi_2(x_1)\dot x_1^2, \quad \ddot x_2 = \Psi_1(x_1) + \Psi_2(x_1)\dot x_1^2, \tag{11}$$

where $\Psi_1(x_1) = m_1 g l_1 \sin\phi(x_1)/(m_1 l^2 \phi'(x_1) + m_1 l \cos\phi(x_1))$ and $\Psi_2(x_1) = -m_1 l^2 \phi''(x_1)/(m_1 l^2 \phi'(x_1) + m_1 l \cos\phi(x_1))$. The structure of (11) prompts two observations. First, the $\ddot x_1$ equation has the structure of (4). This is due to the fact that the constraint $\theta_1 = \phi(x_1)$ only involves the state of pendulum 1, which has degree of underactuation 1. Since x_1 is a real variable, the $\ddot x_1$ subsystem is guaranteed to be Euler-Lagrange, and its energy function $E(x_1, \dot x_1)$ is given by (5)-(7). Moreover, as we will see in a moment, the level sets of E near $(x_1, \dot x_1) = (0,0)$ correspond to the type of oscillations we wish to stabilize in part (i) of SOP. The second observation is that since $\ddot x_1 - \ddot x_2 = 0$, solutions on the constraint manifold are such that $x_1(t) - x_2(t) \to \pm\infty$ whenever $\dot x_1(0) - \dot x_2(0) \neq 0$. This is obviously undesirable because we want $x_2(t) - x_1(t) \to d_1$. Motivated by these observations, we will turn VHC (8) into a dynamic constraint so as to stabilize a level set of $E(x_1, \dot x_1)$, and stabilize the equilibrium $x_2 - x_1 = d_1$. The level sets of $E(x_1, \dot x_1)$ are shown in Figure 2. It can be seen that the level sets in a neighborhood of $(x_1, \dot x_1) = (0,0)$ are closed curves. In particular, the level set $\{E(x_1, \dot x_1) = 0\}$ is the stable equilibrium $(x_1, \dot x_1) = (0,0)$. Since $\phi(0) = 0$, oscillations of $(x_1, \dot x_1)$ near $(0,0)$ correspond to oscillations of $(\theta_1, \dot\theta_1)$ near $(0,0)$. Physically this means that when the VHC $\theta_1 = \phi(x_1)$ is enforced on pendulum 1, with appropriate initialization the cart will oscillate around $x_1 = 0$, while the pendulum will oscillate around its unstable inverted configuration. This is precisely the type of oscillation we wish to stabilize in part (i) of SOP. Accordingly, let $\{E(x_1, \dot x_1) = E_0\}$ be a desired level set of the energy for pendulum 1. Then, the closed orbit we wish to stabilize is

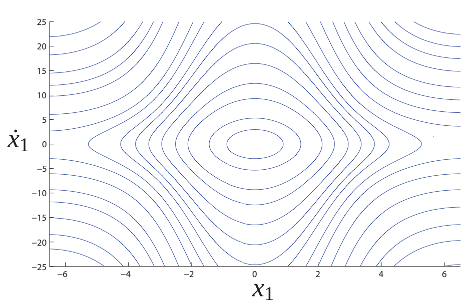

Fig. 2. Phase portraits of the reduced motion of the master cart-pendulum for the VHC's with $\phi(x_1) = -\frac{\pi}{3}\sin(\frac{1.5}{\pi}x_1)$. All physical parameters are assumed to be unity.

$$\begin{aligned}\mathscr{C} = \{(\theta_1, x_1, \dot\theta_1, \dot x_1) \in Q_1 : E(x_1, \dot x_1) = E_0, \\ \theta_1 = \phi(x_1), \dot\theta_1 = \phi'(x_1)\dot x_1\}.\end{aligned} \tag{12}$$

Stabilizing $\mathscr{C}$ with $E_0 = 0$ will correspond to swinging up the pendulum. The stabilization of $\mathscr{C}$ can be performed using the tools of [3] reviewed in Section II. In particular, notice that if a function $\phi(x_1)$ satisfies relations (9), then so does the function $\phi(x_1 - s_1)$ for all $s_1 \in \mathbb{R}$. In light of this observation, we enforce the dynamic VHC $\theta_1 = \phi^{s_1}(x_1) = \phi(x_1 - s_1)$ on pendulum 1, with ϕ given in (10), through the input-output linearizing feedback

$$\begin{aligned} u_1^\star = &\left\{[1 \ -\partial_{x_1}\phi^{s_1}]D_1^{-1}B_1\right\}^{-1}\left[[1 \ -\partial_{x_1}\phi^{s_1}]D_1^{-1}(C_1\dot q_1 + \nabla P_1)\right. \\ &+ (\partial^2_{x_1}\phi^{s_1})\dot x_1^2 + (2\partial_{x_1}\partial_{s_1}\phi^{s_1})\dot x_1\dot s_1 + (\partial^2_{s_1}\phi^{s_1})\dot s_1^2 + (\partial_{s_1}\phi^{s_1})v_1 \\ &\left. - k_1(\theta_1 - \phi^{s_1}) - k_2(\dot\theta_1 - (\partial_{x_1}\phi^{s_1})\dot x_1 - (\partial_{s_1}\phi^{s_1})\dot s_1)\right], \end{aligned} \tag{13}$$

where $k_1, k_2 > 0$ are design parameters. As described in Section II, the evolution of s_1 is governed by

$$\ddot s_1 = K_1(E - E_0)\dot x_1 + K_2 s_1 + K_3 \dot s_1. \tag{14}$$

With an appropriate design of K_1, K_2, K_3, the dynamic feedback $u_1^\star$ in (13)-(14) stabilizes the closed curve $\mathscr{C} \times \{s_1 = \dot s_1 = 0\}$. Now we turn to the stabilization of the equilibrium $x_2 - x_1 = d_1$. For this, we will modify the second constraint in (8) as $\theta_2 = \theta_1 - s_2$, where $\ddot s_2 = v_2$, and v_2 will be designed so that $x_2 - x_1 \to d_1$ and $s_2 \to 0$. This modification introduces a small offset on the oscillations of the slave system, see Figure 3, which can be controlled so that the slave is driven to stay at a fixed distance from the master. One can check that this dynamic VHC is regular for all $|s_2| < \frac{\pi}{2} - \theta_l$. To enforce

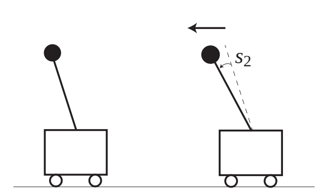

Fig. 3. Use of a dynamic VHC to introduce a driving force on the slave cart-pendulum system to remain at a fixed distance from the master.

the VHC $\theta_2 = \theta_1 - s_2$ we use the input-output linearizing feedback

$$u_2^\star = \left\{[1\ 0]D_2^{-1}B_2\right\}^{-1}\big[[1\ 0]D_1^{-1}(B_1u_1 - C_1\dot{q}_1 - \nabla P_1) + \big[1\ 0]D_2^{-1}(C_2\dot{q}_2 + \nabla P_2) - k_3(\theta_2-\theta_1+s_2) - k_4(\dot\theta_2 - \dot\theta_1 + \dot{s}_2) - v_2\big], \tag{15}$$

where k_3, $k_4 > 0$ are design parameters. Summarizing, we have replaced the VHC in (8) with the dynamic VHC

$$\theta_1 = \phi^{s_1}(x_1) = \phi(x_1 - s_1), \quad \theta_2 = \theta_1 - s_2, \tag{16}$$

where ϕ is given in (10). This VHC is regular for all $s_1 \in \mathbb{R}$ and all $|s_2| < \frac{\pi}{2} - \theta_l$, and the dynamic feedbacks $u_1^\star$, $u_2^\star$ in (13)-(14) and (15) stabilize the associated constraint manifold

$$\bar\Gamma = \{(q_1,q_2,\dot q_1,\dot q_2,s_1,s_2,\dot s_1,\dot s_2) : \theta_1 = \phi^{s_1}(x_1), \theta_2 = \theta_1 - s_2, \dot\theta_1 = \partial_{x_1}\phi^{s_1}(x_1)\dot x_1 + \partial_{s_1}\phi^{s_1}(x_1)\dot s_1, \dot\theta_2 = \dot\theta_1 - \dot s_2\}. \tag{17}$$

What is left to do is to design the control input v_2 in the dynamic compensator $\ddot s_2 = v_2$ so as to stabilize the set $\{x_2 - x_1 - d_1 = 0, \dot x_2 - \dot x_1 = 0, s_2 = \dot s_2 = 0\}$. To this end, we will consider the reduced dynamics on $\bar\Gamma$ assuming that compensator (14) has made the master system converge to the desired energy level set E_0, and that $s_1 = \dot s_1 = 0$. This corresponds to investigating the reduced dynamics on $\bar\Gamma' = \bar\Gamma \cap \{E(x_1,\dot x_1) = E_0, s_1 = \dot s_1 = 0\}$. By left-multiplying the equations in (1) for $i = 1,2$ with a left annihilator of B and evaluating the resulting expressions on $\bar\Gamma'$, we obtain

$$\begin{aligned} \ddot x_2 &= \frac{g\sin(\phi(x_1)-s_2) - l\phi'(x_1)\Psi_1(x_1)}{\cos(\phi(x_1)-s_2)} \\ &\quad - l\frac{(\phi''(x_1)+\phi'(x_1)\Psi_2(x_1))\dot x_1^2 - v_2}{\cos(\phi(x_1)-s_2)} \\ \ddot s_2 &= v_2, \end{aligned} \tag{18}$$

where Ψ_1, Ψ_2 are as before. In the above, $(x_1(t), \dot x_1(t))$ is the periodic solution of the $\ddot x_1$ subsystem in (11) satisfying $E(x_1(t),\dot x_1(t)) = E_0$, and it is viewed as an exogenous signal. In what follows, we will let T_{E_0} be the period of this solution. Consider the error coordinates $e_1 = x_1 - x_2 + d_1$, $e_2 = \dot x_1 - \dot x_2$, $e_3 = s_2$ and $e_4 = \dot s_2$. We have

$$\dot e_1 = e_2,\ \dot e_2 = f(t,e_3) + g(t,e_3)v_2,\ \dot e_3 = e_4,\ \dot e_4 = v_2, \tag{19}$$

where $f(t,e_3)$ and $g(t,e_3)$ are suitable smooth functions. We linearize (19) at the origin $e = 0$, obtaining the linear periodic control system

$$\dot e_1 = e_2,\ \dot e_2 = a_{E_0}(t)e_3 + b_{E_0}(t)v_2,\ \dot e_3 = e_4,\ \dot e_4 = v_2, \tag{20}$$

where

$$\begin{aligned} a_{E_0}(t) &= \frac{g(1+l\phi'(x_1)\cos\phi(x_1)) - l\phi''(x_1)\sin\phi(x_1)\dot x_1^2}{(\cos\phi(x_1))(\cos\phi(x_1)+l\phi'(x_1))} \\ b_{E_0}(t) &= \frac{-l}{\cos\phi(x_1)}, \end{aligned} \tag{21}$$

and where, once again, $(x_1(t),\dot x_1(t))$ is a T_{E_0}-periodic exogenous signal associated to the energy level set E_0 that has been asymptotically stabilized for the master system. The control input v_2 must be designed to stabilize the origin of (20). Consider the linear feedback $v_2 = Le$, where $L = [L_1\ L_2\ L_3\ L_4]$ and $e = [e_1\ e_2\ e_3\ e_4]^\top$. If we find L making the origin of (20) exponentially stable, then the origin $e = 0$ of (19) is exponentially stable as well. To assess the stability of (20) we use averaging theory. Consider the transformations $e \mapsto E$, $L \mapsto C$, defined as

$$\begin{aligned} e &= \mathrm{diag}(1,\varepsilon,\varepsilon^2,\varepsilon^3)E \\ L &= \mathrm{diag}(\varepsilon^4,\varepsilon^3,\varepsilon^2,\varepsilon)C \end{aligned}$$

where $E = [E_1\ E_2\ E_3\ E_4]^\top$, $C = [C_1\ C_2\ C_3\ C_4]^\top$, and $\varepsilon > 0$ is a small parameter. In original coordinates we are using the compensator

$$\ddot s_2 = \varepsilon^4 C_1(x_1 - x_2 + d_1) + \varepsilon^3 C_2(\dot x_1 - \dot x_2) + \varepsilon^2 C_3 s_2 + \varepsilon C_4 \dot s_2. \tag{22}$$

With these definitions, (20) takes on the normal form of averaging theory [5], $\dot E = \varepsilon A_{E_0}(t)E + \varepsilon^3[\star]E$, where

$$A_{E_0}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_{E_0}(t) & 0 \\ 0 & 0 & 0 & 1 \\ C_1 & C_2 & C_3 & C_4 \end{bmatrix}. \tag{23}$$

The averaged system is given by

$$\dot{\bar E} = \varepsilon \bar A_{E_0}\bar E, \tag{24}$$

where $\bar A_{E_0} = (1/T_{E_0})\int_0^{T_{E_0}} A_{E_0}(\tau)\mathrm{d}\tau$ reads as

$$\bar A_{E_0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \bar a_{E_0} & 0 \\ 0 & 0 & 0 & 1 \\ C_1 & C_2 & C_3 & C_4 \end{bmatrix}, \tag{25}$$

with $\bar a_{E_0} = (1/T_{E_0})\int_0^{T_{E_0}} a(\tau)\mathrm{d}\tau$. Note that $\bar a_{E_0}$ depends on the energy level E_0 of the master system corresponding to the periodic orbit $(x_1(t),\dot x_1(t))$ that is rendered asymptotically stable by compensator (14). With the choice of VHC $\phi(x_1)$ in (10), it holds that $\bar a_{E_0} > 0$ as shown with the following lemma.

Lemma 3.1: $\bar a_{E_0} \geq g$, where g is the acceleration due to gravity, for all E_0 such that the set $\{(x_1,\dot x_1) : E(x_1,\dot x_1) = E_0\}$ is a closed curve.

The proof is omitted for brevity. With this lemma $\bar a_{E_0} \neq 0$, it follows that the LTI system

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \bar a_{E_0} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right). \tag{26}$$

is controllable, and therefore for all E_0 there exists $C \in \mathbb{R}^4$ such that $\bar A_{E_0}$ in (25) is Hurwitz. In particular, we have the following result.

Lemma 3.2: The matrix $\bar A_{E_0}$ in (25) is Hurwitz for all $C \in \mathbb{R}^4$ such that $C_1, \ldots, C_4 < 0$ and

$$0 < \bar a_{E_0} < \min\{-C_3C_4/C_2, (C_1C_4^2 - C_2C_3C_4)/C_2^2\}. \tag{27}$$

The proof of the above lemma is straightforward. It relies on the fact, established in Lemma 3.1, that $\bar a_{E_0} > 0$, and the application of the Routh-Hurwitz criterion to the characteristic polynomial of $\bar A_{E_0}$, $\lambda^4 - C_4\lambda^3 - C_3\lambda^2 - \bar a C_2\lambda - \bar a C_1$.

One can show that the function $E_0 \mapsto \bar{a}_{E_0}$ is monotonically increasing. Therefore, in light of inequality (27), if $C \in \mathbb{R}^4$ is such that $\bar{A}_{E_0}$ is Hurwitz, then with the same C it holds that $\bar{A}_{E_1}$ is Hurwitz for all $E_1 \leq E_0$. The next proposition summarizes our results so far.

Proposition 3.3: Consider $N = 2$ pendulums in (1). Given a feasible energy level set E_0, pick $C \in \mathbb{R}^4$ to stabilize the LTI system (26). Then, there exists $\varepsilon^\star > 0$ such that for all $\varepsilon \in (0, \varepsilon^\star)$, the feedbacks $u_1^\star$, $u_2^\star$ in (13), (15) with compensators (14), (22), asymptotically stabilize the sets $\bar{\Gamma}$ in (17) and $\mathscr{S} = \{(q_1,q_2,\dot{q}_1,\dot{q}_2,s_1,s_2,\dot{s}_1,\dot{s}_2) \in \bar{\Gamma} : E(x_1,\dot{x}_1) = E_0, x_1 - x_2 + d_1 = \dot{x}_1 - \dot{x}_2 = 0, s_1 = s_2 = \dot{s}_1 = \dot{s}_2 = 0\}$, therefore solving SOP for $N = 2$. Moreover, the same gain vector C can be used to stabilize any energy level $E_1 \leq E_0$ for the master system.

Sketch of the proof: The feedbacks $u_1^\star$, $u_2^\star$ exponentially stabilize[2] the constraint manifold $\bar{\Gamma}$ in (17). The averaging theorem [5] guarantees the existence of $\varepsilon^\star > 0$ such that for all $\varepsilon \in (0, \varepsilon^\star)$ the averaged system (24) is exponentially stable, implying that the linear periodic system (20) is exponentially stable, and therefore the set $\{x_2 = x_1(t) + d_1, \dot{x}_2 = \dot{x}_1(t), s_2 = \dot{s}_2 = 0\}$ is exponentially stable for system (18) representing the reduced dynamics on $\bar{\Gamma}'$. In other words, $\mathscr{S}$ is exponentially stable relative to $\bar{\Gamma}'$. Moreover, by the theory in [3], $\bar{\Gamma}'$ is exponentially stable relative to $\bar{\Gamma}$ for suitable choices of K_1, K_2, K_3 in (14). By the Seibert-Florio reduction theorem for asymptotic stability of sets in [6], $\mathscr{S}$ is asymptotically stable. ■

B. Design for N cart-pendulums

Here we show that the result for the 2 cart-pendulum systems can be extended to the general N cart-pendulums system *without any additional control design*. The dynamic VHC in (16) is generalized as follows:

$$\theta_1 = \phi^{s_1}(x_1),\ \theta_i = \theta_{i-1} - s_i,\ i \in \{2,\ldots,N\}, \tag{28}$$

where, as before, $\phi^{s_1}(x_1) = \phi(x_1 - s_1)$, and ϕ is given in (10), and the evolution of the parameters s_i is governed by N compensators $\ddot{s}_i = v_i$, $i = 1,\ldots,N$. The dynamic VHC in (28) can be shown to be regular for all $(s_1,\ldots,s_N)$ such that $|\sum_{j=2}^N s_j| < \pi/2 - \theta_l$. The constraint manifold is

$$\begin{aligned}\bar{\Gamma} = \{(q,\dot{q},s,\dot{s}) : \theta_1 = \phi^{s_1}(x_1), \theta_i = \theta_{i-1} - s_i, \dot{\theta}_1 = \partial_{x_1}\phi^{s_1}(x_1)\dot{x}_1 \\ + \partial_{s_1}\phi^{s_1}(x_1)\dot{s}_1, \dot{\theta}_i = \dot{\theta}_{i-1} - \dot{s}_i,\ i \in \{2,\ldots,N\}\},\end{aligned} \tag{29}$$

and the input-output linearizing feedbacks stabilizing $\bar{\Gamma}$ in conjunction with $u_1^\star$ in (13) are

$$\begin{aligned} u_i^\star = \{[1\ 0]D_i^{-1}B_i\}^{-1}\big[[1\ 0]D_{i-1}^{-1}(B_{i-1}u_{i-1} - C_{i-1}\dot{q}_{i-1} \\ -\nabla P_{i-1}) + [1\ 0]D_i^{-1}(C_i\dot{q}_i + \nabla P_i) - k_3(\theta_i - \theta_{i-1} \\ + s_i) - k_4(\dot{\theta}_i - \dot{\theta}_{i-1} + \dot{s}_i) - v_i\big],\ i = 2,\ldots,N. \end{aligned} \tag{30}$$

As before, we are interested in investigating the reduced dynamics on $\bar{\Gamma}' = \bar{\Gamma} \cap \{E(x_1,\dot{x}_1) = E_0, s_1 = \dot{s}_1 = 0\}$. For this, we use the fact that $\ddot{x}_i = -(l/\cos\theta_i)\ddot{\theta}_i + g\sin\theta_i/\cos\theta_i$ and the fact that, on $\bar{\Gamma}$, $\theta_i = \theta_1 - \sum_{j=2}^i s_j$ and $\ddot{\theta}_i = \ddot{\theta}_1 - \sum_{j=2}^i v_j$, so that

$$\ddot{x}_i = \frac{-l(\ddot{\theta}_1 - \sum_{j=2}^i v_j) + g\sin(\theta_1 - \sum_{j=2}^i s_j)}{\cos(\theta_1 - \sum_{j=2}^i s_j)}\Bigg|_{\substack{\theta_1 = \phi(x_1) \\ \ddot{\theta}_1 = \phi''(x_1)\dot{x}_1^2 + \phi'(x_1)\ddot{x}_1}}$$

$$\ddot{s}_i = v_i, \quad i \in \{2,\ldots,N\}.$$

We see from the above that the reduced dynamics on $\bar{\Gamma}'$ are given by $N-1$ decoupled identical subsystems, each of dimension 4, driven by the exogenous signal $(x_1(t),\dot{x}_1(t))$, the periodic solution of the $\ddot{x}_1$ subsystem in (11) with energy level E_0. Defining the error coordinates $e_1^i = x_{i-1} - x_i + d_{i-1}$, $e_2^i = \dot{x}_{i-1} - \dot{x}_i$, $e_3^i = s_i$, $e_4^i = \dot{s}_i$, $i \in \{2,\ldots,N\}$, the error dynamics can be found to be a linear-time varying system which when linearized about the origin and letting $e^i = [e_1^i,\ldots,e_4^i]^\top$, one has that the e^i dynamics are governed by a linear periodic system with input v_i identical to the system in (20), (21). This remarkable fact implies that the analysis performed in the previous section directly carries over to this setting. Thus, for $i \in \{2,\ldots,N\}$, the compensator

$$\ddot{s}_i = \varepsilon^4 C_1(x_{i-1} - x_i + d_{i-1}) + \varepsilon^3 C_2(\dot{x}_{i-1} - \dot{x}_i) + \varepsilon^2 C_3 s_i + \varepsilon C_4 \dot{s}_i, \tag{31}$$

where $C_1,\ldots,C_4$ are designed as in Proposition 3.3 and are independent of i, exponentially stabilizes the origin of the error subsystem.

Proposition 3.4: Consider the N cart-pendulums in (1). Given a feasible energy level set E_0, pick $C \in \mathbb{R}^4$ to stabilize the LTI system (26). Then, there exists $\varepsilon^\star > 0$ such that for all $\varepsilon \in (0,\varepsilon^\star)$, the feedbacks $u_1^\star$ in (13) and $u_i^\star$, $i = 2,\ldots,N$, in (30) with compensators (14), (31) asymptotically stabilize the set $\bar{\Gamma}$ in (29) and the set $\mathscr{S} = \{(q.\dot{q},s,\dot{s}) \in \bar{\Gamma} : E(x_1,\dot{x}_1) = E_0, x_{i-1} - x_i + d_{i-1} = 0, \dot{x}_{i-1} - \dot{x}_i = 0, s_i = \dot{s}_i = 0, i = 1,\ldots,N\}$, therefore solving SOP. Moreover, the same gain vector $C \in \mathbb{R}^4$ can be used to stabilize any energy level set $E_1 \leq E_0$.

Proof: The proof of this result follows directly from the proof of Proposition 3.3 and the fact that the linearized error system is made of N decoupled subsystems, each identical to (20). ■

IV. Simulation results

Here we present simulation results for the case $N = 3$. All physical parameters were taken to be unity. For the master cart-pendulum system we chose to enforce the VHC $\phi(x_1) = -(\pi/3)\sin((1.5/\pi)x_1)$. The gains $k_1,\ldots,k_4$ in the feedbacks $u_1^\star$ in (13) and $u_2^\star, u_3^\star$ in (30) are chosen to be $k_1 = k_3 = 25$, $k_2 = k_4 = 10$. The gains in for the $\ddot{s}_1$ compensator in (14) were chosen to be $K_1 = -0.1, K_2 = -1, K_3 = -1$ The desired cart separations are $d_1 = d_2 = 1$. Finally, for the $\ddot{s}_2$ and $\ddot{s}_3$ compensators in (31) we let $\varepsilon = 1$ and pick $C \in \mathbb{R}^4$ to place all poles of (26) with $\bar{a}_{E_0} = 10$ at -1. This gives $C = [-.11\ \ -.43\ \ -6.33\ \ -4.11]$. Finally, the initial conditions chosen were as follows $q_1(0) = [-.3\ 0]^\top$, $q_2(0) = [.3\ 4]^\top$, $q_3(0) = [.1\ 6]^\top$ and $\dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = [0\ \ 0]^\top$ and $s_i = \dot{s}_i = 0$, $i = 1,2,3$. To illustrate the role of the three compensators $\ddot{s}_1, \ddot{s}_2, \ddot{s}_3$, in Figure 4 we show the positions and angles of the three cart-pendulums when

[2]Actually, $u_1^\star$, $u_2^\star$ exponentially stabilize an open subset of $\bar{\Gamma}$, since the dynamic VHC (16) is regular for small values of s_2.

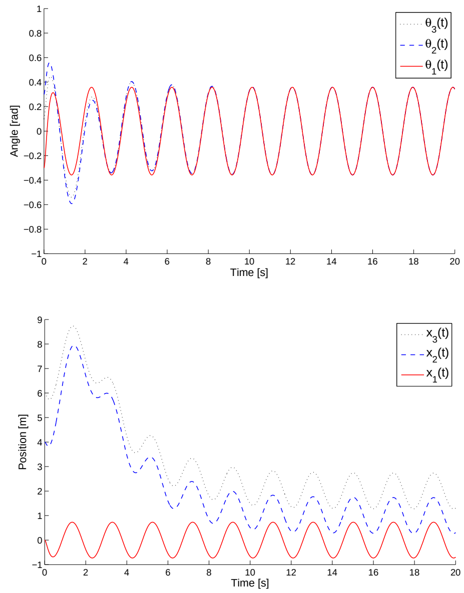

Fig. 4. Full synchronization of 3 cart-pendulums without energy stabilization: angle of each pendulum (top) and cart position (bottom). Note that during transient the three pendulums remain in the upper half-plane.

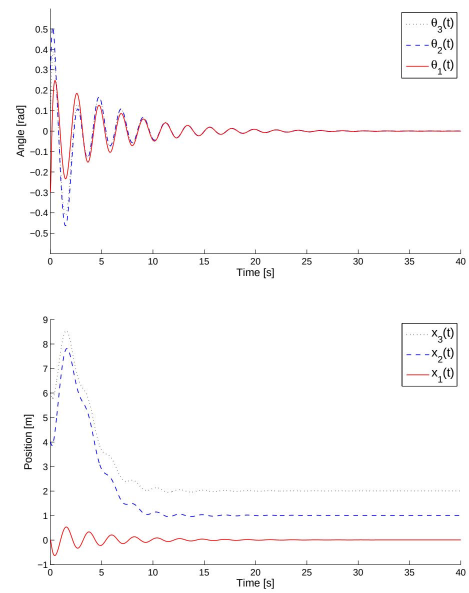

Fig. 5. Full synchronization of 3 cart-pendulums with simultaneous energy level stabilization for the master cart-pendulum. Here we are stabilizing the energy level $E_0 = 0$, which corresponds to stabilizing the upright equilibrium position of the pendulums. Note that during transient the three pendulums remain in the upper half-plane.

the energy stabilization mechanism of compensator $\ddot{s}_1$ is turned off (i.e., $K_1 = K_2 = K_3 = 0$). As Figure 4 shows, the enforcement of the VHC $\theta_1 = \phi(x_1)$ makes the master system oscillate about its inverted configuration, while the other two cart-pendulums fully synchronize to it. Next, in Figure 5 we turn on the energy stabilization mechanism of compensator $\ddot{s}_1$ with $E_0 = 0$, which corresponds to stabilizing the inverted configuration of the three cart-pendulums. We see from Figure 5 that, indeed, the three cart-pendulums fully synchronize, and that the cart positions converge to desired constants while the angles converge to zero.

V. Conclusion

We have presented a technique to fully synchronize N cart-pendulum systems while simultaneously stabilizing a desired oscillation of the pendulums about their inverted configurations. As mentioned in the introduction, SOP has first been presented and solved in [1]. The authors in [1] use VHCs to select a closed orbit, and then apply the transverse linearization technique to stabilize said orbit. Specifically, one needs to compute the $2N-1$-dimensional transverse linearization of (1) around the desired closed orbit, and then use techniques for the stabilization of linear time-varying (LTV) systems to design the feedback. This step can be challenging in practice. Our approach is fundamentally different. First, we require the constraint manifold to be controlled invariant, while in [1] controlled invariance is not imposed since virtual holonomic constraints are only used for planning a specific orbit with desired properties. The controller in [1] is static and time-varying, and it does not make the constraint manifold invariant for the closed-loop system. As a result, the constraint manifold is actually unstable for the closed-loop system. Our controller in Proposition 3.4 is dynamic, with $2N$ states, and it stabilizes the manifold $\bar{\Gamma}$ in (29), thus enforcing the dynamic VHC (28). While in [1] one needs to stabilize an LTV system of order $2N-1$, our control design relies on the stabilization of a fourth-order LTI system, no matter how large N is. Thus, our control design is much simpler. Additionally, the benefit of stabilizing $\bar{\Gamma}$ while simultaneously meeting the specifications of SOP is that one has better control over the transient performance of the system. For instance, if a small disturbance affects the master pendulum so that the relation $\theta_1 = \phi^{s_1}(x_1)$ is violated, the controller will guarantee a graceful recovery, in that during the ensuing transient the quantity $\theta_1 - \phi^{s_1}(x_1)$ will remain small. This is desirable because $|\phi^{s_1}(x_1)| < \pi/2$ for all $s_1 \in \mathbb{R}$, and therefore the master pendulum remains in the upper half-plane for small enough perturbations. Similar considerations hold for all other pendulums in the chain.

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