Reduction Theorems for Hybrid Dynamical Systems

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***Abstract*—This paper presents reduction theorems for stability, attractivity, and asymptotic stability of compact subsets of the state space of a hybrid dynamical system. Given two closed sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$, with Γ_1 compact, the theorems presented in this paper give conditions under which a qualitative property of Γ_1 that holds relative to Γ_2 (stability, attractivity, or asymptotic stability) can be guaranteed to also hold relative to the state space of the hybrid system. As a consequence of these results, sufficient conditions are presented for the stability of compact sets in cascade-connected hybrid systems. We also present a result for hybrid systems with outputs that converge to zero along solutions. If such a system enjoys a detectability property with respect to a set Γ_1, then Γ_1 is globally attractive. The theory of this paper is used to develop a hybrid estimator for the period of oscillation of a sinusoidal signal.**

I. Introduction

OVER the past ten to fifteen years, research in hybrid dynamical systems theory has intensified following the work of A.R. Teel and co-authors (e.g., [9], [10]), which unified previous results under a common framework, and produced a comprehensive theory of stability and robustness. In the framework of [9], [10], a hybrid system is a dynamical system whose solutions can flow and jump, whereby flows are modelled by differential inclusions, and jumps are modelled by update maps. Motivated by the fact that many challenging control specifications can be cast as problems of set stabilization, the stability of sets plays a central role in hybrid systems theory.

For continuous nonlinear systems, a useful way to assess whether a closed subset of the state space is asymptotically stable is to exploit hierarchical decompositions of the stability problem. To illustrate this fact, consider the continuous-time cascade-connected system

$$\begin{aligned} \dot{x}^1 &= f_1(x^1, x^2) \\ \dot{x}^2 &= f_2(x^2), \end{aligned} \tag{1}$$

with state $(x^1, x^2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and assume that $f_1(0,0) = 0$, $f_2(0) = 0$. To determine whether or not the equilibrium $(x^1, x^2) = (0,0)$ is asymptotically stable for (1), one may equivalently determine whether or not $x^1 = 0$ is asymptotically stable for $\dot{x}^1 = f_1(x^1, 0)$ and $x^2 = 0$ is asymptotically stable for $\dot{x}^2 = f_2(x^2)$ (see, e.g., [29], [33]). This way the stability problem is decomposed into two simpler subproblems.

For dynamical systems that do not possess the cascade-connected structure (1), the generalization of the decomposition just described is the focus of the so-called *reduction problem*, originally formulated by P. Seibert in [26], [27]. Consider a dynamical system on $\mathbb{R}^n$ and two closed sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. Assume that Γ_1 is either stable, attractive, or asymptotically stable relative to Γ_2, i.e., when solutions are initialized on Γ_2. What additional properties should hold in order that Γ_1 be, respectively, stable, attractive, or asymptotically stable? The global version of this reduction problem is formulated analogously. For continuous dynamical systems, the reduction problem was solved in [28] for the case when Γ_1 is compact, and in [7] when Γ_1 is a closed set. In particular, the work in [7] linked the reduction problem with a hierarchical control design viewpoint, in which a hierarchy of control specifications corresponds to a sequence of sets $\Gamma_1 \subset \cdots \subset \Gamma_l$ to be stabilized. The design technique of backstepping can be regarded as one such hierarchical control design problem. Other relevant literature for the reduction problem is found in [13], [15].

In the context of hybrid dynamical systems, the reduction problem is just as important as its counterpart for continuous nonlinear systems. To illustrate this fact, we mention three application areas of the theorems presented in this paper. Additional theoretical implications are discussed in Section III.

Recent literature on stabilization of hybrid limit cycles for bipedal robots (e.g., [23]) relies on the stabilization of a set Γ_2 (the so-called hybrid zero dynamics) on which the robot satisfies “virtual constraints.” The key idea in this literature is that, with an appropriate design, one may ensure the existence of a hybrid limit cycle, $\Gamma_1 \subset \Gamma_2$, corresponding to stable walking for the dynamics of the robot on the set Γ_2. In this context, the theorems presented in this paper can be used to show that the hybrid limit cycle is asymptotically stable for the closed-loop hybrid system, without Lyapunov analysis.

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Furthermore, as we show in Section V, the problem of estimating the unknown frequency of a sinusoidal signal can be cast as a reduction problem involving three sets $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3$. More generally, we envision that the theorems in this paper may be applied in hybrid estimation problems as already done in [2], whose proof would be simplified by the results of this paper.

Finally, it was shown in [14], [19], [24] that for underactuated VTOL vehicles, leveraging reduction theorems one may partition the position control problem into a hierarchy of two control specifications: position control for a point-mass, and attitude tracking. Reduction theorems for hybrid dynamical systems enable employing the hybrid attitude trackers in [18], allowing one to generalize the results in [14], [24] and obtain global asymptotic position stabilization and tracking.

Contributions of this paper. The goal of this paper is to extend the reduction theorems for continuous dynamical systems found in [7], [28] to the context of hybrid systems modelled in the framework of [9], [10]. We assume throughout that Γ_1 is a compact set and develop reduction theorems for stability of Γ_1 (Theorem 4.1), local/global attractivity of Γ_1 (Theorem 4.4), and local/global asymptotic stability of Γ_1 (Theorem 4.7). The conditions of the reduction theorem for asymptotic stability are necessary and sufficient. Our results generalize the reduction theorems found in [9, Corollary 19] and [10, Corollary 7.24], which were used in [31] to develop a local hybrid separation principle.

We explore a number of consequences of our reduction theorems. In Proposition 3.1 we present a novel result characterizing the asymptotic stability of compact sets for cascade-connected hybrid systems. In Proposition 3.3 we consider a hybrid system with an output function, and present conditions guaranteeing that boundedness of solutions and convergence of the output to zero imply attractivity of a subset of the zero level set of the output. These conditions give rise to a notion of detectability for hybrid systems that had already been investigated in slightly different form in [25]. Finally, in the spirit of the hierarchical control viewpoint introduced in [7], we present a recursive reduction theorem (Theorem 4.9) in which we consider a chain of closed sets $\Gamma_1 \subset \cdots \subset \Gamma_l \subset \mathbb{R}^n$, with Γ_1 compact, and we deduce the asymptotic stability of Γ_1 from the asymptotic stability of Γ_i relative to Γ_{i+1} for all i. Finally, the theory developed in this paper is applied to the problem of estimating the frequency of oscillation of a sinusoidal signal. Here, the hierarchical viewpoint simplifies an otherwise difficult problem by decomposing it into three separate sub-problems involving a chain of sets $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3$.

Organization of the paper. In Section II we present the class of hybrid systems considered in this paper and various notions of stability of sets. The concepts of this section originate in [7], [9], [10], [28]. In Section III we formulate the reduction problem and its recursive version, and discuss links with the stability of cascade-connected hybrid systems and the output zeroing problem with detectability. In Section IV we present novel reduction theorems for hybrid systems and their proofs. The results of Section IV are employed in Section V to design an estimator for the frequency of oscillation of a sinusoidal signal. Finally, in Section VI we make concluding remarks.

Notation. We denote the set of positive real numbers by $\mathbb{R}_{>0}$, and the set of nonnegative real numbers by $\mathbb{R}_{\geq 0}$. We let $\mathbb{S}^1$ denote the set of real numbers modulo 2π. If $x \in \mathbb{R}^n$, we denote by $|x|$ the Euclidean norm of x, i.e., $|x| = (x^\top x)^{1/2}$. We denote by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^n$, i.e., $\mathbb{B} := \{x \in \mathbb{R}^n : |x| \leq 1\}$. If $\Gamma \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote by $|x|_\Gamma$ the point-to-set distance of x to Γ, i.e., $|x|_\Gamma = \inf_{y \in \Gamma} |x - y|$. If $c > 0$, we let $B_c(\Gamma) := \{x \in \mathbb{R}^n : |x|_\Gamma < c\}$, and $\bar{B}_c(\Gamma) := \{x \in \mathbb{R}^n : |x|_\Gamma \leq c\}$. If U is a subset of $\mathbb{R}^n$, we denote by $\bar{U}$ its closure and by $\operatorname{int} U$ its interior. Given two subsets U and V of $\mathbb{R}^n$, we denote their Minkowski sum by $U + V := \{u + v : u \in U, v \in V\}$. The empty set is denoted by $\emptyset$.

II. Preliminary notions

In this paper we use the notion of hybrid system defined in [9], [10] and some notions of set stability presented in [7]. In this section we review the essential definitions that are required in our development.

Following [9], [10], a hybrid system is a 4-tuple $\mathcal{H} = (C, F, D, G)$ satisfying the

Basic Assumptions ([9], [10])

A1) C and D are closed subsets of $\mathbb{R}^n$.

A2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded on C, and such that $F(x)$ is nonempty and convex for each $x \in C$.

A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded on D, and such that $G(x)$ is nonempty for each $x \in D$.

A *hybrid time domain* is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ which is the union of infinitely many sets $[t_j, t_{j+1}] \times \{j\}$, $j \in \mathbb{N}$, or of finitely many such sets, with the last one of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty) \times \{j\}$. A *hybrid arc* is a function $x : \mathsf{dom}(x) \to \mathbb{R}^n$, where $\mathsf{dom}(x)$ is a hybrid time domain, such that for each j, the function $t \mapsto x(t, j)$ is locally absolutely continuous on the interval $I_j = \{t : (t, j) \in \mathsf{dom}(x)\}$. A *solution* of $\mathcal{H}$ is a hybrid arc $x : \mathsf{dom}(x) \to \mathbb{R}^n$ satisfying the following two conditions.

Flow condition. For each $j \in \mathbb{N}$ such that I_j has nonempty interior,

$$\begin{aligned} &\dot{x}(t,j) \in F(x(t,j)) && \text{for almost all } t \in I_j, \\ &x(t,j) \in C && \text{for all } t \in [\min I_j, \sup I_j). \end{aligned}$$

Jump condition. For each $(t, j) \in \mathsf{dom}(x)$ such that $(t, j+1) \in \mathsf{dom}(x)$,

$$\begin{aligned} &x(t, j+1) \in G(x(t,j)), \\ &x(t,j) \in D. \end{aligned}$$

A solution of $\mathcal{H}$ is *maximal* if it cannot be extended. In this paper we will only consider maximal solutions, and therefore the adjective "maximal" will be omitted in what follows. If $(t_1, j_1), (t_2, j_2) \in \mathsf{dom}(x)$ and $t_1 \leq t_2, j_1 \leq j_2$, then we write $(t_1, j_1) \preceq (t_2, j_2)$. If at least one inequality is strict, then we write $(t_1, j_1) \prec (t_2, j_2)$.

A solution x is *complete* if $\mathsf{dom}(x)$ is unbounded or, equivalently, if there exists a sequence $\{(t_i, j_i)\}_{i \in \mathbb{N}} \subset \mathsf{dom}(x)$ such that $t_i + j_i \to \infty$ as $i \to \infty$.

The set of all maximal solutions of $\mathcal{H}$ originating from $x_0 \in \mathbb{R}^n$ is denoted $\mathcal{S}_{\mathcal{H}}(x_0)$. If $U \subset \mathbb{R}^n$, then

$$\mathcal{S}_{\mathcal{H}}(U) := \bigcup_{x_0 \in U} \mathcal{S}_{\mathcal{H}}(x_0).$$

We let $\mathcal{S}_{\mathcal{H}} := \mathcal{S}_{\mathcal{H}}(\mathbb{R}^n)$. The *range* of a hybrid arc $x : \mathsf{dom}(x) \to \mathbb{R}^n$ is the set

$$\mathsf{rge}(x) := \big\{ y \in \mathbb{R}^n : \big(\exists (t,j) \in \mathsf{dom}(x)\big)\ y = x(t,j) \big\}.$$

If $U \subset \mathbb{R}^n$, we define

$$\mathsf{rge}(\mathcal{S}_{\mathcal{H}}(U)) := \bigcup_{x_0 \in U} \mathsf{rge}\big(\mathcal{S}_{\mathcal{H}}(x_0)\big).$$

Definition 2.1 (Forward invariance): A set $\Gamma \subset \mathbb{R}^n$ is *strongly forward invariant* for $\mathcal{H}$ if

$$\mathsf{rge}(\mathcal{S}_{\mathcal{H}}(\Gamma)) \subset \Gamma.$$

In other words, every solution of $\mathcal{H}$ starting in Γ remains in Γ. The set Γ is *weakly forward invariant* if for every $x_0 \in \Gamma$ there exists a complete $x \in \mathcal{S}_{\mathcal{H}}(x_0)$ such that $x(t,j) \in \Gamma$ for all $(t,j) \in \mathsf{dom}(x)$.

If $\Gamma \subset \mathbb{R}^n$ is closed, then the *restriction* of $\mathcal{H}$ to Γ is the hybrid system $\mathcal{H}|_\Gamma := (C \cap \Gamma, F, D \cap \Gamma, G)$. Whenever Γ is strongly forward invariant, solutions that start in Γ cannot flow out or jump out of Γ. Thus, in this specific case, restricting $\mathcal{H}$ to Γ corresponds to considering only solutions to $\mathcal{H}$ originating in Γ, i.e., $\mathcal{S}_{\mathcal{H}|\Gamma} = \mathcal{S}_{\mathcal{H}}(\Gamma)$.

Definition 2.2 (stability and attractivity): Let $\Gamma \subset \mathbb{R}^n$ be compact.

- Γ is *stable* for $\mathcal{H}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\mathsf{rge}(\mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma))) \subset B_\varepsilon(\Gamma).$$
- The *basin of attraction* of Γ is the largest set of points $p \in \mathbb{R}^n$ such that each $x \in \mathcal{S}_{\mathcal{H}}(p)$ is bounded and, if x is complete, then $|x(t,j)|_\Gamma \to 0$ as $t+j \to \infty$, $(t,j) \in \mathsf{dom}(x)$.
- Γ is *attractive* for $\mathcal{H}$ if the basin of attraction of Γ contains Γ in its interior.
- Γ is *globally attractive* for $\mathcal{H}$ if its basin of attraction is $\mathbb{R}^n$.
- Γ is *asymptotically stable* for $\mathcal{H}$ if it is stable and attractive, and Γ is *globally asymptotically stable* if it is stable and globally attractive.

Let $\Gamma \subset \mathbb{R}^n$ be closed.

- Γ is *stable* for $\mathcal{H}$ if for every $\varepsilon > 0$ there exists an open set U containing Γ such that
$$\mathsf{rge}(\mathcal{S}_{\mathcal{H}}(U)) \subset B_\varepsilon(\Gamma).$$
- The *basin of attraction* of Γ is the largest set of points $p \in \mathbb{R}^n$ such that for each $x \in \mathcal{S}_{\mathcal{H}}(p)$, $|x|_\Gamma$ is bounded and, if x is complete, then $|x(t,j)|_\Gamma \to 0$ as $t+j \to \infty$, $(t,j) \in \mathsf{dom}(x)$.
- Γ is *attractive* if the basin of attraction of Γ contains Γ in its interior.
- Γ is *globally attractive* if its basin of attraction is $\mathbb{R}^n$.
- Γ is *asymptotically stable* if it stable and attractive, and *globally asymptotically stable* if it is stable and globally attractive.

Remark 2.3: When $C \cup D$ is closed, the properties of stability and attractivity hold trivially for compact sets Γ on which there are no solutions. More precisely, if $\Gamma \subset \mathbb{R}^n \setminus (C \cup D)$, then Γ is automatically stable and attractive (and hence asymptotically stable). Moreover, all points outside $C \cup D$ trivially belong to its basin of attraction. △

Remark 2.4: In [10, Definition 7.1], the notions of attractivity and asymptotic stability of compact sets defined above are referred to as local pre-attractivity and local pre-asymptotic stability. The prefix "pre" refers to the fact that the attraction property is only assumed to hold for complete solutions. Recent literature on hybrid systems has dropped this prefix, and in this paper we follow the same convention. △

Remark 2.5: For the case of closed, non-compact sets, [10] adopts notions of uniform global stability, uniform global pre-attractivity, and uniform global pre-asymptotic stability (see [10, Definition 3.6]) that are stronger than the notions presented in Definition 2.2, but they allow the authors of [10] to give Lyapunov characterizations of asymptotic stability. In this paper we use weaker definitions to obtain more general results. Specifically, the results of this paper whose assumptions concern asymptotic stability of closed sets (assumptions (ii) and (ii') in Corollary 4.8, assumptions (i) and (i') in Theorem 4.9) continue to hold when the stronger stability properties of [10] are satisfied.

To illustrate the differences between the above mentioned stability and attractivity notions for closed sets, in [10, Definition 3.6] the uniform global stability property requires that for every $\varepsilon > 0$, the open set U of Definition 2.2 be of the form $B_\delta(\Gamma)$, i.e., a neighborhood of Γ of constant diameter, hence the adjective "uniform." Moreover, [10, Definition 3.6] requires that $\delta \to \infty$ as $\varepsilon \to \infty$, hence the adjective "global." On the other hand, Definition 2.2 only requires the existence of a neighborhood U of Γ, not necessarily of constant diameter, and without the "global" requirement. In particular, the diameter of U may shrink to zero near points of Γ that are infinitely far from the origin, even as $\varepsilon \to \infty$. Similarly, the notion of uniform global pre-attractivity in [10, Definition 3.6] is much stronger than that of global attractivity in Definition 2.2, for it requires solutions not only to converge to Γ, but to do so with a rate of convergence which is uniform over sets of initial conditions of the form $B_r(\Gamma)$. △

Definition 2.6 (local stability and attractivity near a set): Consider two sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$, and assume that Γ_1 is compact. The set Γ_2 is *locally stable near* Γ_1 for $\mathcal{H}$ if there exists $r > 0$ such that the following property holds. For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $x \in \mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma_1))$ and for each $(t,j) \in \mathsf{dom}(x)$, it holds that if $x(s,k) \in B_r(\Gamma_1)$ for all $(s,k) \in \mathsf{dom}(x)$ with $(s,k) \preceq (t,j)$, then $x(t,j) \in B_\varepsilon(\Gamma_2)$. The set Γ_2 is *locally attractive near* Γ_1 for $\mathcal{H}$ if there exists $r > 0$ such that $B_r(\Gamma_1)$ is contained in the basin of attraction of Γ_2.

Remark 2.7: The notions in Definition 2.6 originate in [28]. It is an easy consequence of the definition, and it is shown rigorously in the proof of Theorem 4.7, that local stability of

Γ_2 near Γ_1 is a necessary condition for Γ_1 to be stable. In particular, if Γ_1 is stable, then Γ_2 is locally stable near Γ_1 for arbitrary values[1] of $r > 0$. Moreover, local attractivity of Γ_2 near Γ_1 is a necessary condition for Γ_1 to be attractive. Finally, it is easily seen that if Γ_2 is stable, then Γ_2 is locally stable near Γ_1, thus local stability of Γ_2 near Γ_1 is a necessary condition for both the stability of Γ_1 and the stability of Γ_2. $\triangle$

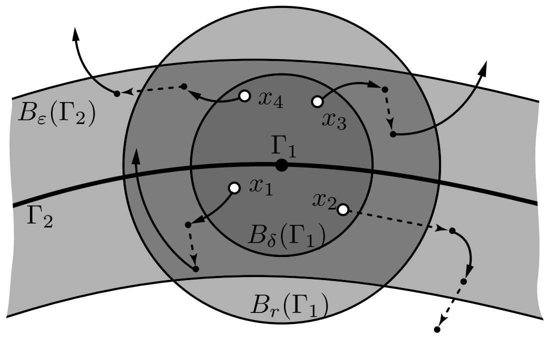

Fig. 1. An illustration of local stability of $\mathbf{\Gamma_2}$ near $\mathbf{\Gamma_1}$. Continuous lines denote flow, while dashed lines denote jumps. All solutions starting sufficiently close to $\mathbf{\Gamma_1}$ remain close to $\mathbf{\Gamma_2}$ so long as they remain in $\boldsymbol{B_r(\Gamma_1)}$. In the figure, the solution from $\boldsymbol{x_1}$ remains in $\boldsymbol{B_r(\Gamma_1)}$ and therefore also in $\boldsymbol{B_\varepsilon(\Gamma_2)}$. The solution from $\boldsymbol{x_2}$ jumps out of $\boldsymbol{B_r(\Gamma_1)}$, then jumps out of $\boldsymbol{B_\varepsilon(\Gamma_2)}$. The solution from $\boldsymbol{x_3}$ flows out of $\boldsymbol{B_r(\Gamma_1)}$, then flows out of $\boldsymbol{B_\varepsilon(\Gamma_2)}$. Finally, the solution from $\boldsymbol{x_4}$ jumps out of $\boldsymbol{B_r(\Gamma_1)}$, then flows out of $\boldsymbol{B_\varepsilon(\Gamma_2)}$.

According to Definition 2.6, the set Γ_2 is locally attractive near Γ_1 if all solutions starting near Γ_1 converge to Γ_2. Thus Γ_2 might be locally attractive near Γ_1 even when it is not attractive in the sense of Definition 2.2. On the other hand, the set Γ_2 is locally stable near Γ_1 if solutions starting close to Γ_1 remain close to Γ_2 so long as they are not too far from Γ_1. This notion is illustrated in Figure 1.

Definition 2.8 (relative properties): Consider two closed sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$. We say that Γ_1 is, respectively, stable, (globally) attractive, or (globally) asymptotically stable *relative to* Γ_2 if Γ_1 is stable, (globally) attractive, or (globally) asymptotically stable for $\mathcal{H}|_{\Gamma_2}$.

Example 2.9: To illustrate the definition, consider the linear time-invariant system

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = x_2,$$

and the sets $\Gamma_1 = \{(0,0)\}$, $\Gamma_2 = \{(x_1, x_2) : x_2 = 0\}$. Even though Γ_1 is an unstable equilibrium, Γ_1 is globally asymptotically stable relative to Γ_2. Now consider the planar system expressed in polar coordinates $(\rho, \theta) \in \mathbb{R}_{>0} \times \mathbb{S}^1$ as

$$\dot{\theta} = \sin^2(\theta/2) + (1-\rho)^2$$
$$\dot{\rho} = 0.$$

Let Γ_1 be the point on the unit circle $\Gamma_1 = \{(\theta, \rho) : \theta = 0, \rho = 1\}$, and Γ_2 be the unit circle, $\Gamma_2 = \{(\theta, \rho) : \rho = 1\}$. On Γ_2, the motion is described by $\dot{\theta} = \sin^2(\theta/2)$. We see that $\dot{\theta} \geq 0$, and $\dot{\theta} = 0$ if and only if $\theta = 0$ modulo 2π. Thus Γ_1 is globally attractive relative to Γ_2, even though it is not an attractive equilibrium. $\triangle$

[1]For this reason, in [7], local stability of Γ_2 near Γ_1 is defined by requiring that the property holds for any $r > 0$.

The next two results will be useful in the sequel (see also [10, Proposition 3.32]).

Lemma 2.10: For a hybrid system $\mathcal{H} := (C, F, D, G)$, if $\Gamma_1 \subset \mathbb{R}^n$ is a closed set which is, respectively, stable, attractive, or globally attractive for $\mathcal{H}$, then for any closed set $\Gamma_2 \subset \mathbb{R}^n$, Γ_1 is, respectively, stable, attractive, or globally attractive for $\mathcal{H}|_{\Gamma_2}$.

Proof: The result is a consequence of the fact that each solution of $\mathcal{H}|_{\Gamma_2}$ is also a solution of $\mathcal{H}$. ■

The next result is a partial converse to Lemma 2.10.

Lemma 2.11: For a hybrid system $\mathcal{H} := (C, F, D, G)$, if $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$ are two closed sets such that Γ_1 is compact and $\Gamma_1 \subset \operatorname{int}\Gamma_2$, then:

(a) Γ_1 is stable for $\mathcal{H}$ if and only if it is stable for $\mathcal{H}|_{\Gamma_2}$.

(b) If Γ_1 is stable for $\mathcal{H}$, then Γ_1 is attractive for $\mathcal{H}$ if and only if Γ_1 is attractive for $\mathcal{H}|_{\Gamma_2}$.

Proof: Part (a). By Lemma 2.10, if Γ_1 is stable for $\mathcal{H}$, then it is also stable for $\mathcal{H}|_{\Gamma_2}$. Next assume that Γ_1 is stable for $\mathcal{H}|_{\Gamma_2}$. Since Γ_1 is compact and contained in the interior of Γ_2, there exists $r > 0$ such that $B_r(\Gamma_1) \subset \Gamma_2$. For any $\varepsilon > 0$, let $\varepsilon' := \min(\varepsilon, r)$. By the definition of stability of Γ_1, there exists $\delta > 0$ such that

$$\mathsf{rge}(\mathcal{S}_{\mathcal{H}|_{\Gamma_2}}(B_\delta(\Gamma_1))) \subset B_{\varepsilon'}(\Gamma_1). \tag{2}$$

Since $B_{\varepsilon'}(\Gamma_1) \subset B_r(\Gamma_1) \subset \Gamma_2$, we have that solutions of $\mathcal{H}$ and $\mathcal{H}|_{\Gamma_2}$ originating in $B_\delta(\Gamma_1)$ coincide, i.e.,

$$\mathcal{S}_{\mathcal{H}|_{\Gamma_2}}(B_\delta(\Gamma_1)) = \mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma_1)). \tag{3}$$

Substituting (3) into (2) and using the fact that $\varepsilon' \leq \varepsilon$ we get

$$\mathsf{rge}(\mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma_1))) \subset B_{\varepsilon'}(\Gamma_1) \subset B_\varepsilon(\Gamma_1),$$

which proves that Γ_1 is stable for $\mathcal{H}$.

Part (b). By Lemma 2.10, if Γ_1 is attractive for $\mathcal{H}$ then it is also attractive for $\mathcal{H}|_{\Gamma_2}$. For the converse, assume that Γ_1 is attractive for $\mathcal{H}|_{\Gamma_2}$. Since Γ_1 is compact and contained in the interior of Γ_2, there exists $\varepsilon > 0$ such that $B_\varepsilon(\Gamma_1) \subset \Gamma_2$. Since Γ_1 is stable for $\mathcal{H}$, there exists $\delta > 0$ such that

$$\mathsf{rge}(\mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma_1))) \subset B_\varepsilon(\Gamma_1) \subset \Gamma_2.$$

The above implies that solutions of $\mathcal{H}$ and $\mathcal{H}|_{\Gamma_2}$ originating in $B_\delta(\Gamma_1)$ coincide, i.e.,

$$\mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma_1)) = \mathcal{S}_{\mathcal{H}|_{\Gamma_2}}(B_\delta(\Gamma_1)). \tag{4}$$

Since Γ_1 is attractive for $\mathcal{H}|_{\Gamma_2}$, the basin of attraction of Γ_1 is a neighborhood of Γ_1, and therefore there exists $\delta > 0$ small enough to ensure (4) and to ensure that $B_\delta(\Gamma_1)$ is contained in the basin of attraction. By (4), $B_\delta(\Gamma_1)$ is also contained in the basin of attraction of Γ_1 for system $\mathcal{H}$, from which it follows that Γ_1 is attractive for $\mathcal{H}$. ■

III. The reduction problem

In this section we formulate the reduction problem, discuss its relevance, and present two theoretical applications: the stability of compact sets for cascade-connected hybrid systems, and a result concerning global attractivity of compact sets for hybrid systems with outputs that converge to zero.

Reduction Problem. Consider a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, and two sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$, with Γ_1 compact and Γ_2 closed. Suppose that Γ_1 enjoys property P relative to Γ_2, where $P \in \{$stability, attractivity, global attractivity, asymptotic stability, global asymptotic stability$\}$. We seek conditions under which property P holds relative to $\mathbb{R}^n$. △

As mentioned in the introduction, this problem was first formulated by Paul Seibert in 1969-1970 [26], [27]. The solution in the context of hybrid systems is presented in Theorems 4.1, 4.4, 4.7 in the next section.

To illustrate the reduction problem, suppose we wish to determine whether a compact set Γ_1 is asymptotically stable, and suppose that Γ_1 is contained in a closed set Γ_2, as illustrated in Figure 2. In the reduction framework, the stability question is decomposed into two parts: (1) Determine whether Γ_1 is asymptotically stable relative to Γ_2; (2) determine whether Γ_2 satisfies additional suitable properties (Theorem 4.7 in Section IV states precisely the required properties). In some cases, these two questions might be easier to answer than the original one, particularly when Γ_2 is strongly forward invariant, since in this case question (1) would typically involve a hybrid system on a state space of lower dimension. This sort of decomposition occurs frequently in control theory, either for convenience or for structural necessity, as we now illustrate.

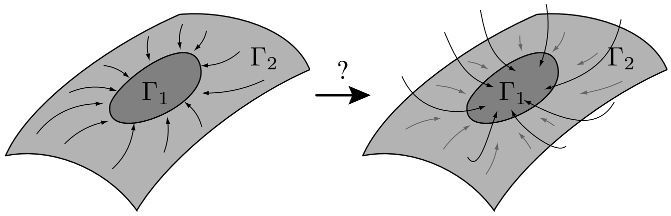

Fig. 2. Illustration of the reduction problem when $\mathbf{\Gamma_2}$ is strongly forward invariant.

In the context of control systems, the sets $\Gamma_1 \subset \Gamma_2$ might represent two control specifications organized hierarchically: the specification associated with set Γ_2 has higher priority than that associated with set Γ_1. Here, the reduction problem stems from the decomposition of the control design into two steps: meeting the high-priority specification first, i.e., stabilize Γ_2; then, assuming that the high-priority specification has been achieved, meet the low-priority specification, i.e., stabilize Γ_1 relative to Γ_2. This point of view is developed in [7], and has been applied to the almost-global stabilization of VTOL vehicles [24], distributed control [6], [32], virtual holonomic constraints [17], robotics [20], [21], and static or dynamic allocation of nonlinear redundant actuators [22]. Similar ideas have also been adopted in [19], where the concept of local stability near a set, introduced in Definition 2.6, is key to ruling out situations where the feedback stabilizer may generate solutions that blow up to infinity. In the hybrid context, the hierarchical viewpoint described above has been adopted in [2] to deal with unknown jump times in hybrid observation of periodic hybrid exosystems, while discrete-time results are used in the proof of GAS reported in [1] for so-called stubborn observers in discrete time. In the case of more than two control specifications, one has the following.

Recursive Reduction Problem. Consider a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, and l closed sets $\Gamma_1 \subset \cdots \subset \Gamma_l \subset \mathbb{R}^n$, with Γ_1 compact. Suppose that Γ_i enjoys property P relative to Γ_{i+1} for all $i \in \{1, \dots, l\}$, where $P \in \{$stability, attractivity, global attractivity, asymptotic stability, global asymptotic stability$\}$. We seek conditions under which the set Γ_1 enjoys property P relative to $\mathbb{R}^n$.

The solution of this problem is found in Theorem 4.9 in the next section. It is shown in [7] that the backstepping stabilization technique can be recast as a recursive reduction problem. △

As mentioned earlier, the reduction problem may emerge from structural considerations, such as when the hybrid system is the cascade interconnection of two subsystems.

Cascade-connected hybrid systems. Consider a hybrid system $\mathcal{H} = (C, F, D, G)$, where $C = C_1 \times C_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $D = D_1 \times D_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ are closed sets, and $F : \mathbb{R}^{n_1+n_2} \rightrightarrows \mathbb{R}^{n_1+n_2}$, $G : \mathbb{R}^{n_1+n_2} \rightrightarrows \mathbb{R}^{n_1+n_2}$ are maps satisfying the Basic Assumptions. Suppose that F and G have the upper triangular structure

$$F(x^1, x^2) = \begin{bmatrix} F_1(x^1, x^2) \\ F_2(x^2) \end{bmatrix}, \; G(x^1, x^2) = \begin{bmatrix} G_1(x^1, x^2) \\ G_2(x^2) \end{bmatrix}, \quad (5)$$

where $(x^1, x^2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Define $\hat{F}_1 : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_1}$ and $\hat{G}_1 : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_1}$ as

$$\hat{F}_1(x^1) := F_1(x^1, 0), \; \hat{G}_1(x^1) := G_1(x^1, 0). \quad (6)$$

With these definitions, we can view $\mathcal{H}$ as the cascade connection of the hybrid systems

$$\mathcal{H}_1 = (C_1, \hat{F}_1, D_1, \hat{G}_1), \; \mathcal{H}_2 = (C_2, F_2, D_2, G_2),$$

with $\mathcal{H}_2$ driving $\mathcal{H}_1$. The following result is a corollary of Theorem 4.7 in Section IV. It generalizes to the hybrid setting classical results for continuous time-invariant dynamical systems in, e.g., [29], [33]. Using Theorems 4.1 and 4.4, one may formulate analogous results for the properties of attractivity and stability.

Proposition 3.1: Consider the hybrid system $\mathcal{H} := (C_1 \times C_2, F, D_1 \times D_2, G)$, with maps F, G given in (5), and the two hybrid subsystems $\mathcal{H}_1 := (C_1, \hat{F}_1, D_1, \hat{G}_1)$ and $\mathcal{H}_2 := (C_2, F_2, D_2, G_2)$ satisfying the Basic Assumptions, with maps $\hat{F}_1, \hat{G}_1$ given in (6). Let $\hat{\Gamma}_1 \subset \mathbb{R}^{n_1}$ be a compact set, and denote

$$\Gamma_1 = \{(x^1, x^2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : x^1 \in \hat{\Gamma}_1, \, x^2 = 0\}. \quad (7)$$

Suppose that $0 \in C_2 \cup D_2$. Then the following holds:

(i) Γ_1 is asymptotically stable for $\mathcal{H}$ if $\hat{\Gamma}_1$ is asymptotically stable for $\mathcal{H}_1$ and $0 \in \mathbb{R}^{n_2}$ is asymptotically stable for $\mathcal{H}_2$.

(ii) Γ_1 is globally asymptotically stable for $\mathcal{H}$ if $\hat{\Gamma}_1$ is globally asymptotically stable for $\mathcal{H}_1$, $0 \in \mathbb{R}^{n_2}$ is globally asymptotically stable for $\mathcal{H}_2$, and all solutions of $\mathcal{H}$ are bounded.

The result above is obtained from Theorem 4.7 in Section IV setting Γ_1 as in (7), and $\Gamma_2 = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : x_2 = 0\}$. The restriction $\mathcal{H}|_{\Gamma_2}$ is given by

$$\mathcal{H}|_{\Gamma_2} = \left(C_1 \times \{0\}, \begin{bmatrix} F_1(x_1, 0) \\ F_2(0) \end{bmatrix}, D_1 \times \{0\}, \begin{bmatrix} G_1(x_1, 0) \\ G_2(0) \end{bmatrix} \right),$$

from which it is straightforward to see that Γ_1 is (globally) asymptotically stable relative to Γ_2 if and only if $\hat{\Gamma}_1$ is (globally) asymptotically stable for $\mathcal{H}_1$. It is also clear that if $0 \in \mathbb{R}^{n_2}$ is (globally) asymptotically stable for $\mathcal{H}_2$, then Γ_2 is (globally) asymptotically stable for $\mathcal{H}$. The converse, however, is not true. Namely, the (global) asymptotic stability of Γ_2 for $\mathcal{H}$ does not imply that $0 \in \mathbb{R}^{n_2}$ is (globally) asymptotically stable for $\mathcal{H}_2$, which is why Proposition 3.1 states only sufficient conditions. The reason is that the set of hybrid arcs $x_2(t, j)$ generated by solutions of $\mathcal{H}$ is generally smaller than the set of solutions of $\mathcal{H}_2$. This phenomenon is illustrated in the next example.

Example 3.2: Consider the cascade connected system $\mathcal{H} = (C_1 \times C_2, F, D_1 \times D_2, G)$, with $C_1 = \{1\}$, $C_2 = \mathbb{R}$, $D_1 = D_2 = \emptyset$, and

$$F(x_1, x_2) = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}.$$

All solutions of $\mathcal{H}$ have the form $(1, x_2(0, 0))$, and are defined only at $(t, j) = (0, 0)$. Since the origin $(x_1, x_2) = (0, 0)$ is not contained in $C \cup D$, it is trivially asymptotically stable for $\mathcal{H}$ (see Remark 2.3). Moreover, there are no complete solutions, and all solutions are constant, hence bounded, which implies that the basin of attraction of the origin is the entire $\mathbb{R}^2$. Hence the origin is globally asymptotically stable for $\mathcal{H}$. On the other hand, $\mathcal{H}_2$ is the linear time-invariant continuous-time system on $\mathbb{R}$ with dynamics $\dot{x}_2 = x_2$, clearly unstable. This example shows that the condition, in Proposition 3.1, that 0 be (globally) asymptotically stable for $\mathcal{H}_2$ is not necessary. △

Proposition 3.1 is to be compared to [31, Theorem 1], where the author presents an analogous result for a different kind of cascaded hybrid system. The notion of cascaded hybrid system used in Proposition 3.1 is one in which a jump is possible only if the states x^1 and x^2 are simultaneously in their respective jump sets, D_1 and D_2, and a jump event involves both states, simultaneously. On the other hand, the notion of cascaded hybrid system proposed in [31] is one in which jumps of x^1 and x^2 occur independently of one another, so that when x^1 jumps nontrivially, x^2 remains constant, and vice versa. Moreover, in [31] the jump and flow sets are not expressed as Cartesian products of sets in the state spaces of the two subsystems.

Another circumstance in which the reduction problem plays a prominent role is the notion of detectability for systems with outputs.

Output zeroing with detectability. Consider a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, with a continuous output function $h : \mathbb{R}^n \to \mathbb{R}^k$, and let Γ_1 be a compact, strongly forward invariant subset of $h^{-1}(0)$. Assume that all solutions on Γ_1 are complete. Suppose that all $x \in \mathcal{S}_{\mathcal{H}}$ are bounded. Under what circumstances does the property $h(x(t, j)) \to 0$ for all complete $x \in \mathcal{S}_{\mathcal{H}}$ imply that Γ_1 is globally attractive? This question arises in the context of passivity-based stabilization of equilibria [3] and closed sets [5] for continuous control systems. In the hybrid systems setting, a similar question arises when using virtual constraints to stabilize hybrid limit cycles for biped robots (e.g., [23], [34], [35]). In this case the zero level set of the output function is the virtual constraint.

Let Γ_2 denote the maximal weakly forward invariant subset contained in $h^{-1}(0)$. Using the sequential compactness of the space of solutions of $\mathcal{H}$ [11, Theorem 4.4], one can show that the closure of a weakly forward invariant set is weakly forward invariant. This fact and the maximality of Γ_2 imply that Γ_2 is closed. Furthermore, since Γ_1 is strongly forward invariant, contained in $h^{-1}(0)$, and all solutions on it are complete, necessarily $\Gamma_1 \subset \Gamma_2$. It turns out (see the proof of Proposition 3.3 below) that any bounded complete solution x such that $h(x(t, j)) \to 0$ converges to Γ_2.

In light of the discussion above, the question we asked earlier can be recast as a reduction problem: Suppose that Γ_2 is globally attractive. What stability properties should Γ_1 satisfy relative to Γ_2 in order to ensure that Γ_1 is globally attractive for $\mathcal{H}$? The answer, provided by Theorem 4.4 in Section IV, is that Γ_1 should be globally asymptotically stable relative to Γ_2 (attractivity is not enough, as shown in Example 4.6 below).

Following[2] [5], the hybrid system $\mathcal{H}$ is said to be Γ_1-*detectable from* h if Γ_1 is globally asymptotically stable relative to Γ_2, where Γ_2 is the maximal weakly forward invariant subset contained in $h^{-1}(0)$.

Using the reduction theorem for attractivity in Section IV (Theorem 4.4), we get the answer to the foregoing output zeroing question.

Proposition 3.3: Let $\mathcal{H}$ be a hybrid system satisfying the Basic Assumptions, $h : \mathbb{R}^n \to \mathbb{R}^k$ a continuous function, and $\Gamma_1 \subset h^{-1}(0)$ be a compact set which is strongly forward invariant for $\mathcal{H}$, such that all solutions from Γ_1 are complete. If 1) $\mathcal{H}$ is Γ_1-detectable from h, 2) each $x \in \mathcal{S}_{\mathcal{H}}$ is bounded, and 3) all complete $x \in \mathcal{S}_{\mathcal{H}}$ are such that $h(x(t, j)) \to 0$, then Γ_1 is globally attractive.

Proof: Let Γ_2 be the maximal weakly forward invariant subset of $h^{-1}(0)$. This set is closed by sequential compactness of the space of solutions of $\mathcal{H}$ [11, Theorem 4.4]. By assumption, any $x \in \mathcal{S}_{\mathcal{H}}$ is bounded. If $x \in \mathcal{S}_{\mathcal{H}}$ is complete, by [25, Lemma 3.3], the positive limit set $\Omega(x)$ is nonempty, compact, and weakly invariant. Moreover, $\Omega(x)$ is the smallest closed set approached by x. Since $h(x(t, j)) \to 0$ and h is continuous, $\Omega(x) \subset h^{-1}(0)$. Since $\Omega(x)$ is weakly forward invariant and contained in $h^{-1}(0)$, necessarily $\Omega(x) \subset \Gamma_2$. Thus Γ_2 is globally attractive for $\mathcal{H}$. Since Γ_1 is strongly forward invariant, contained in $h^{-1}(0)$, and on it all solutions are complete, Γ_1 is contained in Γ_2, the maximal set with these properties. By the Γ_1-detectability assumption, Γ_1 is globally asymptotically stable relative to Γ_2. By Theorem 4.4, we conclude that Γ_1 is globally attractive. ■

[2] In [25], the authors adopt a different definition of detectability, one that requires Γ_1 to be globally attractive, instead of globally asymptotically stable, relative to Γ_2. When they employ this property, however, they make the extra assumption that Γ_1 be stable relative to Γ_2.

IV. Main results

In this section we solve the reduction problem, presenting reduction theorems for stability, (global) attractivity, and (global) asymptotic stability. We also present the solution of the recursive reduction problem for the property of asymptotic stability.

Theorem 4.1 (Reduction theorem for stability): For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, consider two sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$, with Γ_1 compact and Γ_2 closed. If

(i) Γ_1 is asymptotically stable relative to Γ_2,
(ii) Γ_2 is locally stable near Γ_1,

then Γ_1 is stable for $\mathcal{H}$.

Remark 4.2: As argued in Remark 2.7, local stability of Γ_2 near Γ_1 (assumption (ii)) is a necessary condition in Theorem 4.1. In place of this assumption, one may use the stronger assumption that Γ_2 be stable, which might be easier to check in practice but is not a necessary condition (see for example system (12) in Example 4.3). There are situations, however, when the local stability property is essential and emerges quite naturally from the context of the problem. This occurs, for instance, when solutions far from Γ_1 but near Γ_2 have finite escape times. For examples of such situations, refer to [12] and [19]. △

Proof: Hypotheses (i) and (ii) imply that there exists a scalar $r > 0$ such that:

(a) Set Γ_1 is globally asymptotically stable for system $\mathcal{H}_{r,0} := (C \cap \Gamma_2 \cap \bar{B}_r(\Gamma_1), F, D \cap \Gamma_2 \cap \bar{B}_r(\Gamma_1), G)$,
(b) Given system $\mathcal{H}_r := \mathcal{H}|_{\bar{B}_r(\Gamma_1)}$ for each $\varepsilon > 0$, $\exists \delta > 0$ such that all solutions to $\mathcal{H}_r$ satisfy:

$$|x(0,0)|_{\Gamma_1} \leq \delta \;\Rightarrow\; |x(t,j)|_{\Gamma_2} \leq \varepsilon,\; \forall (t,j) \in \mathsf{dom}(x).$$

Since Γ_1 is contained in the interior of $\bar{B}_r(\Gamma_1)$, by Lemma 2.11 to prove stability of Γ_1 for $\mathcal{H}$ it suffices to prove stability of Γ_1 for system $\mathcal{H}_r$ introduced in (b). The rest of the proof follows similar steps to the proof of stability reported in [10, Corollary 7.24].

From item (a) and due to [10, Theorem 7.12], there exists a class $\mathcal{KL}$ bound $\beta \in \mathcal{KL}$ and, due to [10, Lemma 7.20] applied with a constant perturbation function $x \mapsto \rho(x) = \bar{\rho}$ and with $\mathcal{U} = \mathbb{R}^n$, for each $\varepsilon > 0$ there exists $\bar{\rho} > 0$ such that defining

$$\begin{aligned} C_{\bar{\rho},r} &:= C \cap \bar{B}_{\bar{\rho}}(\Gamma_2) \cap \bar{B}_r(\Gamma_1) \\ &\subset \{x \in \mathbb{R}^n :\; (x + \bar{\rho}\mathbb{B}) \cap (C \cap \Gamma_2 \cap \bar{B}_r(\Gamma_1)) \neq \emptyset\} \\ D_{\bar{\rho},r} &:= D \cap \bar{B}_{\bar{\rho}}(\Gamma_2) \cap \bar{B}_r(\Gamma_1) \\ &\subset \{x \in \mathbb{R}^n :\; (x + \bar{\rho}\mathbb{B}) \cap (D \cap \Gamma_2 \cap \bar{B}_r(\Gamma_1)) \neq \emptyset\} \end{aligned} \tag{8}$$

and introducing system $\mathcal{H}_{\bar{\rho},r} := (C_{\bar{\rho},r}, F, D_{\bar{\rho},r}, G)$, we have[3]

$$|x(t,j)|_{\Gamma_1} \leq \beta(|x(0,0)|_{\Gamma_1}, t+j) + \tfrac{\varepsilon}{2}, \quad \forall (t,j) \in \mathsf{dom}(x), \forall x \in \mathcal{S}_{\mathcal{H}_{\bar{\rho},r}} \tag{9}$$

Let $\varepsilon > 0$ be given. Let $\bar{\rho} > 0$ be such that (9) holds. Due to item (b) above, there exists a small enough $\delta > 0$ such that $\beta(\delta, 0) \leq \frac{\varepsilon}{2}$ and

$$(x \in \mathcal{S}_{\mathcal{H}_r}, |x(0,0)|_{\Gamma_1} \leq \delta) \Rightarrow |x(t,j)|_{\Gamma_2} \leq \bar{\rho}, \forall (t,j) \in \mathsf{dom}(x). \tag{10}$$

Then the solutions considered in (10) are also solutions of $\mathcal{H}_{\bar{\rho},r}$ because they remain in $B_{\bar{\rho}}(\Gamma_2)$. Since these are solutions of $\mathcal{H}_{\bar{\rho},r}$, we may apply (9) to get

$$|x(t,j)|_{\Gamma_1} \leq \beta(\delta, 0) + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\; \forall (t,j) \in \mathsf{dom}(x), \tag{11}$$

which completes the proof. ■

Example 4.3: Assumption (i) in the above theorem cannot be replaced by the weaker requirement that Γ_1 be stable relative to Γ_2. To illustrate this fact, consider the linear time-invariant system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0, \end{aligned}$$

with $\Gamma_1 = \{(0,0)\}$ and $\Gamma_2 = \{(x_1, x_2) : x_2 = 0\}$. Although Γ_1 is stable relative to Γ_2 and Γ_2 is stable, Γ_1 is an unstable equilibrium. On the other hand, consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= 0, \end{aligned}$$

with the same definitions of Γ_1 and Γ_2. Now Γ_1 is asymptotically stable relative to Γ_2, and Γ_2 is stable. As predicted by Theorem 4.1, Γ_1 is a stable equilibrium. Finally, let $\sigma : \mathbb{R} \to [0,1]$ be a C^1 function such that $\sigma(s) = 0$ for $|s| \leq 1$ and $\sigma(s) = 1$ for $|s| \geq 2$, and consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1(1 - \sigma(x_1)) + x_2^2 \\ \dot{x}_2 &= \sigma(x_1) x_2, \end{aligned} \tag{12}$$

with the earlier definitions of Γ_1 and Γ_2. One can see that Γ_1 is asymptotically stable relative to Γ_2, and Γ_2 is unstable. For the former property, note that the motion on Γ_2 is described by $\dot{x}_1 = -x_1(1 - \sigma(x_1))$, a C^1 differential equation which near $\{x_1 = 0\}$ reduces to $\dot{x}_1 = -x_1$. To see that Γ_2 is an unstable set, note that if $x_1(0) \geq 2$, then $x_1(t) \geq x_1(0)$ and $\dot{x}_2 = x_2$. Namely, solutions move away from Γ_2. On the other hand, Γ_2 is locally stable near Γ_1, because as long as $|x_1| \leq 1$, $\dot{x}_2 = 0$. By Theorem 4.1, Γ_1 is a stable equilibrium. △

Theorem 4.4 (Reduction theorem for attractivity): For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, consider two sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$, with Γ_1 compact and Γ_2 closed. Assume that

(i) Γ_1 is globally asymptotically stable relative to Γ_2,
(ii) Γ_2 is globally attractive,

then the basin of attraction of Γ_1 is the set

$$\mathcal{B} := \{x_0 \in \mathbb{R}^n : \text{all } x \in \mathcal{S}_{\mathcal{H}}(x_0) \text{ are bounded}\}. \tag{13}$$

In particular, if $\mathcal{B}$ contains Γ_1 in its interior, then Γ_1 is attractive. If all solutions of $\mathcal{H}$ are bounded, then Γ_1 is globally attractive.

Proof: By definition, any bounded non complete solution belongs to the basin of attraction of Γ_1. The proof amounts then to showing that any bounded and complete solution $x \in \mathcal{S}_{\mathcal{H}}$ converges to Γ_1, so that all points in $\mathcal{B}$ defined in (13) are contained in its basin of attraction. Conversely, any solution

[3]Note that for a constant perturbation $\rho(x) = \bar{\rho}$ the inflated flow and jump sets in [10, Definition 6.27] are exactly $\bar{\rho}$ inflations of the original ones.

in the basin of attraction of Γ_1 is bounded by definition, so it belongs to $\mathcal{B}$. Hypothesis (i) corresponds to the following fact:

(a) Set Γ_1 is globally asymptotically stable for system $\mathcal{H}|_{\Gamma_2} := (C \cap \Gamma_2, F, D \cap \Gamma_2, G)$.

The rest of the proof follows similar steps to the proof of attractivity reported in [10, Corollary 7.24]. Given any bounded and complete solution $x \in \mathcal{S}_{\mathcal{H}}$, define $M := \max_{(t,j)\in \mathsf{dom}(x)} |x(t,j)|_{\Gamma_1}$. Convergence of x to Γ_1 is established by showing that for each ε, there exists $T \geq 0$ such that

$$|x(t,j)|_{\Gamma_1} \leq \varepsilon, \quad \forall (t,j) \in \mathsf{dom}(x) : t+j \geq T. \tag{14}$$

From item (a) above, and applying [10, Theorem 7.12], there exists a uniform class $\mathcal{KL}$ bound $\beta \in \mathcal{KL}$ on the solutions to system $\mathcal{H}|_{\Gamma_2}$. Fix an arbitrary $\varepsilon > 0$. To establish (14), due to [10, Lemma 7.20] applied to $\mathcal{H}|_{\Gamma_2}$ with $\mathcal{U} = \mathbb{R}^n$, with a constant perturbation function $x \mapsto \rho(x) = \bar{\rho}$ and with the compact set $K = \bar{B}_M(\Gamma_1)$ (to be used in the definition of semiglobal practical $\mathcal{KL}$ asymptotic stability of [10, Definition 7.18]), there exists a small enough $\bar{\rho} > 0$ such that defining [4]

$$\begin{array}{rcl} C_{\bar{\rho}} & := & \bar{B}_M(\Gamma_1) \cap C \cap \bar{B}_{\bar{\rho}}(\Gamma_2) \\ & \subset & \bar{B}_M(\Gamma_1) \cap \{x \in \mathbb{R}^n : (x+\bar{\rho}\mathbb{B}) \cap ((C \cap \Gamma_2) \neq \emptyset\} \\ D_{\bar{\rho}} & := & \bar{B}_M(\Gamma_1) \cap D \cap \bar{B}_{\bar{\rho}}(\Gamma_2) \\ & \subset & \bar{B}_M(\Gamma_1) \cap \{x \in \mathbb{R}^n : (x+\bar{\rho}\mathbb{B}) \cap ((D \cap \Gamma_2) \neq \emptyset\} \end{array} \tag{15}$$

and introducing system $\mathcal{H}_{\bar{\rho}} := (C_{\bar{\rho}}, F, D_{\bar{\rho}}, G)$, we have

$$\begin{aligned} |\bar{x}(t,j)|_{\Gamma_1} &\leq \beta(|\bar{x}(0,0)|_{\Gamma_1}, t+j) + \frac{\varepsilon}{2}, \qquad (16) \\ &\leq \beta(M, t+j) + \frac{\varepsilon}{2}, \;\; \forall (t,j) \in \mathsf{dom}(\bar{x}), \forall \bar{x} \in \mathcal{S}_{\mathcal{H}_{\bar{\rho}}}. \end{aligned}$$

Define now $T_2 > 0$ satisfying $\beta(M, T_2) \leq \frac{\varepsilon}{2}$, and obtain:

$$\bar{x} \in \mathcal{S}_{\mathcal{H}_{\bar{\rho}}} \quad \Rightarrow \quad |\bar{x}(t,j)|_{\Gamma_1} \leq \varepsilon, \forall (t,j) \in \mathsf{dom}(\bar{x}) : t+j \geq T_2. \tag{17}$$

Moreover, from hypothesis (ii), there exists $T_1 > 0$ such that $|x(t,j)|_{\Gamma_2} \leq \bar{\rho}$ for all $(t,j) \in \mathsf{dom}(x)$ satisfying $t+j \geq T_1$. As a consequence, the tail of solution x (after $t+j \geq T_1$) is a solution to $\mathcal{H}_{\bar{\rho}}$. By virtue of (17), equation (14) is established with $T = T_1 + T_2$ and the proof is completed. ■

Example 4.5: Consider a hybrid system with continuous states $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and a discrete state $q \in \{1, -1\}$. The dynamics are defined as

$$\begin{array}{ll} \dot{x}_1 = q x_2 & x_1^+ = x_1 \\ \dot{x}_2 = -q x_1 & x_2^+ = x_2 \\ \dot{x}_3 = x_1^2 - x_3 & x_3^+ = x_3/2 \\ \dot{q} = 0, & q^+ = -q, \end{array}$$

and the flow and jump sets are selected as closed sets ensuring that along flowing solutions we have $x_1(t,j) > 0 \Rightarrow q(t,j) = 1$ and $x_1(t,j) < 0 \Rightarrow q(t,j) = -1$. To this end, when the solution hits the set $\{x_1 = 0\}$, the discrete state is toggled, $q^+ = -q$, and the state x_3 is halved, $x_3^+ = x_3/2$. In particular, we select

$$\begin{aligned} C &= \{(x,q) : x_1 \geq 0, q = 1\} \cup \{(x,q) : x_1 \leq 0, q = -1\}, \\ D &= \{(x,q) : x_1 = 0, q = 1\} \cup \{(x,q) : x_1 = 0, q = -1\}, \end{aligned}$$

For any flowing solution starting in C, the states (x_1, x_2) describe an arc of a circle centered at $(x_1, x_2) = (0, 0)$. The direction of motion is clockwise on the half-space $x_1 > 0$, and counter-clockwise on $x_1 < 0$. Each solution reaches the set $\{(x,q) : x_1 = 0\}$ in finite time. On this set, the only complete solutions are Zeno, namely, the discrete state q persistently toggles. The set

$$\Gamma_2 := \{(x,q) : x_1 = 0\}$$

is, therefore, globally attractive for $\mathcal{H}$. It is, however, unstable, as solutions of the (x_1, x_2)-subsystem starting arbitrarily close to Γ_2 with $x_2 > 0$ evolve along arcs of circles that move away from Γ_2. On Γ_2, the flow is described by the differential equation $\dot{x}_3 = -x_3$, while the jumps are described by the difference equation $x_3^+ = x_3/2$. Thus the x_2 axis

$$\Gamma_1 := \{(x,q) \in \Gamma_2 : x_3 = 0\},$$

is globally asymptotically stable relative to Γ_2. Since the states (x_1, x_2) are bounded, so is the x_3 state. By Theorem 4.4, Γ_1 is globally attractive for $\mathcal{H}$. On the other hand, Γ_1 is unstable for $\mathcal{H}$. △

Example 4.6: In Theorem 4.4, one may not replace assumption (i) by the weaker requirement that Γ_1 be attractive relative to Γ_2. We illustrate this fact with an example taken from [4]. Consider the smooth differential equation

$$\begin{aligned} \dot{x}_1 &= (x_2^2 + x_3^2)(-x_2) \\ \dot{x}_2 &= (x_2^2 + x_3^2)(x_1) \\ \dot{x}_3 &= -x_3^3, \end{aligned}$$

and the sets $\Gamma_1 = \{(x_1, x_2, x_3) : x_2 = x_3 = 0\}$ and $\Gamma_2 = \{(x_1, x_2, x_3) : x_3 = 0\}$. One can see that Γ_2 is globally asymptotically stable, and the motion on Γ_2 is described by the system

$$\begin{aligned} \dot{x}_1 &= -x_2(x_2^2) \\ \dot{x}_2 &= x_1(x_2^2). \end{aligned}$$

On $\Gamma_1 \subset \Gamma_2$, every point is an equilibrium. Phase curves on Γ_2 off of Γ_1 are concentric semicircles $\{x_1^2 + x_2^2 = c\}$, and therefore Γ_1 is a global, but unstable, attractor relative to Γ_2. As shown in Figure 3, for initial conditions not in Γ_2 the trajectories are bounded and their positive limit set is a circle inside Γ_2 which intersects Γ_1 at equilibrium points. Thus Γ_1 is not attractive. △

Theorem 4.7 (Reduction theorem for asymptotic stability): For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, consider two sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$, with Γ_1 compact and Γ_2 closed. Then Γ_1 is asymptotically stable if, and only if

(i) Γ_1 is asymptotically stable relative to Γ_2,
(ii) Γ_2 is locally stable near Γ_1,
(iii) Γ_2 is locally attractive near Γ_1.

Moreover, Γ_1 is globally asymptotically stable for $\mathcal{H}$ if, and only if,

[4] Note that the set inclusions in (15) always hold for a small enough $\bar{\rho}$. Indeed, even in the peculiar case when $C \cap \Gamma_2$ is empty, since C and Γ_2 are closed, it is possible to pick $\bar{\rho}$ small enough so that $C \cap \bar{B}_{\bar{\rho}}(\Gamma_2)$ is empty too, and then the inclusions (15) hold because both sides are empty sets. Similar arguments apply when $D \cap \Gamma_2$ is empty.

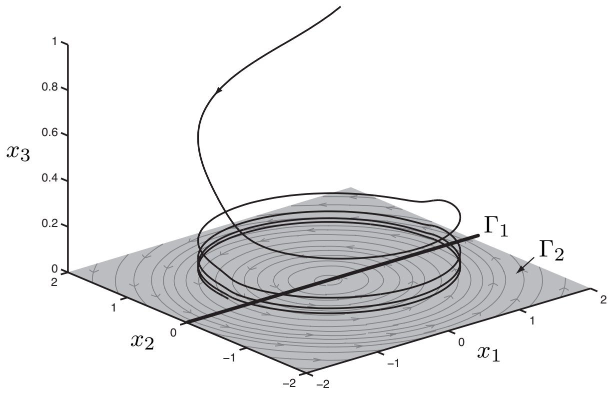

Fig. 3. Example 4.6: $\mathbf{\Gamma_1}$ is globally attractive relative to $\mathbf{\Gamma_2}$, $\mathbf{\Gamma_2}$ is globally asymptotically stable, and yet $\mathbf{\Gamma_1}$ is not attractive.

(i') Γ_1 is globally asymptotically stable relative to Γ_2,
(ii') Γ_2 is locally stable near Γ_1,
(iii') Γ_2 is globally attractive,
(iv') all solutions of $\mathcal{H}$ are bounded.

Proof: ($\Leftarrow$) We begin by proving the local version of the theorem.

By assumption (i), there exists $r > 0$ such that Γ_1 is globally asymptotically stable relative to the set $\Gamma_{2,r} := \Gamma_2 \cap \bar{B}_r(\Gamma_1)$ for $\mathcal{H}$. By Lemma 2.10, the same property holds for the restriction $\mathcal{H}_r := \mathcal{H}|_{\bar{B}_r(\Gamma_1)}$.

By assumption (iii) and by making, if necessary, r smaller, $\Gamma_{2,r}$ is globally attractive for $\mathcal{H}_r$.

By Theorem 4.4, the basin of attraction of Γ_1 for $\mathcal{H}_r$ is the set of initial conditions from which solutions of $\mathcal{H}_r$ are bounded. Since the flow and jump sets of $\mathcal{H}_r$ are compact, all solutions of $\mathcal{H}_r$ are bounded, and thus Γ_1 is attractive for $\mathcal{H}_r$.

Assumptions (i) and (ii) and Theorem 4.1 imply that Γ_1 is stable for $\mathcal{H}$. Since Γ_1 is contained in the interior of $\bar{B}_r(\Gamma_1)$, by Lemma 2.11 the attractivity of Γ_1 for $\mathcal{H}_r$ implies the attractivity of Γ_1 for $\mathcal{H}$. Thus Γ_1 is asymptotically stable for $\mathcal{H}$.

For the global version, it suffices to notice that assumptions (i'), (iii'), and (iv') imply, by Theorem 4.4, that Γ_1 is globally attractive for $\mathcal{H}$.

($\Rightarrow$) Suppose that Γ_1 is asymptotically stable. By Lemma 2.10, Γ_1 is asymptotically stable for $\mathcal{H}|_{\Gamma_2}$, and thus condition (i) holds. By [11, Proposition 6.4], the basin of attraction of Γ_1 is an open set $\mathcal{B}$ containing Γ_1, each solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{B})$ is bounded and, if it is complete, it converges to Γ_1. Since $\Gamma_1 \subset \Gamma_2$, such a solution converges to Γ_2 as well. Thus the basin of attraction of Γ_2 contains $\mathcal{B}$, proving that Γ_2 is locally attractive near Γ_1 and condition (iii) holds. To prove that Γ_2 is locally stable near Γ_1, let $r > 0$ and $\varepsilon > 0$ be arbitrary. Since Γ_1 is stable, there exists $\delta > 0$ such that each $x \in \mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma_1))$ remains in $B_\varepsilon(\Gamma_1)$ for all hybrid times in its hybrid time domain. Since $\Gamma_1 \subset \Gamma_2$, $B_\varepsilon(\Gamma_1) \subset B_\varepsilon(\Gamma_2)$. Thus each $x \in \mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma_1))$ remains in $B_\varepsilon(\Gamma_2)$ for all hybrid times in its hybrid time domain. In particular, it also does so for all the hybrid times for which it remains in $B_r(\Gamma_1)$. This proves that condition (ii) holds.

Suppose that Γ_1 is globally asymptotically stable. The proof that conditions (i'), (ii'), (iii') hold is a straightforward adaptation of the arguments presented above. Since Γ_1 is globally attractive, its basin of attraction is $\mathbb{R}^n$. Since Γ_1 is compact, by definition all solutions originating in its basin of attraction are bounded. Thus condition (iv') holds. ■

Theorems 4.1 and 4.7 generalize to the hybrid setting analogous results for continuous systems in [7], [28], [30]. The following corollary is of particular interest.

Corollary 4.8: For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, consider two sets $\Gamma_1 \subset \Gamma_2 \subset \mathbb{R}^n$, with Γ_1 compact and Γ_2 closed. If

(i) Γ_1 is asymptotically stable relative to Γ_2,
(ii) Γ_2 is asymptotically stable,

then Γ_1 is asymptotically stable. Moreover, if

(i') Γ_1 is globally asymptotically stable relative to Γ_2,
(ii') Γ_2 is globally asymptotically stable,

then Γ_1 is asymptotically stable with basin of attraction given by the set of initial conditions from which all solutions are bounded. In particular, if all solutions are bounded, then Γ_1 is globally asymptotically stable.

Proof: If Γ_2 is asymptotically stable then Γ_2 is locally attractive near Γ_1. Moreover, for each $\varepsilon > 0$ there exists an open set U containing Γ_2 such that each $x \in \mathcal{S}_{\mathcal{H}}(U)$ remains in $B_\varepsilon(\Gamma_2)$ for all hybrid times in its hybrid time domain. Since $\Gamma_1 \subset \Gamma_2$, Γ_1 is contained in U. Since Γ_1 is compact, there exists $\delta > 0$ such that $B_\delta(\Gamma_1) \subset U$. Thus each solution $x \in \mathcal{S}_{\mathcal{H}}(B_\delta(\Gamma_1))$ remains in $B_\varepsilon(\Gamma_2)$ for all hybrid times in its hybrid time domain, implying that Γ_2 is locally stable near Γ_1. By Theorem 4.7, Γ_1 is asymptotically stable. An analogous argument holds for the global version of the corollary. ■

If in Theorems 4.1, 4.4, and 4.7 one replaces $\mathbb{R}^n$ by a closed subset $\mathcal{X}$ of $\mathbb{R}^n$, then the conclusions of the theorems hold relative to $\mathcal{X}$, for one can apply the theorems to the restriction $\mathcal{H}|_{\mathcal{X}}$. This allows one to apply the theorems inductively to finite sequences of nested subsets $\Gamma_1 \subset \cdots \subset \Gamma_l$ to solve the recursive reduction problem.

Theorem 4.9 (Recursive reduction theorem for asymptotic stability): For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, consider l sets $\Gamma_1 \subset \cdots \subset \Gamma_l \subset \Gamma_{l+1} := \mathbb{R}^n$, with Γ_1 compact and all Γ_i closed. If

(i) Γ_i is asymptotically stable relative to Γ_{i+1}, $i = 1, \ldots, l$,

then Γ_1 is asymptotically stable for $\mathcal{H}$. On the other hand, if

(i') Γ_i is globally asymptotically stable relative to Γ_{i+1}, $i = 1, \ldots, l$,
(ii') all $x \in \mathcal{S}_{\mathcal{H}}$ are bounded,

then Γ_1 is globally asymptotically stable for $\mathcal{H}$.

Analogous statements hold, *mutatis mutandis*, for the properties of stability and attractivity (see [7, Proposition 14]). The proof of the theorem above is contained in that of [7, Proposition 14] and is therefore omitted.

V. Adaptive hybrid observer for uncertain internal models

Consider a LTI system described by equations of the form

$$\dot{\chi} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \chi := S\chi, \tag{18a}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \chi := H\chi, \tag{18b}$$

with $\omega \in \mathbb{R}$ not precisely known, for which however lower and upper bounds are assumed to be available, namely $\omega_m < \omega < \omega_M$, $\omega_m, \omega_M \in \mathbb{R}_+$. Note that (18) can be considered a hybrid system with empty jump set and jump map. Suppose in addition that the norm of the initial condition $\chi(0,0)$ is upper and lower bounded, namely $\chi_m \le |\chi(0,0)| \le \chi_M$, for some known positive constants χ_m and χ_M. By the nature of the dynamics in (18), the bounds above imply the existence of a compact set $\mathcal{W} := \{\chi \in \mathbb{R}^2 : |\chi| \in [\chi_m, \chi_M]\}$ that is strongly forward invariant for (18) and where solutions to (18) are constrained to evolve.

The objective of this section consists in estimating the period of oscillation, namely $2\pi/\omega$ with ω unknown, and in (asymptotically) reconstructing the state of the system (18) via the measured output y. It is shown that this task can be reformulated in terms of the results discussed in the previous sections. Towards this end, let

$$\begin{cases} \dot{\hat{\chi}} &= \hat{S}(T)\hat{\chi} + \hat{L}(T)(y - H\hat{\chi}), \\ \dot{q} &= 0, \\ \dot{T} &= 0, \\ \dot{\tau} &= 1, \end{cases} \quad \begin{cases} \hat{\chi}^+ &= \hat{\chi}, \\ q^+ &= \operatorname{sign}(y), \\ T^+ &= \lambda T + (1-\lambda)2\tau, \\ \tau^+ &= 0, \end{cases} \tag{19}$$

with $\lambda \in [0,1)$, denote the *flow* and *jump* maps, respectively, of the proposed hybrid estimator, where the matrices $\hat{S}(T)$ and $\hat{L}(T)$ are defined as

$$\hat{S}(T) := \begin{bmatrix} 0 & -\dfrac{2\pi}{T} \\ \dfrac{2\pi}{T} & 0 \end{bmatrix}, \quad \hat{L}(T) := \begin{bmatrix} \dfrac{4\pi}{T} \\ 0 \end{bmatrix}, \tag{20}$$

which are such that $(\hat{S}(T) - \hat{L}(T)H)$ is Hurwitz. Note that the lower bound T_m on T specified below guarantees that matrix $\hat{S}(T)$ is well-defined.

Intuitively, the rationale behind the definition of flow and jump sets for the hybrid estimator given below is that the system is forced to jump whenever the sign of the logic variable q is different from the sign of the output y. Therefore, homogeneity of the dynamics implies that τ is eventually upper-bounded by some value $\bar{\tau} = \pi/\omega_m$. Moreover, note that the lower and upper bounds on ω induce similar bounds on the possible values of T, namely $2\pi/\omega_M = T_m < T < T_M = 2\pi/\omega_m$. Denoting by Ξ the space where state $\xi := (\chi, \hat{\chi}, q, T, \tau)$ evolves,

$$\Xi := \mathcal{W} \times \mathbb{R}^2 \times \{-1, 1\} \times [T_m, T_M] \times [0, \pi/\omega_m],$$

the closed-loop system (18)-(19) is then completed by the flow set

$$\mathcal{C} := \{(\chi, \hat{\chi}, q, T, \tau) \in \Xi : qy \ge -\sigma\}, \tag{21}$$

and by the jump set

$$\mathcal{D} := \{(\chi, \hat{\chi}, q, T, \tau) \in \Xi : |y| \ge \sigma, qy \le -\sigma\} \tag{22}$$

for some $\sigma > 0$ that should be selected smaller than χ_m to guarantee that the output trajectory, under the assumptions for the initial conditions of (18), intersects the line $qy = -\sigma$. Note that $\mathcal{C}$ and $\mathcal{D}$ depend only on the output y.

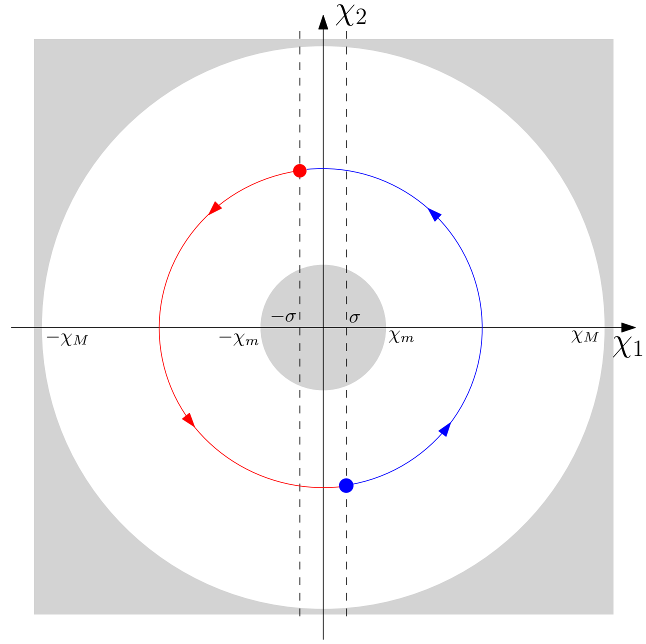

Fig. 4. The white *doughnut* represents the set $\mathcal{W}$. The red/blue curve is a solution $\chi(t,j)$ where the dots represents jump instants. The solution is blue in regions where $\mathfrak{h}(\chi(t,j)) = -1$ and is red in regions where $\mathfrak{h}(\chi(t,j)) = 1$

Adopting the notation introduced in the previous sections, define the functions $\mathfrak{h} : \mathbb{R}^2 \to \{-1, 1\}$ as

$$\mathfrak{h}(\chi) := \begin{cases} -1, & \text{if} \quad \chi_1 \ge \sigma \ \vee \ (|\chi_1| < \sigma \wedge \chi_2 > 0) \\ \ \ 1, & \text{if} \quad \chi_1 \le -\sigma \ \vee \ (|\chi_1| < \sigma \wedge \chi_2 < 0) \end{cases} \tag{23}$$

and $\varrho : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ as $\varrho(\chi, \tau) := He^{S(\pi/\omega - \tau)}\chi - \mathfrak{h}(\chi)\sigma$, which is constant along flowing solutions because

$$\dot{\varrho}(\chi, \tau) = -He^{S(\pi/\omega - \tau)}S\chi + He^{S(\pi/\omega - \tau)}\dot{\chi} = 0, \tag{24}$$

which is zero if and only if τ is suitably synchronized with χ, namely such that $\tau^+ = \pi/\omega$: this would in turn guarantee that $T^+ = 2\pi/\omega$ at the next jump provided that also $T = 2\pi/\omega$. Then, consider the sets

$$\Gamma_3 := \left\{\xi \in \Xi : \varrho(\chi, \tau) = 0\right\}, \tag{25}$$

$$\Gamma_2 := \left\{\xi \in \Gamma_3 : T = \frac{2\pi}{\omega}\right\} \tag{26}$$

and

$$\Gamma_1 := \left\{\xi \in \Gamma_2 : \chi = \hat{\chi}\right\} \tag{27}$$

with $\xi := (\chi, \hat{\chi}, q, T, \tau)$, which clearly satisfy $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3$. Roughly speaking, on the set Γ_1 the state $\hat{\chi}$ of the hybrid estimator (19) is perfectly synchronized with that of system (18), Γ_2 consists of the set of states that ensure $T^+ = 2\pi/\omega$ at the next jump, while Γ_3 prescribes the correct value of the initial timer τ, depending on the initial phase of χ, such that at jumps τ coincides with π/ω. Note that Γ_1 is compact, by the hypothesis on $\mathcal{W}$, while Γ_2 and Γ_3 are closed.

Let us now show GAS of Γ_1 by using reductions theorems. To this end, we apply the recursive version of Theorem 4.7 given in Theorem 4.9. In particular, we show GAS of Γ_1

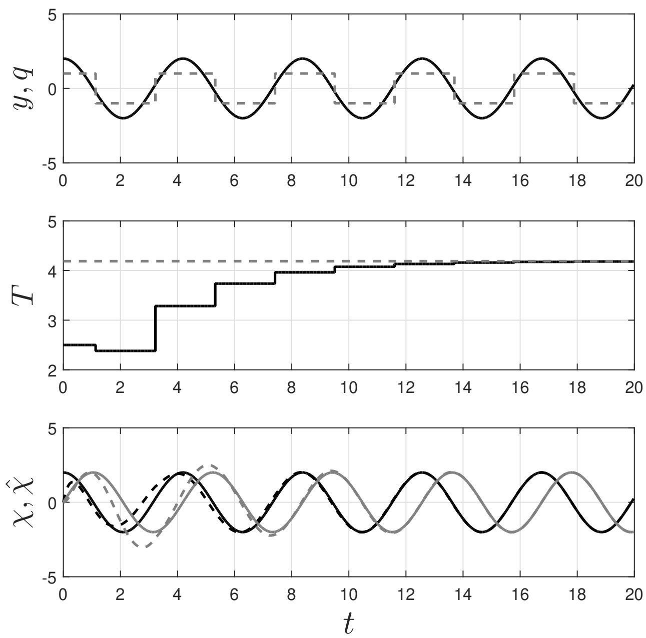

Fig. 5. Top Graph: time histories of the function y generated by (18) and of the state $q(t,k)$, solid and dashed lines, respectively. Middle Graph: time histories of the estimate $T(t,k)$, converging to the correct value of the period of oscillation $2\pi/\omega$. Bottom Graph: time histories of $\hat{\chi}_1(t,k)$ (dark) and $\hat{\chi}_2(t,k)$ (gray), solid lines, converging to the actual states $\chi_1(t,k)$ and $\chi_2(t,k)$, dashed lines.

relative to Γ_2, GAS of Γ_2 relative to Γ_3, GAS of Γ_3 and finally boundedness of solutions. To begin with, it can be shown that Γ_1 is globally asymptotically stable relative to Γ_2. In fact, letting $\eta_1 = \chi - \hat{\chi}$ denote the estimation error, then its dynamics restricted to Γ_2, due to the trivial jumps of χ and $\hat{\chi}$, is described by the hybrid system defined by the flow dynamics

$$\dot{\eta}_1 = S\chi - \hat{S}(T)\hat{\chi} - \hat{L}(T)H\eta_1 = (S - \hat{L}(T)H)\eta_1\,, \quad (28)$$

which is obtained by considering that, on the set Γ_2, $\hat{S}(T) = S$, for $\xi \in \mathcal{C}$, and the jump dynamics $\eta_1^+ = \eta_1$ for $\xi \in \mathcal{D}$. The claim follows by recalling that $\hat{L}(T)$ is such that $(S - \hat{L}(T)H)$ is Hurwitz and by *persistent flowing* conditions of stability [10, Proposition 3.27].

Moreover, Γ_2 is globally asymptotically stable relative to Γ_3. To show this, let $\eta_2 = T - 2\pi/\omega$ and recall that all the trajectories of (19) that remain in Γ_3 are characterized by the property that $\tau = \pi/\omega$ at the time of jump. Therefore, the dynamics of η_2 restricted to Γ_3 is described by the hybrid system defined by the flow dynamics $\dot{\eta}_2 = 0$, for $\xi \in \mathcal{C}$ and the jump dynamics

$$\eta_2^+ = T^+ - \frac{2\pi}{\omega} = \lambda\left(T - \frac{2\pi}{\omega}\right) = \lambda\eta_2\,, \quad (29)$$

for $\xi \in \mathcal{D}$. Asymptotic stability of Γ_2 relative to Γ_3 then follows by *persistent jumping* stability conditions [10, Proposition 3.24], which applies because $\sigma > \chi_m$, and by recalling that $0 \leq \lambda < 1$. In addition, global attractivity of Γ_3 can be shown by relying on the fact that $\tau(t_2, 1)$, namely the value of τ before the second jump, is equal to π/ω, hence implying that $\varrho(\chi(t,k), \tau(t,k)) = 0$ for $(t,k) \in \operatorname{dom}\varrho$ with $k > 1$. Stability of Γ_3, on the other hand, follows by noting that a perturbation δ on $\tau(0,0)$ with respect to the values in Γ_3, *i.e.* values that satisfy $\varrho(\chi, \tau) = 0$, results in $\tau(t_1, 0) = \pi/\omega + \varepsilon(\delta)$, with ε a class-$\mathcal{K}$ function of δ.

Finally, boundedness of the trajectories of the state χ and of q, T and τ follows by the existence of the strongly forward invariant set $\mathcal{W}$ - described by the lower, χ_m, and upper, χ_M, bounds - and by definition of the flow and jump sets, respectively. Therefore, to conclude global asymptotic stability of the set Γ_1 it only remains to show that the trajectories of $\hat{\chi}$ are bounded. Towards this end, recall the flow dynamics of $\hat{\chi}$ in (19), namely

$$\dot{\hat{\chi}} = (S(T) - \hat{L}(T)C_o)\hat{\chi} + \hat{L}(T)C_o\chi := M(T)\hat{\chi} + \hat{L}(T)C_o\chi\,, \quad (30)$$

with $M(T)$, and its derivative with respect to T, uniformly bounded in T, since $T \in [T_m, T_M]$, and Hurwitz uniformly in T by definition of $\hat{L}(T)$, whereas the jump dynamics is described by $\hat{\chi}^+ = \hat{\chi}$. Thus, by applying [16, Lemma 5.12], it follows that there exists a unique positive definite solution $P(T)$ to the Lyapunov equation $P(T)M(T) + M(T)^\top P(T) = -I$, with the additional property that $c_1|\hat{\chi}|^2 \leq \hat{\chi}^\top P(T)\hat{\chi} \leq c_2|\hat{\chi}|^2$, for some positive constants c_1 and c_2. Boundedness of the trajectories of $\hat{\chi}$ then follows by standard manipulations on the time derivative of the functions $\hat{\chi}^\top P(T)\hat{\chi}$ along the trajectories of (30) and by noting that $\hat{L}(T)$ is uniformly bounded, by the definition of $\hat{L}$ and of T, and by recalling that $|\chi|$ is uniformly bounded by definition of the strongly forward invariant compact set $\mathcal{W}$.

In the following numerical simulations, we suppose that $\omega = 1.5$ and we let $\sigma = 0.25$ and $\lambda = 0.5$. Moreover, we let $\chi(0,0) = [2,\ 0]'$ and $\hat{\chi}(0,0) = [0,\ 0]'$, while the remaining components of the estimator are initialized as $q(0,0) = 1$, $T(0,0) = 2.5$ and $\tau(0,0) = 0$. The top graph of Figure 5 depicts the time histories of the function y generated by (18) and of the state $q(t,k)$, solid and dashed lines, respectively. The middle graph of Figure 5 shows the time histories of the estimate $T(t,k)$, converging to the correct value of the period of oscillation $2\pi/\omega$, while the bottom graph displays the time histories of $\hat{\chi}_1(t,k)$ (dark) and $\hat{\chi}_2(t,k)$ (gray), solid lines, converging to the actual states $\chi_1(t,k)$ and $\chi_2(t,k)$, dashed lines.

VI. Conclusion

In this paper we presented three reduction theorems for stability, local/global attractivity, and local/global asymptotic stability of compact sets for hybrid dynamical systems, along with a number of their consequences. The proofs of these results rely crucially on the $\mathcal{KL}$ characterization of robustness of asymptotic stability of compact sets found in [10, Theorem 7.12]. A different proof technique is possible which generalizes the proofs found in [7]. As a future research direction, we conjecture that, similarly to what was done in [7] for continuous dynamical systems, it may be possible to state reduction theorems for hybrid systems in which the set Γ_1 is only assumed to be closed, not necessarily bounded.

In addition to the applications listed in the introduction, the reduction theorems presented in this paper may be employed to generalize the position control laws for VTOL vehicles presented in [19], [24], by replacing continuous attitude stabilizers with hybrid ones, such as the one found in [18]. Furthermore, the results of this paper may be used to generalize the allocation techniques of [22], possibly following similar ideas to those in [8].

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