Robust Output Tracking: The VTOL Aircraft Example

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Abstract

We study the tracking problem in the presence of smooth, bounded uncertainty and show that, if the uncertainty satisfies a suitable matching condition, one can design a partial information controller (i.e., an output feedback controller) achieving arbitrarily small steady-state tracking error without employing high-gain feedback. We illustrate a preliminary application of these results to the control of the (simplified) model of a VTOL aircraft affected by uncertainty.

1 Introduction

In [1] and [2] the notion of a practical internal model was introduced as a paradigm to solve the output feedback (or partial information) tracking problem for nonlinear systems. The word *practical internal model* was chosen to indicate the fact that this paradigm allows to solve the tracking problem *practically* (i.e., to an arbitrary degree of accuracy), rather than asymptotically, and that its solution relies on the existence of a compensator (the practical internal model) which has a conceptually similar role to a nonlinear internal model in output regulation theory (see, e.g., [3] for an introduction to the output regulation problem and the definition of nonlinear internal model). In [2] it was also showed that, when the tracking problem is posed within an output regulation framework with appropriate restrictions, the practical internal model can be replaced by an internal model and the paradigm can still be employed. As pointed out in [1] and [2], this theory is still far from being self-contained and leaves several open questions. One of them is the extension of the results in [1, 2] to the case when the system is affected by uncertainty. The present paper represents a first step in this direction.

Consider the VTOL (vertical take-off and landing) aircraft model introduced in [4]

$$\begin{aligned}
\ddot{\chi} &= -u_1 \sin\theta + \epsilon u_2 \cos\theta \\
\ddot{z} &= u_1 \cos\theta + \epsilon u_2 \sin\theta - g \\
\ddot{\theta} &= \lambda u_2
\end{aligned} \tag{1}$$

In this model χ and z are the coordinates of the center of mass of the aircraft on a fixed inertial frame, and θ is its inclination with respect to the vertical axis. Let $x = [\chi, \dot{\chi}, z, \dot{z}, \theta, \dot{\theta}]^\top$ and rewrite (1) in state-variable form

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -u_1 \sin x_5 + \epsilon u_2 \cos x_5 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u_1 \cos x_5 + \epsilon u_2 \sin x_5 - g \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \lambda u_2.
\end{aligned} \tag{2}$$

As shown in [5], system (2) is dynamic feedback linearizable (differentially flat) with respect to the output given by the *Huygens center of oscillation* $y = (x_1 - \frac{\epsilon}{\lambda}\sin x_5, x_3 + \frac{\epsilon}{\lambda}\cos x_5)$. The linearizing compensator for (2) was found in [5] to be

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -v_1 \sin x_5 + v_2 \cos x_5 + \xi_1 x_6 \\
u_1 &= \xi_1 + \frac{\epsilon}{\lambda} x_6^2 \\
u_2 &= \frac{1}{\lambda \xi_1}(-v_1 \cos x_5 - v_2 \sin x_5 - 2\xi_2 x_6^2).
\end{aligned} \tag{3}$$

It is easily verified, indeed, that the composite system (2), (3) is equivalent to the trivial system $y_1^{(4)} = v_1$, $y_2^{(4)} = v_2$. The linearizing transformation is well-defined on the set $\{[x^\top, \xi^\top]^\top \in \mathbb{R}^8 \,|\, \xi_1 \neq 0\}$.

Now, as in [6], we consider the case when (2) is affected by additive uncertainties as follows

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -u_1 \sin x_5 + \epsilon u_2 \cos x_5 + \Delta_1(t) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u_1 \cos x_5 + \epsilon u_2 \sin x_5 - g + \Delta_2(t) \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \lambda u_2,
\end{aligned} \tag{4}$$

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where $\Delta_1(t)$ and $\Delta_2(t)$ are unknown smooth, bounded functions of time. Assuming that the output y is measurable, the problem we want to solve in this paper entails designing a controller making y track a desired smooth reference trajectory $r(t)$. More specifically, we seek to find a *partial information controller*, i.e., a controller using only the information given by y and r, without assuming the state x or any other signal to be available for feedback. To do that, we employ the idea of practical internal models introduced in [1] for systems without uncertainties, and we introduce an extension allowing us to deal with the uncertainty in (4).

2 Robust Tracking

Here, we present a preliminary extension to the theory developed in [1], [2] which handles the presence of certain types of uncertainties. Given the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u, \Delta(t)) \\ y &= h(x), \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurable output, and $\Delta(t) : \mathbb{R}^+ \to \mathbb{R}^d$ is an unknown smooth function of its arguments belonging to[1] $C^l \cap \mathcal{L}_\infty$, we seek to find a tracking controller solving the following problem

Problem 1 (Output Feedback Practical Tracking): *Given the dynamical system (5) and a sufficiently smooth reference trajectory $r(t) = [r_1(t), \ldots, r_p(t)]^\top$, design a dynamic output feedback controller*

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y, r) \\ u &= h_c(x_c, y) \end{aligned} \tag{6}$$

where f_c and h_c are sufficiently smooth, such that the closed-loop system (5)-(6) has the property that there exists a $T > 0$ such that $\|e(t)\| \leq e_0$ for all $t \geq T$, and such that the internal states x and x_c are bounded for all $t \geq 0$, and for all initial conditions $[x(0)^\top, x_c(0)^\top]^\top \in \mathcal{A}$, for some closed set $\mathcal{A}$.

In [1], we have showed that, when no uncertainty affects the system, if there exists a practical internal model then Problem 1 has a solution. We start by introducing some basic assumptions.

Assumption A1 (Stable Inverse): Given $r(t)$, for all $\Delta \in C^l \cap \mathcal{L}_\infty$ there exist sufficiently smooth and bounded functions $x_r(t)$ and $c_r(t)$ such that

$$\begin{aligned} \dot{x}_r(t) &= f(x_r(t), c_r(t), \Delta(t)) \\ r(t) &= h(x_r(t)) \end{aligned} \tag{7}$$

for some initial condition $x_r(0), c_r(0)$, and all $t \geq 0$.

Next, consider the change of coordinates $\tilde{x} = x - x_r(t)$, rewrite (5) in new coordinates as

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, u, \Delta(t)), \tag{8}$$

and notice that the asymptotic stability of the origin of (8) is equivalent to the stability of the trajectory $x_r(t)$. Next, we assume that in the ideal case when the x, x_r, c_r, and Δ are available for feedback, one can find a smooth controller stabilizing the $\tilde{x}$ dynamics.

Assumption A2 (Stabilizability of the Trajectory $x_r(t)$): There exists a smooth function $\bar{u}(x, x_r, c_r, \Delta(t))$ such that $\bar{u}(x_r, x_r, c_r, \Delta) = c_r$ and the origin is a uniformly asymptotically stable equilibrium point of $\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}, \bar{u}(x, x_r, c_r), \Delta(t))$, with domain of attraction a closed set $\tilde{\mathcal{D}} \subset \mathbb{R}^n$, i.e., there exists (see [7]) a function $V(\tilde{x}, t)$, defined for $\tilde{x} \in \tilde{\mathcal{D}}$, which is continuous with continuous partial derivatives, and continuous positive definite functions $\alpha_1(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}_\infty$, $\alpha_2(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}$, and $\alpha_3(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \in \mathcal{K}$ such that

$$(i) \quad \alpha_1(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \leq V(\tilde{x}, t) \leq \alpha_2(\|\tilde{x}\|_{\tilde{\mathcal{D}}}) \tag{9}$$

$$(ii) \quad \frac{\partial V}{\partial \tilde{x}} \tilde{f}(t, \tilde{x}, \bar{u}(x, x_r, c_r)) + \frac{\partial V}{\partial t} \leq -\alpha_3(\|\tilde{x}\|_{\tilde{\mathcal{D}}}), \tag{10}$$

for $\tilde{x} \in \tilde{\mathcal{D}}$, $\tilde{x} \neq 0$, for all Δ and all $t \geq 0$, where

$$\|\tilde{x}\|_{\tilde{\mathcal{D}}} \triangleq \max\left\{\|\tilde{x}\|, \frac{1}{\rho(\tilde{x}, \tilde{\mathcal{D}}^o)} - \frac{2}{\rho(0, \tilde{\mathcal{D}}^o)}\right\},$$

$\tilde{\mathcal{D}}^o$ is the complement of $\tilde{\mathcal{D}}$ in $\mathbb{R}^n$, and $\rho(\tilde{x}, \tilde{\mathcal{D}}^o)$ denotes the distance of $\tilde{x}$ from the set $\tilde{\mathcal{D}}^o$ (i.e., $\rho(\tilde{x}, \tilde{\mathcal{D}}^o) = \inf_{z \in \tilde{\mathcal{D}}^o} \|\tilde{x} - z\|$).

Now extend the dynamics of (8) with m integrators - one for every input channel,

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(t, \tilde{x}, s, \Delta(t)), \\ \dot{s} &= u'. \end{aligned} \tag{11}$$

The $m \times 1$ vector u' is the new control input after dynamic extension. Using integrator backstepping, we can find a smooth controller $\bar{u}'(x, s, x_r, c_r, \Delta(t))$ such that setting $u' = \bar{u}'(x, s, x_r, c_r, \Delta(t))$ the origin of (11) is uniformly asymptotically stable. Let $\tilde{\mathcal{D}}'$ be the domain of attraction of the origin of (11) when $u' = \bar{u}'$ and $V'(\tilde{x}, s, t)$ be the Lyapunov function resulting from V when applying integrator backstepping. Note that V' has properties analogous to those in V in (9) and (10).

Remark 1: The dynamic extension in (11) is used in the proof of Theorem 1 to eliminate the presence of an algebraic loop in the controller solving Problem

[1] The degree of continuity l is assumed to be "sufficiently large," in other words we assume the disturbance $\Delta(t)$ to be sufficiently smooth and bounded.

1. When the stabilizer in Assumption A2' rather than being static is dynamic, the dynamic extension (11) is not needed.

Next, we assume that the uncertainty $\Delta(t)$ satisfies a matching condition

Assumption A3 (Matching Condition): There exists a smooth function $m(x,u,\Delta(t)) : \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{R}^d \to \mathbb{R}^m$ such that, setting $\tilde{u} = m(x,u,\Delta(t))$, (5) can be rewritten as

$$\begin{aligned} \dot{x} &= f(x,\tilde{u},0) \\ y &= h(x), \end{aligned} \tag{12}$$

and the function $m(x,u,\Delta)$ is a diffeomorphism with respect to its second and third argument, i.e., there exist smooth functions $m_\Delta^{-1}(x,u,\tilde{u})$ and $m_u^{-1}(x,\tilde{u},\Delta)$ such that

$$\Delta = m_\Delta^{-1}(x,u,\tilde{u}),\ u = m_u^{-1}(x,\tilde{u},\Delta). \tag{13}$$

We now introduce a condition to estimate the functions $x_r(t)$ and $c_r(t)$ on-line. Before stating the assumption, let $\tilde{c}_r(t) = m(x_r, c_r, \Delta(t))$ and us A3 to rewrite (7) as

$$\begin{aligned} \dot{x}_r &= f(x_r,\tilde{c}_r,0) \\ r(t) &= h(x_r). \end{aligned} \tag{14}$$

It is useful to think of (14) as a copy of the plant with unknown state x_r, unknown input $\tilde{c}_r$, but a known output which is the reference trajectory $r(t)$. Consider a compensator of the type

$$\begin{aligned} \dot{\zeta}_r &= a(\zeta_r, x_r, v_r) \\ \tilde{c}_r &= b(\zeta_r, x_r), \end{aligned} \tag{15}$$

where $\zeta_r \in \mathbb{R}^q$ $(q \geq m)$, $v_r \in \mathbb{R}^m$, a and b are sufficiently smooth, and v_r is the new input of the composite system (14)-(15). Let $X_1 = [{x_r}^\top, {\zeta_r}^\top]^\top$ and rewrite (14)-(15) as

$$\begin{aligned} \dot{X}_1 &= f^X(X_1, v_r) \\ r &= h^X(X_1) \end{aligned} \tag{16}$$

(with obvious definition of f^X and h^X). Define the observability mapping associated with X_1 in (16) as

$$\begin{aligned} y_{X_1} &\triangleq [r_1,\ldots,r_1^{(\bar{k}_1-1)},\ldots,r_p,\ldots,r_p^{(\bar{k}_p-1)}]^\top \\ &\triangleq \mathcal{H}_X\left(X_1, v_r, \ldots, {v_r}^{(\bar{n}_u-1)}\right), \end{aligned}$$

where $\sum_{i=1}^p \bar{k}_i = n+q$, $0 \leq \bar{n}_u \leq \max\{\bar{k}_1,\ldots,\bar{k}_p\}-1$.

Assumption A4 (Practical Internal Model): There exists a compensator of the form (15), which we call a *practical internal model*, which is regular (i.e., for each $x(0)$ and $u(t)$ there exist $\zeta(0)$ and $v(t)$ such that $b(\zeta,x) = u$, for all $t \geq 0$) and such that the following two properties hold for the composite system (14)-(15).
(i) $\mathcal{H}_X$ does not depend on v_r and its derivatives, i.e., $\mathcal{H}_X = \mathcal{H}_X(X_1)$.
(ii) There exists a set of indices $\{\bar{k}_1,\ldots,\bar{k}_p\}$ such that the mapping $y_{X_1} = \mathcal{H}_X(X_1)$ is invertible with respect to X_1, and its inverse is sufficiently smooth, for all $X_1 \in \mathcal{X}_a \subset \mathbb{R}^{n+q}$.

Notice that, by replacing x_r, ζ_r, $\tilde{c}_r$, and v_r in (14), (15) by x, ζ, $\tilde{u} = m(x,u,\Delta(t))$, and v, we get an observability assumption for a copy of the plant with state x and input u, augmented by a practical internal model with state ζ and input v. Thus, letting $X_2 = [x^\top, \zeta^\top]^\top$, the dynamics associated with X_2 have identical structure to (16),

$$\begin{aligned} \dot{X}_2 &= f^X(X_2, v) \\ y &= h^X(X_2), \end{aligned} \tag{17}$$

and A4 guarantees that from y and its time derivatives (i.e., the vector $y_{X_2} = [y_1,\ldots,y_1^{(\bar{k}_1-1)},\ldots,y_p,\ldots, y_p^{(\bar{k}_p-1)}]^\top$) one can get X_2, i.e., x and ζ, and thus also $\tilde{u} = b(\zeta,x)$. We will use this fact, together with A3, to estimate x and $\Delta(t)$. Next, we need to guarantee that the reference trajectory is contained in within an observable region.

Assumption A5 (Reference Trajectory): The reference trajectory $r(t)$ is such that, for all $t \geq 0$,

$$y_{X_1} \in \mathcal{C}_r \subset \mathcal{H}_X(\mathcal{X}_a),$$

for some convex compact set $\mathcal{C}_r$ with C^1 boundary.

Finally, we need to make sure that the state and input trajectories of the closed-loop system travel within the observable domain of the plant (at least in the ideal case when the state feedback controller is employed). To this end, in the following assumption we characterize a subset of the domain of attraction $\tilde{\mathcal{D}}'$ which is contained within an observable region of (17). Given any scalar $c > 0$ let

$$\Omega_c \triangleq \{[x^\top, s^\top]^\top \in \mathbb{R}^{n+m} \,|\, V'(x - x_r, s, t) \leq c, \forall t \geq 0\}$$

and note that, by the properness of V' and the definition of $\tilde{\mathcal{D}}'$, given any set $\Pi \subset \tilde{\mathcal{D}}'$, there exists a sufficiently large scalar $c^* > 0$ such that $\Pi \subset \Omega_{c^*} \subset \tilde{\mathcal{D}}'$. From A2, when $u' = \bar{u}'$ in (11), the set Ω_c is positively invariant, for any $c > 0$. In other words,

$$[x(0)^\top, s(0)^\top]^\top \in \Omega_c \Rightarrow [x(t)^\top, s(t)^\top]^\top \in \Omega_c,\ \forall t \geq 0,$$

From A3 we can rewrite (5) as (12) where, from the previous discussion and the boundedness of $\Delta(t)$, $\tilde{u}(t) = m(x(t), s(t), \Delta(t))$ is a uniformly bounded time signal with bound depending on c. Consider now (17), i.e., (12) augmented with a practical internal model.

From the regularity property of the practical internal model, for all $x(t)$ there exists an initial condition $\zeta(0)$ and a bounded control input $v(t)$ such that $b(\zeta(t), x(t)) = \tilde{u}(t)$. In particular, the uniform boundedness of $\tilde{u}(t)$ and $x(t)$ implies the existence of a compact set Ω_c^ζ such that $\zeta(t) \in \Omega_c^\zeta$ for all $t \geq 0$ whenever $[x(t)^\top, s(t)^\top]^\top \in \Omega_c$. Let Ω_c^x be the projection of Ω_c on the x coordinates, i.e., $\Omega_c^x = \{x \in \mathbb{R}^n \mid [x^\top, s^\top]^\top \in \Omega_c\}$ and consider the following assumption.

Assumption A6 (Topology of $\mathcal{O}$): There exists a positive scalar $\bar{c}$ such that

$$\mathcal{H}_X(\Omega_{\bar{c}}^x \times \Omega_{\bar{c}}^\zeta) \subset \mathcal{C} \subset \mathcal{H}_X(\mathcal{X}_a),$$

for some convex compact $\mathcal{C}$ with C^1 boundary.

Theorem 1 *Suppose that A1-A6 hold. Then, for any $\Delta(t) \in C^l \cap \mathcal{L}_\infty$, Problem 1 has a solution on a set $\mathcal{A}$ whose size depends on the size of the sets $\mathcal{C}$, $\mathcal{C}_r$, and $\mathcal{D}$. If A2 and A4 hold globally (i.e., $\tilde{\mathcal{D}}' = \mathbb{R}^{n+m}$ and $\mathcal{X}_a = \mathbb{R}^{n+q}$) and $\mathcal{H}_X(\mathbb{R}^{n+q})$ is convex, then the solution of Problem 1 is semiglobal and $\mathcal{A}$ can be chosen to be an arbitrarily large compact set.*

Sketch of the proof. Recall the definition of X_1 and X_2, and let $v_1 = v_r$, $v_2 = v$, $y_1 = r = h^X(X_1)$, $y_2 = y = h^X(X_2)$, $\mathcal{C}^1 = \mathcal{C}$, $\mathcal{C}^2 = \mathcal{C}_r$, so that (16), (17) can be rewritten as

$$\begin{aligned} \dot{X}_i &= f^X(X_i, v_i) \\ y_i &= h^X(X_i),\ i = 1, 2. \end{aligned} \tag{18}$$

For $i = 1, 2$, consider the estimator in (19), (20). The $(n+q) \times (n+q)$ matrix $\mathcal{E}^i$ is defined as $\mathcal{E}^i = \textit{block-diag}[\mathcal{E}_1^i, \ldots, \mathcal{E}_p^i]$, where $\mathcal{E}_j^i = \textit{diag}[\rho_i, \rho_i^2, \ldots, \rho_i^{\bar{k}_j}]$, $j = 1, \ldots, p$, and ρ_i is a positive design parameter. The $(n+q) \times 1$ vector $N^i(\hat{y}^P_{X_i})$ represents the normal to $\partial\mathcal{C}^i$ at $\hat{y}^P_{X_i}$. The $(n+q) \times p$ matrix L^i is defined as $L^i = \textit{block-diag}[L_1^i, \ldots, L_p^i]$, where each L_j^i, $j = 1, \ldots, p$, is a $\bar{k}_j \times 1$ Hurwitz vector. Finally, $\Gamma^i = (S^i\bar{\mathcal{E}}^i)^{-1}(S^i\bar{\mathcal{E}}^i)^{-1}$, where $\bar{\mathcal{E}}^i = \textit{block-diag}[\bar{\mathcal{E}}_1^i, \ldots, \mathcal{E}_p^i]$, with $\bar{\mathcal{E}}_j^i = \textit{diag}[1/\rho_i^{\bar{k}_j - 1}, \ldots, 1]$, $j = 1, \ldots, p$, and S^i is the symmetric matrix square root of P^i, the solution of the Lyapunov equation ${A^i}^\top P^i + P^i A^i = -I_{(n+q)\times(n+q)}$, with

$$A_i = \begin{bmatrix} 0_{(n+q-1)\times 1} & I_{(n+q-1)\times(n+q-1)} \\ & 0_{1\times(n+q)} \end{bmatrix} - L^i[1, 0_{1\times n+q-1}].$$

The estimator (19) incorporates a high-gain component to guarantee convergence, and a dynamic projection to avoid peaking and confine the estimator state to within the observable region $\mathcal{X}_a$. Its properties are summarized in the following result, which is essentially identical to a result found in [8] and is reported without proof.

Lemma 1 *Consider (18) and (19), and assume that A4 and A5 ($i = 1$) or A6 ($i = 2$) hold. Then the estimates $\hat{X}_i^P$ enjoy the following properties*
(i) Boundedness: *if $\hat{X}_i^P(0) \in \mathcal{H}_X^{-1}(\mathcal{C}^i)$, then $\hat{X}_i^P(t) \in \mathcal{H}_X^{-1}(\mathcal{C}^i)$ for all t.*
(ii) Uniform Ultimate Boundedness of the Estimation Error: *For all $\delta > 0$, there exist $\bar{\rho}_i$, $\bar{\rho}_i \in (0, 1]$, and $T(\rho_i) > 0$ such that $\hat{X}_i^P(t) - X_i(t) \leq \delta$ for all $t \geq T(\rho_i)$, whenever $\rho_i \in (0, \bar{\rho}_i)$.*
(iii) Arbitrarily fast rate of convergence: *$T(\rho_i)$ in part (ii) has the property that $T(\rho_i) \to 0$ as $\rho_i \to 0$.*

For the estimator obtained setting $i = 2$, parts (ii) and (iii) hold provided that $X_2(t) \in \Omega_{\bar{c}}^x \times \Omega_{\bar{c}}^\zeta$, for all $t \geq 0$.

The idea used to solve Problem 1 in the presence of uncertainties is illustrated in Figure 1. We start by defining a *full information* controller which, if x, ζ, x_r, ζ_r were known, would yield asymptotic tracking. Then, using A6 and the existence of a practical internal model for (5), we utilize a separation principle to find a partial information controller using y and r to recover the performance of the full information controller.

Consider the plant (5) and its copy (7), and use A3 to rewrite them as

$$\begin{aligned} \dot{x} &= f(x, \tilde{u}, 0) & \dot{x}_r &= f(x_r, \tilde{c}_r, 0) \\ y &= h(x) & r &= h(x_r). \end{aligned} \tag{21}$$

Now use A4 and augment both systems with two practical internal models with states ζ and ζ_r, respectively, so that, recalling that $X_1 = [{x_r}^\top, {\zeta_r}^\top]^\top$ and $X_2 = [x^\top, \zeta^\top]^\top$, the augmented systems can be written as (18). Assume that X_1 and X_2 (i.e., x_r, ζ_r, x, and ζ) are available for feedback. Then, from A3, we have that $\Delta = m_\Delta^{-1}(x, u, b(\zeta, x)) \stackrel{\triangle}{=} \gamma_1(X_2, u)$, $c_r = m_u^{-1}(x_r, b(\zeta_r, x_r), \Delta) \stackrel{\triangle}{=} \gamma_2(X_1, X_2)$. In conclusion, back to (5) and (7), the knowledge of X_1 and X_2 allows to specify the following full information controller

$$\begin{aligned} u &= s \\ \dot{s} &= \bar{u}'(x, s, x_r, \gamma_2(X_1, X_2), \gamma_1(X_2, s)) \end{aligned} \tag{22}$$

which, by A2, achieves asymptotic stability of (11) for all $[x(0)^\top, s(0)^\top]^\top \in \tilde{\mathcal{D}}'$ and, thus, asymptotic tracking. Note here the role of the m integrators with state s introduced in A2 and appearing in (22): without s the full information controller

$$u = \bar{u}(x, x_r, \gamma_2(X_1, X_2), \gamma_1(X_2, u))$$

would be implicitly defined or even not defined at all. The integrators with state s eliminate this problem.

Consider now the partial information controller derived by replacing X_1 and X_2 in (22) by their estimates gen-

$$\dot{\hat{X}}_i^P = \begin{cases} \left[\dfrac{\partial \mathcal{H}_X}{\partial \hat{X}_i^P}\right]^{-1} \left\{ \dot{\hat{y}}_{X_i}|_{\hat{X}_i^P} - \Gamma^i \dfrac{N^i(\hat{y}_{X_i}^P) N^i(\hat{y}_{X_i}^P)^\top \dot{\hat{y}}_{X_i}|_{\hat{X}_i^P}}{N^i(\hat{y}_{X_i}^P)^\top \Gamma^i N^i(\hat{y}_{X_i}^P)} \right\} & \text{if } N^i(\hat{y}_{X_i}^P)^\top \dot{\hat{y}}_{X_i}|_{\hat{X}_i^P} \geq 0 \text{ and } \hat{y}_{X_i}^P \in \partial \mathcal{C}^i \\ \hat{f}^X(\hat{X}_i, y_i) = f^X(\hat{X}_i, 0) + \left[\dfrac{\partial \mathcal{H}_X(\hat{X}_i)}{\partial \hat{X}_i}\right]^{-1} (\mathcal{E}^i)^{-1} L^i \left(y_i - h^X(\hat{X}_i)\right) & \text{otherwise} \end{cases} \tag{19}$$

$$\dot{\hat{y}}_{X_i}|_{\hat{X}_i^P} = \frac{\partial \mathcal{H}_X}{\partial \hat{X}_i^P} f^X(\hat{X}_i^P, y_i), \quad \hat{y}_{X_i}^P = \mathcal{H}_X(\hat{X}_i^P). \tag{20}$$

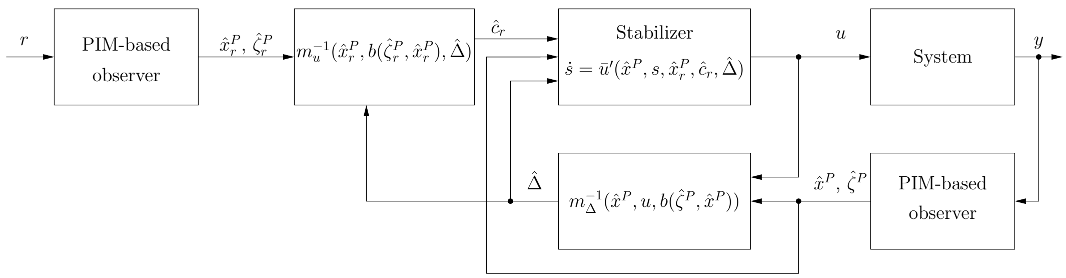

Figure 1: Robust tracking scheme.

erated by (19).

$$\begin{aligned} \hat{u} &= s \\ \dot{s} &= \bar{u}'(\hat{x}^P, s, \hat{x}_r^P, \hat{c}_r, \hat{\Delta}), \end{aligned} \tag{23}$$

where

$$\begin{aligned} \hat{c}_r &= m_u^{-1}(\hat{x}_r^P, b(\hat{\zeta}_r^P, \hat{x}_r^P), \hat{\Delta}) \\ \hat{\Delta} &= m_\Delta^{-1}(\hat{x}^P, s, b(\hat{\zeta}^P, \hat{x}^P)), \end{aligned}$$

and $[\hat{x}_r^{P\top}, \hat{\zeta}_r^{P\top}]^\top = \hat{X}_1^P$, $[\hat{x}^{P\top}, \hat{\zeta}^{P\top}]^\top = \hat{X}_2^P$ are the states of the estimators (19), with $i = 1, 2$. From the convergence properties of the estimators (19), listed in Lemma 1, we can apply the separation principle developed in [8] and conclude that, for any $\bar{c} \in (0, \bar{c})$, there exist sufficient small values of ρ_1 and ρ_2 such that the controller (23) with state $x_c = [s^\top, \hat{x}^{P\top}, \hat{\zeta}^{P\top}, \hat{x}_r^{P\top}, \hat{\zeta}_r^{P\top}]^\top$ solves Problem 1 on the set

$$\mathcal{A} = \Omega_{\underline{c}} \times \mathcal{H}_X^{-1}(\mathcal{C}) \times \mathcal{H}_X^{-1}(\mathcal{C}_r).$$

If A2 and A4 hold globally, then we have that $\tilde{\mathcal{D}} = \mathbb{R}^n$ and $\mathcal{X}_a = \mathbb{R}^{n+q}$. From the integrator backstepping lemma, the global stabilizability of the origin of (8) implies the global stabilizability of the origin of (11) or, in other words, $\tilde{\mathcal{D}}' = \mathbb{R}^{n+m}$. From the fact that $\mathcal{X}_a = \mathbb{R}^{n+q}$ and that $\mathcal{H}_X(\mathbb{R}^{n+q}$ is convex we have that for any bounded reference trajectory with bounded derivatives (i.e., y_{X_1} is bounded), there exists a sufficiently large set $\mathcal{C}_r$ satisfying A5. Further, since V' is proper on $\tilde{\mathcal{D}}'$, A6 is satisfied for any $\bar{c} > 0$ by an arbitrarily large set $\mathcal{C}$. From these observations and the first part of this theorem we conclude that there exist sufficiently small values of ρ_1 and ρ_2 such that (23) solves Problem 1 on an arbitrarily large compact set $\mathcal{A}$. ■

3 Application to the VTOL Model

Go back to the uncertain VTOL aircraft model (4), let $\Delta(t) = [\Delta_1(t), \Delta_2(t)]^\top$, and notice that, setting

$$\tilde{u} = \begin{bmatrix} u_1 + \Delta^\top \begin{pmatrix} -\sin x_5 \\ \cos x_5 \end{pmatrix} \\ u_2 + \dfrac{1}{\epsilon} \Delta^\top \begin{pmatrix} \cos x_5 \\ \sin x_5 \end{pmatrix} \end{bmatrix} = m(x, u, \Delta), \tag{24}$$

we can rewrite (4) as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tilde{u}_1 \sin x_5 + \epsilon u_2 \cos x_5 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \tilde{u}_1 \cos x_5 + \epsilon u_2 \sin x_5 - g \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= \lambda u_2 \\ y &= \left(x_1 - \frac{\epsilon}{\lambda} \sin x_5, x_3 + \frac{\epsilon}{\lambda} \cos x_5\right), \end{aligned} \tag{25}$$

which is in the form (12). Note further that, from (24), we can write

$$\Delta = \begin{bmatrix} -\sin x_5 & \cos x_5 \\ \dfrac{1}{\epsilon} \cos x_5 & \dfrac{1}{\epsilon} \sin x_5 \end{bmatrix}^\top (\tilde{u} - u) = m_\Delta^{-1}(x, u, \tilde{u})$$

$$u = \tilde{u} - \begin{bmatrix} -\sin x_5 & \cos x_5 \\ \dfrac{1}{\epsilon} \cos x_5 & \dfrac{1}{\epsilon} \sin x_5 \end{bmatrix} \Delta = m_u^{-1}(x, \tilde{u}, \Delta),$$

thus showing that A3 is satisfied.

Since (25) is dynamic feedback linearizable (differentially flat), it was shown in [2] that a practical internal model is given by the linearizing compensator (3) augmented with m (in this case $m = 2$) integrators at the input side

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= -\zeta_3 \sin x_5 + \zeta_3 \cos x_5 + \zeta_1 x_6 \\
\dot{\zeta}_3 &= v_1' \\
\dot{\zeta}_4 &= v_2' \\
u_1 &= \zeta_1 + \frac{\epsilon}{\lambda} x_6^2 \\
u_2 &= \frac{1}{\lambda\zeta}(-\zeta_3 \cos x_5 - \zeta_4 \sin x_5 - 2\zeta_2 x_6^2).
\end{aligned} \tag{26}$$

Thus, it is readily seen that A4' is satisfied on the set $\mathcal{X}_a = \{[x^\top, \zeta^\top]^\top \in \mathbb{R}^{10} \,|\, \zeta_1 \neq 0\}$.

Using the fact that (4) is dynamic feedback linearizable with linearizing compensator (3), the stabilizer in A2 is simply given by the feedback linearizing controller for the augmented system (4), (3), parameterized by $\Delta(t)$. From Remark 1, since the stabilizer is dynamic, there is no need to add m integrators at the input side of the system to avoid the presence of an algebraic loop in the final tracking controller.

4 Simulation Results

Consider the problem of making the Huygens center of oscillation of the aircraft follow a circle, $r(t) = [\cos t, \sin t]^\top$. The uncertainty $\Delta(t) = [\Delta_1(t), \Delta_2(t)]^\top$ is chosen to be a sinusoid with frequency $5Hz$ (the two components of Δ are not in phase) with $\|\Delta\| \leq 8 \cdot 10^{-3}$. In Figure 2 we plot the reference trajectory and the output of the VTOL aircraft using the partial information controller introduced in the proof of Theorem 1 and depicted in Figure 1, with the practical internal model (26) and a stabilizer given by the feedback linearizing controller for the extended dynamics (4),(3). Figure 3 depicts the norm of the tracking error as a function of time. The asymptotic tracking error can be made arbitrarily small (thus rejecting the disturbance Δ) without using high-gain control.

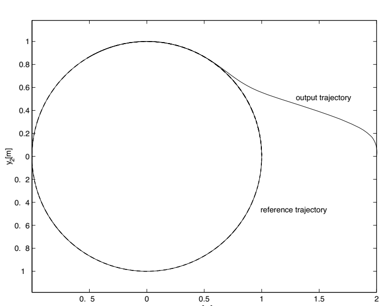

Figure 2: Output and reference trajectories.

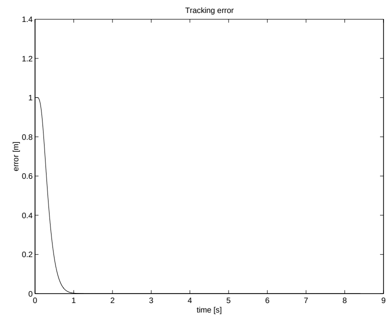

Figure 3: Tracking error.

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