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Reduction Principles for Hierarchical Control Design

Preface

These notes, developed in support of a mini course at the Norwegian Institute of Science and Technology, present an overview of a developing body of work on hierarchical set stabilization for nonlinear control systems. In these notes, a control specification is a closed set that we wish to stabilize by feedback control. We consider specifications that can be broken down hierarchically.

We propose a framework in which a hierarchy of control specifications is a list of nested closed subsets $\Gamma_1 \subset \cdots \subset \Gamma_n$ to be asymptotically stabilized. The key theoretical tool in this context is a reduction theorem for asymptotic stability of nested sets.

We present a number of applications of the theory:

- A circular formation path following problem for a collection of nonholonomic vehicles.

- An almost global position control methodology for underactuated thrust-propelled vehicles.

- A backstepping methodology that doesn't rely on Lyapunov functions.

- A result concerning the stability of sets for cascade-connected systems.

The notes are organized as follows. In Chapter 1 we formulate the hierarchical control problem (HCP) and introduce a number of examples motivating the chosen formulation. In Chapter 2 we introduce notions of set stability, and present reduction theorems for stability, attractivity, and asymptotic stability of nested sets. We use these results to solve HCP. Finally, in Chapter 3 we return to the examples of Chapter 1 and work out their solution using the theoretical tools of Chapter 1.

I am grateful to my Ph.D. students Mohamed I. El-Hawwary and Ashton Roza, who were collaborators on this research. Without their penetrating technical insight these results wouldn't have been possible. I am deeply indebted to Professor Kristin Y. Pettersen for the invitation to hold a mini-course at NTNU and for her kind hospitality.

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Contents

Notation

$\mathbb{R}_+$	The set of nonnegative real numbers
$\mathbb{C}^-$	The set of complex numbers with real part < 0
$\mathbb{S}^1$	The set of real numbers modulo 2π, with the unique manifold structure that makes it diffeomorphic to the unit circle in $\mathbb{R}^2$
$\mathbb{R}^n$	The set of all ordered n-tuples of real numbers
$\mathbb{R}^{n\times m}$	The set of $n \times m$ matrices with real entries
$S_1 \times S_2$	The Cartesian product of sets S_1 and S_2
S^n	The Cartesian product $S \times \cdots \times S$, n times
$\mathrm{cl}(S)$	The closure of a set S
$\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$	The natural basis of $\mathbb{R}^n$
$SO(3)$	The set $\{R \in \mathbb{R}^{3\times 3} : R^\top R = I_3,\ \det R = 1\}$
$\mathbf{0}$	A vector or matrix of zeros, with dimension implied from the context
I_n	The $n \times n$ identity matrix
$\mathbf{n}$	The index set $\{1,\ldots,n\}$
$\mathbf{1}$	The vector $[1\ \cdots\ 1]^\top$ of dimension deduced from the context
$\omega^\times$	The skew symmetric matrix generated by the vector $\omega \in \mathbb{R}^3$,

$$\omega^\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

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$M_\times$	For a skew symmetric $M^\top = -M$ in $\mathbb{R}^{3\times 3}$, $M_\times = \begin{bmatrix} M_{32} & M_{13} & M_{21} \end{bmatrix}^\top$
$\mathrm{blockdiag}(M_1,\ldots,M_k)$	The blockdiagonal matrix with diagonal blocks $M_1,\ldots,M_k$, in this order
$M_1 \otimes M_2$	The Kronecker product of matrices M_1 and M_2
$\|x\|_\Gamma$	The point-to-set distance of a point $x \in \mathbb{R}^n$ to a set $\Gamma \subset \mathbb{R}^n$, $\|x\|_\Gamma := \inf_{y\in\Gamma}\|x-y\|$
$B_\delta(p)$	Open ball of radius δ centred at $p \in \mathbb{R}^n$, $B_\delta(p) = \{x \in \mathbb{R}^m : \|x\| < \delta\}$
$B_\delta(\Gamma)$	δ-neighborhood of a set $\Gamma \subset \mathbb{R}^n$, $B_\delta(\Gamma) = \{x \in \mathbb{R}^m : \|x\|_\Gamma < \delta\}$
$\sigma(A)$	The spectrum of a matrix A

Chapter 1

The Hierarchical Control Problem

In these notes we present a framework for hierarchical control design. Our aim is to formalize one of the design principles commonly used by practitioners to solve complex control problems, namely the idea of breaking down a control problem into a prioritized sequence of elementary sub-problems. The designer addresses each sub-problem separately, hoping that the final control law works for the ensemble. For example, a common approach for controlling the speed of electric motors is to have an inner loop that controls the armature current and an outer loop to control the speed. The proper operation of the current control loop has higher priority because, without it, the speed control loop would not be able to function.

Our first objective is to gain some intuition about hierarchical control design. We will do that through three examples. Then, in Section 1.4 we will formulate the hierarchical control problem.

1.1 POSITION CONTROL OF A THRUST-PROPELLED VEHICLE

Consider a rigid body propelled by a thrust force and equipped with some mechanism to induce torques about the three body axes.
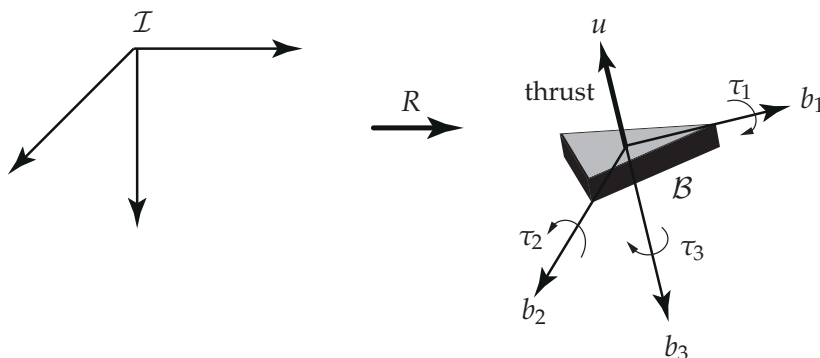
Fix an inertial frame $\mathcal{I}$ and a body frame $\mathcal{B} = \{b_1, b_2, b_3\}$. Let $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ denote the position and velocity of the vehicle's centre of mass in frame $\mathcal{I}$. Let $R \in SO(3)$ be the attitude of the body, i.e., the rotation matrix whose columns are the vectors b_1, b_2, b_3 represented in frame $\mathcal{I}$, and let $\omega \in \mathbb{R}^3$ be the body's angular velocity in frame $\mathcal{B}$. Assume that the vehicle is propelled by a thrust vector with

magnitude u directed along the negative b_3 axis. Let τ be the torque vector in body frame. The equations of motion of the vehicle are

$$\text{Translational:} \begin{cases} \dot{x} = v \\ m\dot{v} = mg\mathbf{e}_3 - uR\mathbf{e}_3 \\ \qquad = mg\mathbf{e}_3 + T. \end{cases} \tag{1.1}$$

$$\text{Rotational:} \begin{cases} \dot{R} = R\omega^\times \\ J\dot{\omega} + \omega \times J\omega = \tau, \end{cases} \tag{1.2}$$

where m denotes the mass of the body and J is its symmetric inertial matrix in frame $\mathcal{B}$. We denote the state of the vehicle by $\chi := (x, v, R, \omega) \in \mathcal{X} := \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3$.

The above model encompasses a large class of VTOL aircrafts such as quadrotor helicopters. There are six degrees-of-freedom and four control inputs, the torque $\tau \in \mathbb{R}^3$ and the thrust magnitude $u \in \mathbb{R}_+$. Thus the rigid body is underactuated, with degree of underactuation equal to two.

The control objective is position stabilization: stabilize the equilibrium $\chi^\star = (x^\star, \mathbf{0}, R^\star, \mathbf{0})$, where $R^\star$, the attitude of the vehicle at equilibrium, corresponds to a hovering configuration, i.e., $R^\star \mathbf{e}_3 = \mathbf{e}_3$ (the thrust axis is vertical).

A commonly used control strategy is illustrated in the block diagram below.
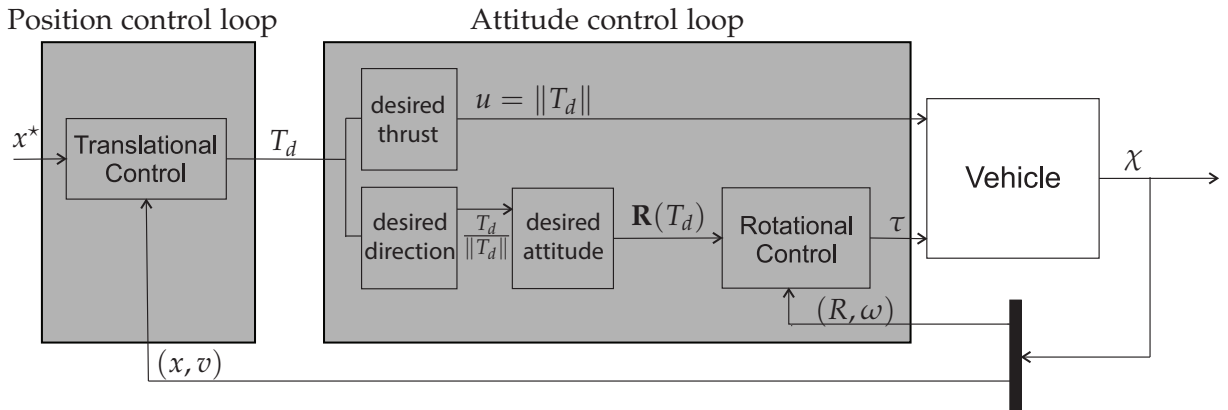
There are two nested loops. The outer loop is a position controller which provides reference signals for the inner loop, an attitude controller. The position controller treats the vehicle as a point-mass and demands a desired thrust vector T_d. The inner loop orients the vehicle so that the actual thrust vector coincides with the desired one. This control topology is appealing due to its simplicity and its modularity. One may redesign the position control module without affecting the attitude control module, and vice versa. The controller structure also conforms to a practical requirement: commonly available commercial autopilots perform attitude control, so it is desirable to leverage such autopilots by adding a position control outer loop. We will now outline the steps that one could perform to produce a position controller that conforms to the above block diagram.

Step 1: Position control design. Consider the translational dynamics in (1.1), repeated below for convenience,

$$\dot{x} = v$$

$$m\dot{v} = mg\mathbf{e}_3 - uR\mathbf{e}_3 = mg\mathbf{e}_3 + T,$$

and view the thrust vector $T = -uR\mathbf{e}_3$ as a control input. Let $T_d(x, v)$ be a feedback such that when $T = T_d(x, v)$, the equilibrium $(x, v) = (x^\star, \mathbf{0})$ is globally asymptotically stable for (1.1). Suppose that the third component of T_d is never zero (i.e., the thrust has a nonzero vertical component) so that $\|T_d\| \neq 0$.

Step 2: Attitude extraction. Define a function $\mathbf{R}(T_d) : \mathbb{R}^3 \to SO(3)$ such that

$$\mathbf{R}(T_d)\mathbf{e}_3 = -\frac{T_d}{\|T_d\|} \text{ for all } T_d \in \mathbb{R}^3 \backslash \{\mathbf{0}\}.$$

In other words, $\mathbf{R}(T_d)$ is a rotation matrix with the property that when $R = \mathbf{R}(T_d)$ and $u = \|T_d\|$, the thrust vector $T = -uR\mathbf{e}_3$ coincides with the desired thrust vector T_d. Thus $\mathbf{R}(T_d)$ represents the desired attitude of the vehicle.

There are infinitely many choices of smooth functions $\mathbf{R}$. Indeed, if $v \in \mathbb{R}^3$ is an arbitrary unit vector, the matrix $\mathbf{R}(T_d) = \begin{bmatrix} b_{1d} & b_{2d} & b_{3d} \end{bmatrix}$ with

$$b_{1d} = \frac{T_d}{\|T_d\|} \times v, \quad b_{2d} = -\frac{T_d}{\|T_d\|} \times b_{1d}, \quad b_{3d} = -\frac{T_d}{\|T_d\|},$$

has the required properties.

Step 3: Attitude stabilization. Consider the attitude dynamics in (1.2),

$$\dot{R} = R\omega^\times$$
$$J\dot{\omega} + \omega \times J\omega = \tau,$$

and let $\tau_d(R, \omega)$ be a smooth feedback that asymptotically stabilizes $(R, \omega) = (I, \mathbf{0})$. There is a multitude of controllers available in the literature to solve this problem either locally or almost-globally. We will present one of them in Chapter 3.

Position control feedback. Having performed the three design steps above, we are ready to present a position control feedback. Recalling the block diagram presented earlier, we set the thrust magnitude equal to the magnitude of the desired thrust T_d,

$$u = \|T_d(x, v)\|. \tag{1.3}$$

Next, we use the attitude stabilizer to define a torque feedback making R converge to the desired attitude $\mathbf{R}(T_d)$. To this end, we define the desired angular velocity $\boldsymbol{\omega}(\chi)$ as

$$\boldsymbol{\omega}(\chi) := \left[\mathbf{R}(T_d(x, v))^{-1}\dot{\mathbf{R}}(\chi)\right]_\times,$$

where $\dot{\mathbf{R}}$ denotes the derivative of the map $(x, v) \mapsto \mathbf{R}(x, v)$ along (1.1) substituting $u = \|T_d(x, v)\|$.

Next, we define error variables

$$\tilde{\mathbf{R}}(\chi) := \mathbf{R}(T_d(x, v))^{-1}R, \text{ and } \tilde{\omega}(\chi) := \omega - \tilde{\mathbf{R}}^{-1}(\chi)\boldsymbol{\omega}(\chi).$$

It is not hard to show that if one sets

$$\tau(\chi) = \tau_d(\tilde{\mathbf{R}}, \tilde{\omega}) - \tilde{\omega} \times J\tilde{\omega} + \omega \times J\omega - J\left(\tilde{\omega}^\times\tilde{\mathbf{R}}^{-1}\boldsymbol{\omega} - \tilde{\mathbf{R}}^{-1}\dot{\boldsymbol{\omega}}\right), \tag{1.4}$$

the dynamics of the error variables read as

$$\dot{\tilde{\mathbf{R}}} = \tilde{\mathbf{R}}\tilde{\omega}^{\times}$$
$$J\dot{\tilde{\omega}} + \tilde{\omega} \times J\tilde{\omega} = \tau_d(\tilde{\mathbf{R}}, \tilde{\omega}).$$

The function $\dot{\omega}$ in (1.4) is the derivative of the map $\chi \mapsto \omega(\chi)$ along (1.1), (1.2) substituting $u = \|T_d(x, v)\|$.

By the design of τ_d, the equilibrium $(\tilde{\mathbf{R}}, \tilde{\omega}) = (I, \mathbf{0})$ of the above error dynamics is asymptotically stable so that R converges to $\mathbf{R}(T_d(x, v))$, as required.

In conclusion, we have arrived at the position control feedback (1.3), (1.4). Does this feedback solve the position control problem? More precisely, suppose we are given:

- a position control feedback $T_d(x, v)$,

- an attitude extraction map $\mathbf{R}(T_d)$, and

- an attitude stabilizer $\tau_d(R, \omega)$,

and we combine the three functions above according to the formulas (1.3), (1.4). Noting that, at the desired equilibrium $(x, v) = (x^{\star}, \mathbf{0})$, the desired torque is $T_d(x^{\star}, \mathbf{0}) = -mg\mathbf{e}_3$, the attitude extraction map at equilibrium gives $\mathbf{R}(T_d(x^{\star}, 0)) = \mathbf{R}(-mg\mathbf{e}_3)$. Denote $R^{\star} := \mathbf{R}(-mg\mathbf{e}_3)$. The question arising in this context is this: Is it true that the equilibrium $(x, v, R, \omega) = (x^{\star}, \mathbf{0}, R^{\star}, \mathbf{0})$ is asymptotically stable?

What we know from the control design is the following:

- Letting $\Gamma_2 := \{\chi \in \mathcal{X} : R = \mathbf{R}(x, v), \omega = \tilde{\mathbf{R}}^{-1}(\chi)\omega(\chi)\}$, the torque feedback (1.4) makes Γ_2 *asymptotically stable.*

- Due to the property of the attitude extraction map $\mathbf{R}$, on the set Γ_2 we have that $T = -uR\mathbf{e}_3 = T_d(x, v)$. Therefore, the motion of the vehicle on the set Γ_2 is described by equation (1.1) with feedback $T = T_d(x, v)$. By the design of T_d, the equilibrium $\Gamma_1 := \{\chi = (x^{\star}, \mathbf{0}, R^{\star}, \mathbf{0})\}$ is asymptotically stable for (1.1). In other words, the equilibrium Γ_1 is *asymptotically stable relative to the set* Γ_2.

In essence, the question we need to answer is this: *Does the fact that the set Γ_2 is asymptotically stable and the fact that the equilibrium Γ_1 is asymptotically stable relative to Γ_2 imply that Γ_1 is asymptotically stable?*

We need some theoretical development to answer this question. We will return to it in Chapter 3. For now, we remark that the nested loop structure of this position controller corresponds to a hierarchical structure in the control specifications. Namely, the design requires *first* the stabilization of the set Γ_2 (attitude control loop) and *then* the stabilization of the required hovering position within Γ_2 (position control loop). The hierarchy of control specifications is reflected in the fact that $\Gamma_1 \subset \Gamma_2$.

1.2   CIRCULAR FORMATION STABILIZATION FOR KINEMATIC VEHICLES

Consider a collection of $n \geq 2$ identical vehicles on a plane, modeled as kinematic unicycles,

$$\dot{x}_1^i = u_1^i \cos x_3^i$$
$$\dot{x}_2^i = u_1^i \sin x_3^i \qquad i \in \mathbf{n}. \tag{1.5}$$
$$\dot{x}_3^i = u_2^i.$$

The pair $(x_1^i, x_2^i) \in \mathbb{R} \times \mathbb{R}$ represents the coordinates of unicycle i in a common planar inertial reference frame, while $x_3^i \in \mathbb{S}^1$ is the ith unicycle's heading angle. The control inputs (u_1^i, u_2^i) represent, respectively, the linear and angular speeds of unicycle i. We denote $\chi^i = (x_1^i, x_2^i, x_3^i)$, and $\chi = (\chi^1, \dots, \chi^n)$. The collective state space of the unicycles is the Cartesian product $\mathcal{X} = (\mathbb{R} \times \mathbb{R} \times \mathbb{S}^1)^n$.

We want to make the vehicles follow a common circle of radius r with unspecified centre, and move in formation along the circle counterclockwise, with desired ordering and spacing, as illustrated in the figure on the right-hand side. The formation specification on the common circle is given in terms of angles θ_i, represented in the side figure. More precisely, θ_i is the desired angle subtended by the arc of the circle between unicycles i and $i+1$. Equivalently, as illustrated in the figure, θ_i is the desired relative heading angle between unicycles i and $i+1$, $\theta_i = (x_3^{i+1} - x_3^i) \bmod 2\pi$.
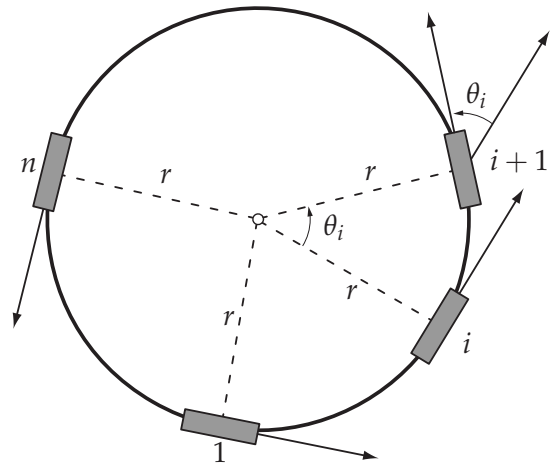
The circular path following specification can be formalized with a little geometric insight. In order to travel around a circle of radius r counterclockwise, the linear and angular speeds of unicycle i must be related as follows: $u_2^i = u_1^i / r$, $u_1^i > 0$. If this is the case, then the centre of the circle has coordinates

$$c^i(\chi^i) = \begin{bmatrix} x_1^i - r \sin x_3^i \\ x_2^i + r \cos x_3^i \end{bmatrix}.$$

(This is easily seen by drawing a vector of length r orthogonal to the velocity of unicycle i). The circular path following specification is the requirement that $c^1(\chi^1) = \cdots = c^n(\chi^n)$ and that $u_2^i = u_1^i / r$, $u_1^i > 0$.

Now we create a hierarchy of control specifications. We will *first* make the unicycles converge to a common circle, and *then* stabilize the required formation on the circle. Accordingly, let

$$\Gamma_2 := \{\chi : c^1(\chi^1) = \cdots = c^n(\chi^n)\},$$

and

$$\Gamma_1 := \{\chi \in \Gamma_2 : x_3^{i+1} - x_3^i = \theta_i \bmod 2\pi, \ i = 1, \dots, n-1\}.$$

As in Section 1.1, we have two subsets $\Gamma_1 \subset \Gamma_2$ of the state space representing two control specifications. The fact that the subsets are nested reflects a hierarchy in the control specifications. We may flip the order of specifications, picking $\Gamma_2 = \{\chi : x_3^{i+1} - x_3^i = \theta_i \bmod 2\pi, \ i = 1, \dots, n-1\}$ and $\Gamma_1 = \{\chi \in \Gamma_2 : c^1(\chi^1) = \cdots = c^n(\chi^n)\}$. This definition would give rise to a different control design and a different physical behavior of the vehicles. The hierarchical control design strategy is to design feedbacks such that

- The set Γ_2 is globally asymptotically stable (the unicycles follow a common circle).

- The set Γ_1 is asymptotically stable relative to Γ_2 (the unicycles move in formation on the circle).

As in Section 1.1, the question is whether or not the two properties above imply that Γ_1 is asymptotically stable. We will return to this problem in Chapter 3, where we will design smooth static feedbacks with the two properties above, and we will show that they do indeed solve the circular formation stabilization problem.

1.3   Backstepping as Hierarchical Control Design

Backstepping [KKK95] is a popular design methodology to stabilize the origin of smooth lower-triangular systems

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2$$

$$\vdots$$

$$\dot{x}_i = f_i(x_1, \cdots, x_i) + g_i(x_1, \cdots, x_i)x_{i+1}$$

$$\vdots$$

$$\dot{x}_n = f_n(x_1, \cdots, x_n) + g_n(x_1, \cdots, x_n)u,$$

where $x_i \in \mathbb{R}$, $f_i(0, \cdots, 0) = 0$, and $g_i \neq 0$, $i \in \mathbf{n}$. The idea is to break down the control design into n steps. At step 1, we view x_2 as control input and design a feedback $x_2 = \mu_1(x_1)$ stabilizing $x_1 = 0$ for the x_1-subsystem. At step 2, we view x_3 as control input and design a feedback stabilizing the error variable $x_2 - \mu_1(x_1)$. The procedure continues recursively in this manner until, at step n, a feedback u is found.

More in detail, let $\mu_1 = g_1^{-1}(-f_1 - K_1 x_1)$ so that, when $x_2 = \mu_1(x_1)$, $\dot{x}_1 = -K_1 x_1$. Then define the error $e_1 = x_2 - \mu_1(x_1)$. Its time derivative is $\dot{e}_1 = f_2 + g_2 x_3 - \dot{\mu}_1$. If x_3 were a control input, we would set $x_3 = \mu_2(x_1, x_2) := g_2^{-1}(-f_2 + \dot{\mu}_1 - K_2 e_1)$. Continuing like this, we define[1] recursively

$$\mu_i = g_i^{-1}\left(-f_i + \dot{\mu}_{i-1} - K_i(x_i - \mu_{i-1})\right), \quad i = 2, \ldots, n-1.$$

Finally, at step n we obtain

$$u = g_n^{-1}\left(-f_n + \dot{\mu}_{n-1} - K_n(x_n - \mu_{n-1})\right).$$

Denote by Γ_1 the equilibrium $(x_1, \ldots, x_n) = (0, \ldots, 0)$ and, for $i = 2, \ldots, n$, define sets

$$\Gamma_i = \{(x_1, \ldots, x_n) : x_j = \mu_{j-1}(x_1, \ldots, x_{j-1}), j = i, \ldots, n\}.$$

We have n nested invariant sets $\Gamma_1 \subset \cdots \subset \Gamma_n$. The above backstepping design has these properties:

- For $i = 1, \ldots, n-1$, the set Γ_i is globally asymptotically stable relative to Γ_{i+1}.

- The set Γ_n is globally asymptotically stable.

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[1]Classical backstepping [KKK95] relies on the recursive definition of a Lyapunov function and results in slightly different definitions of μ_i.

As in Sections 1.1 and 1.2, the question is whether the above properties imply that Γ_1 is globally asymptotically stable. We will see in Chapter 3 that the answer is *yes* for this particular problem.

In light of the above, we can think of backstepping as a hierarchical control design arising from n specifications: design u to globally asymptotically stabilize Γ_n. Then make Γ_{n-1} globally asymptotically stable relative to Γ_n, and so on. Finally, we require Γ_1 to be globally asymptotically stable relative to Γ_2.

1.4   STATEMENT OF THE HIERARCHICAL CONTROL PROBLEM

We now formalize the intuition developed in the previous three examples. Let us review their salient features.

- Each control specification was formulated as stabilization of a closed subset of the state space. This subset is *controlled invariant,* i.e., it can be rendered positively invariant by a suitable feedback. Imposing controlled invariance amounts to requiring that the control specification is *feasible* in that, as we will discuss in Remark 2.6, a necessary condition for a set to be stable is that it be positively invariant. To illustrate the significance of controlled invariance, consider a point mass on the plane actuated by a force u, $m\ddot{x} = u$, $x, u \in \mathbb{R}^2$. Suppose we wish to make the mass converge to the unit circle $\Gamma = \{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : x^\top x = 1\}$. This set is not controlled invariant: if the point-mass is initialized on the circle with velocity not tangent to it, the mass will leave the circle no matter what force is applied to it. It is therefore impossible to stabilize Γ: the specification is not feasible and it needs to be refined. It is sufficient to replace Γ by its maximal controlled invariant subset,

$$\hat{\Gamma} = \{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : x^\top x = 1, x^\top \dot{x} = 0\}.$$

On this set, the particle is on the circle and its velocity is tangent to it. Now $\hat{\Gamma}$ is a feasible specification.

- Multiple control specifications were prioritized, giving rise to nested controlled invariant sets. In the circular formation stabilization problem of Section 1.2, higher priority was given to the stabilization of a common circle; lower priority was given to formation stabilization. In the position control problem of Section 1.1, we gave higher priority to attitude stabilization, and lower priority to position control. In backstepping (Section 1.3) we gave higher priority to the stabilization of the error $x_n - \mu_{n-1}(x_1, \ldots, x_{n-1})$, lower priority to the stabilization of $x_{n-1} - \mu_{n-2}$, and so on.

- The control design was decomposed into simpler sub-problems: The enforcement of the i-th specification was achieved assuming that specification $i - 1$ had already been met. For instance, in the position control problem of Section 1.1, position control was achieved *assuming* that the attitude of the vehicle coincides with the desired attitude, so that the thrust vector $T = -uR\mathbf{e}_3$ coincided with the desired thrust vector $T_d(x, v)$. In backstepping, the stabilization of $x_i - \mu_{i-1}(x_1, \ldots, x_{i-1})$ was achieved assuming that $x_{i+1} = \mu_i(x_1, \ldots, x_i)$.

We are now ready to formalize the hierarchical control problem. The definitions that follow rely on notions of set stability that will be defined in the next chapter.

Consider a locally Lipschitz control system

$$\dot{x} = f(x, u), \quad x \in \mathcal{X}, \tag{1.6}$$

with state space an open subset $\mathcal{X}$ of $\mathbb{R}^n$.

Definition 1.1. A **feasible control specification** SPEC for system (1.6) is a closed set $\Gamma \subset \mathcal{X}$ which is controlled invariant. That is, there exists a locally Lipschitz feedback $\bar{u}(x)$ such that Γ is a positively invariant set for the closed-loop system $\dot{x} = f(x, \bar{u}(x))$.

Now consider a list of feasible control specifications Γ_i for system (1.6), $i \in \mathbf{n}$, labeling the ith specification as SPEC i.

Definition 1.2. We say that specification SPEC i has **lower priority** than specification SPEC j, SPEC $i \preceq$ SPEC j, if $\Gamma_i \subset \Gamma_j$. This notion of priority induces a partial ordering on specifications. A **hierarchy of control specifications** is an ordered list SPEC $1 \preceq \cdots \preceq$ SPEC n of feasible control specifications corresponding to nested sets $\Gamma_1 \subset \cdots \subset \Gamma_n$.

Now we turn to control.

Definition 1.3. Consider a hierarchy of feasible control specifications SPEC $1 \preceq \cdots \preceq$ SPEC n with corresponding sets $\Gamma_1 \subset \cdots \subset \Gamma_n$. A **locally hierarchical feedback** is a locally Lipschitz feedback $\bar{u}(x)$ with the following properties:

 (i) For each $i \in \{1, \ldots, n-1\}$, set Γ_i is asymptotically stable relative to Γ_{i+1} for $\dot{x} = f(x, \bar{u}(x))$.

 (ii) The set Γ_n is asymptotically stable for $\dot{x} = f(x, \bar{u}(x))$.

If $\bar{u}(x)$ induces properties (i) and (ii) (almost) globally, then we call it a **(almost) globally hierarchical feedback.**

We are now ready to state the hierarchical control problem.

Problem 1.4 (Hierarchical Control Problem (HCP)). Given a hierarchy of feasible control specifications SPEC $1 \preceq \cdots \preceq$ SPEC n for system (1.6), find conditions under which a hierarchical feedback (globally) asymptotically stabilizes the set Γ_1.

1.5   BIOGRAPHICAL NOTES

The specific formulation of the position control problem for thrust-propelled vehicles and the solution methodology presented in Section 1.1 are taken from [RM14]. Other papers in the literature use a nested loop architecture. In particular, the idea of attitude extraction map figures prominently in the work of Tayebi and collaborators [AT10, RT11, Rob11]. See also [LLM10].

The circular formation stabilization problem of Section 1.2 is taken from [EHM13a]. A similar problem was investigated in [SPL07] and [SPL08] for planar particles having unit speed whose model coincides with (1.5) putting $u_1^i = 1$.

The backstepping controller presented in Section 1.3 is a minor variation of the basic feedback presented in [KKK95], and it is taken from [EHM13b]. The setup in these notes does not rely on Lyapunov functions, and as such the "virtual controls" μ_i in Section 1.3 omit some terms that are normally used to cancel out certain derivatives of Lyapunov functions.

The formulation of the hierarchical control problem in Section 1.4 is an expanded version of ideas taken from [EHM13b]. Our formulation differs substantially from the hierarchical control theory pioneered by Ramadge and Wonham for discrete-event systems [RW87], and adapted to nonlinear control systems by Pappas and Simic [PS02]. In the context of nonlinear control systems, the notion of hierarchical consistency of [PS02] imposes requirements on the structure of the system's vector fields. In these notes, the hierarchy is embodied in a collection of nested controlled invariant sets, and no structure is assumed or required on the vector fields.

Chapter 2

Reduction Principles and Solution of HCP

In this chapter we solve the hierarchical control problem. We begin by presenting basic notions of set stability, then turn our attention to the so-called *reduction problem*, a fundamental building block towards the solution of HCP.

2.1 STABILITY NOTIONS

Consider a differential equation

$$\Sigma : \dot{x} = f(x) \tag{2.1}$$

with state space a domain $\mathcal{X} \subset \mathbb{R}^n$. Assume that f is locally Lipschitz on $\mathcal{X}$, and let $\phi(t, x_0)$ denote the maximal solution of (2.1) at time t with initial condition $x(0) = x_0$. For each x_0, denote by T_{x_0} the maximal interval of existence of the solution through x_0. Denote $T_{x_0}^+ = T_{x_0} \cap [0, \infty)$ and $T_{x_0}^- = T_{x_0} \cap (-\infty, 0]$.

Definition 2.1. A set $\Gamma \subset \mathcal{X}$ is **positively invariant** for Σ if for all $x_0 \in \Gamma$, $\phi(t, x_0) \in \Gamma$ for all $t \in T_{x_0}^+$. Γ is **negatively invariant** for Σ if for all $x_0 \in \Gamma$, $\phi(t, x_0) \in \Gamma$ for all $t \in T_{x_0}^-$. Γ is **invariant** for Σ if it is both positively and negatively invariant.

Example 2.2. Consider the linear system (a saddle)

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = x_2$$

The sets $\{(x_1, x_2) : x_1 = 0\}$ and $\{(x_1, x_2) : x_2 = 0\}$ are invariant. The set $\{(x_1, x_2) : 0 \leq x_1 \leq 1\}$ is positively invariant and not invariant. The set $\{(x_1, x_2) : 0 \leq x_2 \leq 1\}$ is negatively invariant.

Definition 2.3 (Set stability and attractivity). Let $\Gamma \subset \mathcal{X}$ be a closed positively invariant set for Σ.

(i) Γ is **stable** for Σ if for all $\varepsilon > 0$ there exists a neighbourhood of Γ, $\mathcal{N}(\Gamma)$, such that $\phi(\mathbb{R}_+, \mathcal{N}(\Gamma)) \subset B_\varepsilon(\Gamma)$.

(ii) Γ is an **attractor** for Σ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\lim_{t \to \infty} \|\phi(t, x_0)\|_\Gamma = 0$ for all $x_0 \in \mathcal{N}(\Gamma)$.

(iii) Γ is a **global attractor** for Σ if it is an attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$.

(iv) Γ is [**globally**] **asymptotically stable** for Σ if it is stable and attractive [globally attractive] for Σ.

(v) Γ is **almost globally asymptotically stable** for Σ if it is asymptotically stable with domain of attraction $\mathcal{X} \backslash N$, where N is a set of measure zero.

Example 2.4. Consider again the saddle

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = x_2.$$

The set $\{(x_1, x_2) : x_1 = 0\}$, highlighted in the figure on the right-hand side, is globally asymptotically stable because the dynamics transversal to the subspace are described by $\dot{x}_1 = -x_1$.

More generally, consider a linear time-invariant (LTI) system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n$$

and let $\mathcal{V} \subset \mathbb{R}^n$ be an A-invariant subspace, i.e., $A\mathcal{V} \subset \mathcal{V}$. Questions about stability properties of $\mathcal{V}$ can always be reduced to questions about the stability of an equilibrium. Let $\{v^1, \ldots, v^n\}$ be a basis for $\mathbb{R}^n$ such that $\{v^1, \ldots, v^k\}$, $k \leq n$, is a basis for $\mathcal{V}$. Let T be the matrix whose columns are the basis vectors v^i, $T = [v^1 \cdots v^n]$. The isomorphism $(x^1, x^2) = T^{-1}x$ gives

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}.$$

In new coordinates, the subspace $\mathcal{V}$ is given by

$$T^{-1}\mathcal{V} = \{(x^1, x^2) : x^2 = \mathbf{0}\}.$$

Thus, the subsystem $\dot{x}^2 = A_{22}x^2$ represents the dynamics transversal to $\mathcal{V}$. It is clear that the subspace $\mathcal{V}$ is asymptotically for the LTI system if and only if the eigenvalues of A_{22} are in $\mathbb{C}^-$. Moreover, $\mathcal{V}$ is globally asymptotically stable if and only if it is a global attractor. Finally, $\mathcal{V}$ is stable if and only if the subsystem $\dot{x}^2 = A_{22}x^2$ is stable.
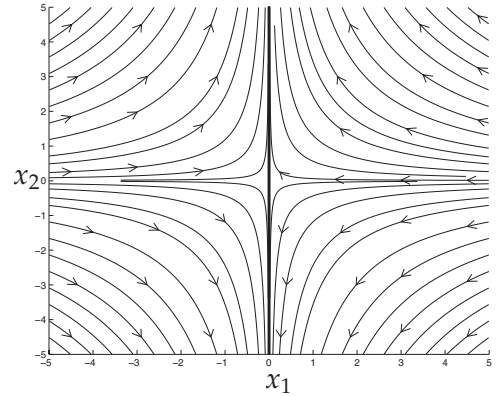
Example 2.5. Consider the nonlinear system

$$\dot{x}_1 = x_2(x_2)^2$$
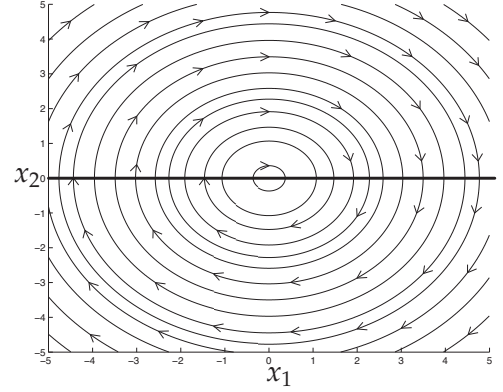$$\dot{x}_2 = -x_1(x_2)^2$$

whose phase portrait is depicted on the side. The x_1 axis $\{(x_1, x_2) : x_2 = 0\}$ is a continuum of equilibria. The differential equation can be rewritten as follows:

$$\dot{x} = x_2^2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x,$$

and we see that this is the vector field of a harmonic oscillator scaled by the nonnegative function x_2^2. We conclude that phase curves are half-circles delimited by the x_1 axis, from which it follows that the x_1 axis is globally attractive but unstable. Indeed, there are initial conditions arbitrarily close to the x_1 axis which give rise to semicircles with arbitrarily large radius, implying that orbits travel far away from the x_1 axis before returning there.

Remark 2.6. In the definition of feasible control specification (Definition 1.1), we imposed that the set Γ embodying a specification be positively invariant. As a matter of fact, positive invariance of Γ is a necessary condition for Γ to be stable. Indeed, consider the definition of stability,

$$(\forall \epsilon > 0)(\exists \mathcal{N}(\Gamma)) \, \phi(\mathbb{R}_+, \mathcal{N}(\Gamma)) \subset B_\epsilon(\Gamma). \tag{2.2}$$

Suppose, by way of contradiction, that the property above holds but Γ is not positively invariant. Then there exists $\bar{x} \in \Gamma$ and $\bar{t} > 0$ such that $\|\phi(\bar{t}, \bar{x})\|_\Gamma = \mu > 0$. Let $\epsilon = \mu/2$. Then for any neighbourhood $\mathcal{N}(\Gamma)$ of Γ, we have $\bar{x} \in \mathcal{N}(\Gamma)$ and $\phi(\bar{t}, \bar{x}) \notin B_\epsilon(\Gamma)$, violating the stability property (2.2). This gives a contradiction and proves that positive invariance of Γ is necessary for (2.2) to hold.

Definition 2.7 (Relative set stability and attractivity). Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be closed positively invariant sets.

(i) Γ_1 is **stable relative to** Γ_2 for Σ if, for any $\varepsilon > 0$, there exists a neighbourhood of Γ_1, $\mathcal{N}(\Gamma_1)$, such that $\phi(\mathbb{R}_+, \mathcal{N}(\Gamma_1) \cap \Gamma_2) \subset B_\varepsilon(\Gamma_1)$.

(ii) Γ_1 is **attractive relative to** Γ_2 for Σ if there exists a neighbourhood of Γ_1, $\mathcal{N}(\Gamma_1)$, such that for all $x_0 \in \mathcal{N}(\Gamma_1) \cap \Gamma_2$, $\lim_{t \to \infty} \|\phi(t, x_0)\|_{\Gamma_1} = 0$. Γ_1 is **globally attractive relative to** Γ_2 if for all $x_0 \in \Gamma_2$, $\lim_{t \to \infty} \|\phi(t, x_0)\|_{\Gamma_1} = 0$.

(iii) Γ_1 is **[globally] asymptotically stable relative to** Γ_2 if it is stable and [globally] attractive relative to Γ_2.

Example 2.8. In the LTI setting, the relative stability properties of invariant subspaces can be easily assessed using an adaptation of the technique presented in Example 2.4. Consider an LTI system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n,$$

and let $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathbb{R}^n$ be two A-invariant subspaces. Select a basis $\{v^1, \ldots, v^n\}$ for $\mathbb{R}^n$ in such a way that $\{v^1, \ldots, v^{k_1}\}$ is a basis for $\mathcal{V}_1$, and $\{v^1, \ldots, v^{k_2}\}$, $k_2 > k_1$, is a basis for $\mathcal{V}_2$. Let T be the matrix whose columns are the basis vectors v^i, $T = \begin{bmatrix} v^1 & \cdots & v^n \end{bmatrix}$, and consider the isomorphism

$$(x^1, x^2, x^3) = T^{-1}x.$$

where $\dim(x^1) = k_1$ and $\dim(x^2) = k_2 - k_1$. The dynamics in new coordinates read as

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix},$$

and the subspaces become

$$T^{-1}\mathcal{V}_1 = \{(x^1, x^2, x^3) : x^2 = 0, x^3 = 0\},$$
$$T^{-1}\mathcal{V}_2 = \{(x^1, x^2, x^3) : x^3 = 0\}.$$

We see from this decomposition that $\mathcal{V}_1$ is globally attractive or globally asymptotically stable relative to $\mathcal{V}_2$ if and only if the eigenvalues of A_{22} are in $\mathbb{C}^-$. Moreover $\mathcal{V}_1$ is stable relative to $\mathcal{V}_2$ if and only if the subsystem $\dot{x}^2 = A_{22}x^2$ is stable.

Example 2.9. Consider the system in polar coordinates (r, θ)

$$\dot{r} = r(r-1)$$
$$\dot{\theta} = \sin^2(\theta/2).$$

The equation for r has two equilibria $r = 0$ and $r = 1$. If $r > 1$ then $\dot{r} > 0$, so solutions with initial conditions outside of the unit disk remain outside of the disk and move away from it. If $0 < r < 1$, then $\dot{r} < 0$, so solutions with nonzero initial conditions inside the unit disk asymptotically move away from the unit circle and converge to the origin. All solutions with $r(0) = 1$ remain on the unit circle - the unit circle is an invariant set. The equation for θ has equilibria at 0 modulo 2π and satisfies $\dot{\theta} \geq 0$, implying that if the angle is initialized at 0 modulo 2π, then it remains constant; otherwise, the angle increases and asymptotically approaches 0 modulo 2π. These considerations justify the phase portrait on the right-hand side.
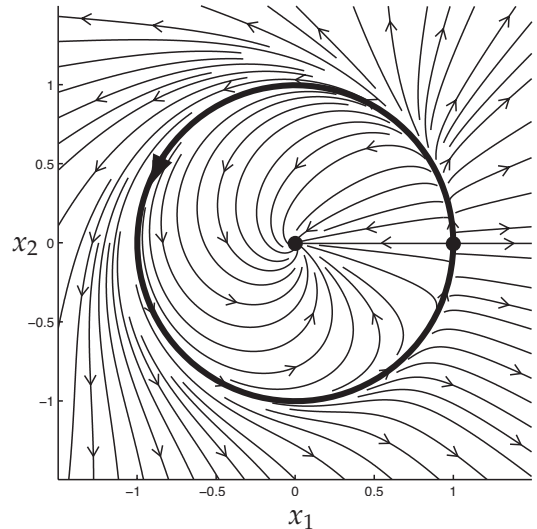
Denote the equilibrium $(x_1, x_2) = (1, 0)$ by Γ_1, and let Γ_2 be the unit circle. Then Γ_1 is globally attractive and unstable relative to Γ_2, and Γ_2 is unstable. To see that Γ_1 is unstable relative to Γ_2, pick an initial condition on the circle arbitrarily close to Γ_1, and lying on the upper-half plane $\{x_2 > 0\}$. The resulting

solution will move away from the equilibrium, and travel once counterclockwise around the circle before returning to the equilibrium.

Definition 2.10 (Local stability and attractivity). Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be closed sets that are positively invariant for Σ.

(i) The set Γ_2 is **locally stable near** Γ_1 for Σ if for all $x \in \Gamma_1$, for all $c > 0$, and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma_1)$ and all $t > 0$, whenever $\phi([0, t], x_0) \subset B_c(x)$ one has that $\phi([0, t], x_0) \subset B_\varepsilon(\Gamma_2)$.

(ii) The set Γ_2 is **locally attractive near** Γ_1 for Σ if there exists a neighbourhood of Γ_1, $\mathcal{N}(\Gamma_1)$, such that, for all $x_0 \in \mathcal{N}(\Gamma_1)$, $\phi(t, x_0) \to \Gamma_2$ at $t \to +\infty$.
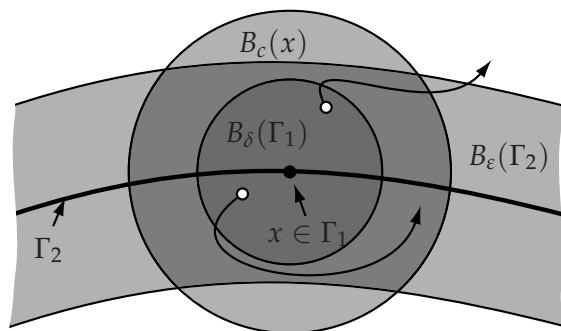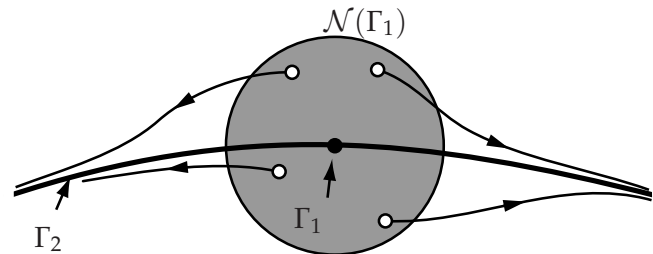
Local stability   Local attractivity

Referring to the figure above, the definition of local stability can be rephrased as follows. Given an arbitrary ball $B_c(x)$ centred at a point x in Γ_1, trajectories originating in $B_c(x)$ sufficiently close to Γ_1 cannot travel far away from Γ_2 before first exiting $B_c(x)$. It is immediate to see that if Γ_1 is stable, then Γ_2 is locally stable near Γ_1, and therefore local stability of Γ_2 near Γ_1 is a necessary condition for the stability of Γ_1.

Example 2.11. Consider the system

$$\dot{x}_1 = -x_1(1 - x_2^2)$$

$$\dot{x}_2 = x_2,$$

let $\Gamma_1 = \{0\}$ be the origin, and let Γ_2 be the x_2 axis. We claim that Γ_2 is unstable, but it is locally stable near Γ_1. Indeed, if $x_2(0) \neq 0$, then $x_2(t) \to \infty$, so that eventually $\dot{x}_1(t)$ will have the same sign as $x_1(t)$ and tend to infinity. Thus Γ_2 is unstable. On the other hand, for any ball $B_c(0)$, let $\epsilon > 0$ be arbitrary. We see from the phase portrait that we can find $\delta > 0$ such that solutions originating in $B_\delta(\Gamma_1)$ do not exit $B_\epsilon(\Gamma_2)$ as long as they remain in $B_c(0)$. Thus, Γ_2 is locally stable near Γ_1.
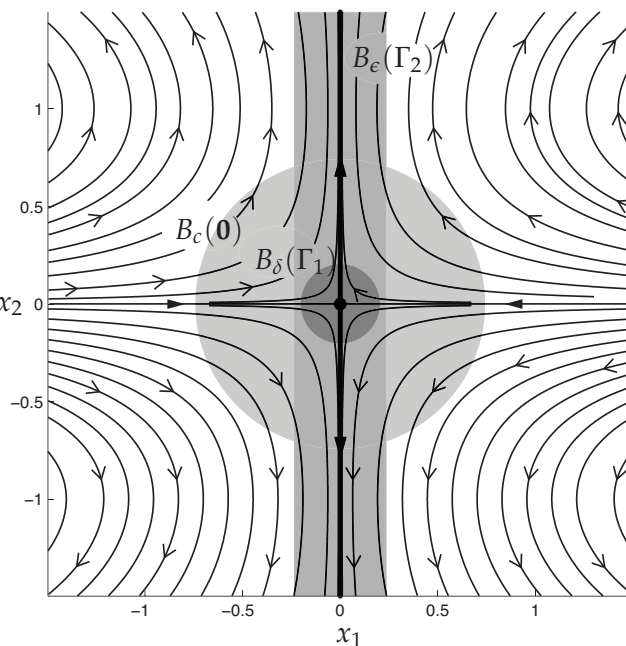
2.2   THE REDUCTION PROBLEM

At the heart of the statement of HCP in Section 1.4 is the following question. Consider two positively invariant sets $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$. If Γ_1 is asymptotically stable relative to Γ_2, and if Γ_2 is asymptotically stable, what conditions (if any) are needed to guarantee that Γ_1 is asymptotically stable? In this section, we will address a generalized version of this problem, investigating not just the property of asymptotic stability, but also stability and attractivity.

Problem 2.12 (Reduction Problem). Consider two closed sets $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$ which are positively invariant for a locally Lipschitz differential equation $\dot{x} = f(x)$. Suppose that Γ_1 is either stable, attractive, or asymptotically stable relative to Γ_2. Find conditions under which Γ_1 enjoys the same property relative to $\mathcal{X}$.

This problem was first formulated by P. Seibert in [Sei69, Sei70]. The biographical notes at the end of this chapter contain more information.

In the next examples we illustrate some of the challenges of the reduction problem.

Example 2.13. (Relative attractivity is a fragile property). Consider the differential equation

$$\dot{x}_1 = (x_2^2 + x_3^2)(-x_2)$$
$$\dot{x}_2 = (x_2^2 + x_3^2)(x_1)$$
$$\dot{x}_3 = -x_3^3,$$

and define $\Gamma_1 = \{x \in \mathbb{R}^3 : x_2 = x_3 = 0\}$, $\Gamma_2 = \{x \in \mathbb{R}^3 : x_3 = 0\}$. We see that Γ_2 is globally asymptotically stable because the motion off Γ_2 is described by the autonomous differential equation $\dot{x}_3 = -x_3^3$, whose equilibrium $x_3 = 0$ is globally asymptotically stable. The motion on Γ_2 is described by the subsystem

$$\dot{x}_1 = (x_2^2)(-x_2)$$
$$\dot{x}_2 = (x_2^2)(x_1)$$

which describes a harmonic oscillator scaled by the nonnegative function x_2^2, a system we encountered in Example 2.5.

Thus, the set Γ_1 is globally attractive relative to Γ_2. On the other hand, Γ_2 is not attractive relative to $\mathbb{R}^3$, as illustrated in the figure on the side. This example illustrates that relative attractivity is a fragile property that cannot be extended to the whole state space even though Γ_2 is globally asymptotically stable. A reduction theorem for attractivity must require stronger assumptions.
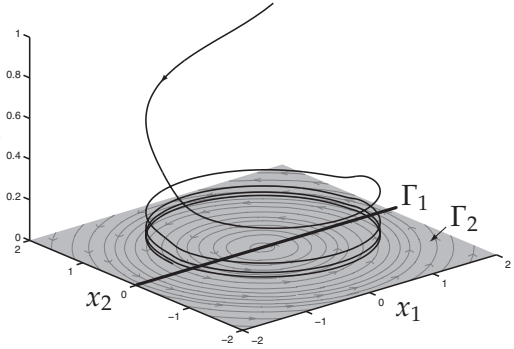
Example 2.14. (Relative stability is a fragile property). Consider the linear time-invariant system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0.$$

Let $\Gamma_1 = \{0\}$ and $\Gamma_2 = \{x_2 = 0\}$. The set Γ_2 is stable. On it, the motion is described by $\dot{x}_1 = 0$, therefore Γ_1 is stable relative to Γ_2. Yet, Γ_1 is unstable. This example illustrates that the stability of Γ_2 is not enough to guarantee that the stability of Γ_1 relative to Γ_2 can be extended outside of Γ_2.

We have seen in the previous example that relative stability is a fragile property even in the LTI case. In the next example, we show that relative attractivity or asymptotic stability of invariant subspaces (two equivalent properties) are not fragile.

Example 2.15. Consider an LTI system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n,$$

and let $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathbb{R}^n$ be two A-invariant subspaces. We have the following result.

Proposition 2.16. *If $\mathcal{V}_1$ is asymptotically stable relative to $\mathcal{V}_2$ and $\mathcal{V}_2$ is asymptotically stable, then $\mathcal{V}_1$ is asymptotically stable.*

Proof. Consider the decomposition of Example 2.8,

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix},$$

$$T^{-1}\mathcal{V}_1 = \{(x^1, x^2, x^3) : x^2 = x^3 = \mathbf{0}\},$$
$$T^{-1}\mathcal{V}_2 = \{(x^1, x^2, x^3) : x^3 = \mathbf{0}\}.$$

The assumption that $\mathcal{V}_1$ is asymptotically stable relative to $\mathcal{V}_2$ implies that $\sigma(A_{22}) \subset \mathbb{C}^-$, and the asymptotic stability of $\mathcal{V}_2$ implies that $\sigma(A_{33}) \subset \mathbb{C}^-$. Thus,

$$\sigma\left(\begin{bmatrix} A_{22} & A_{23} \\ \mathbf{0} & A_{33} \end{bmatrix}\right) \subset \mathbb{C}^-,$$

implying that $\mathcal{V}_1$ is asymptotically stable.   □

2.3   Reduction Theorems

In this section we present reduction theorems for stability, attractivity, and asymptotic stability. The proofs are found in [EHM13b] and are not included in these notes. Consider again the locally Lipschitz differential equation

$$\Sigma : \dot{x} = f(x), \quad x \in \mathcal{X}.$$

Theorem 2.17 (Stability)**.** *Let $\Gamma_1 \subset \Gamma_2$ be two closed positively invariant subsets of $\mathcal{X}$, and assume that Γ_1 is compact. Then, Γ_1 is stable if the following conditions hold:*

  (i) *Γ_1 is asymptotically stable relative to Γ_2,*

  (ii) *Γ_2 is locally stable near Γ_1.*

Since the stability of Γ_1 implies the local stability of Γ_2 near Γ_1, we have the following corollary.

Corollary 2.18. *Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed positively invariant sets, and assume that Γ_1 is compact. If Γ_1 is asymptotically stable relative to Γ_2 and Γ_2 is stable, then Γ_1 is stable.*

Theorem 2.19 (Attractivity)**.** *Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed positively invariant sets. Then, Γ_1 is attractive if the following conditions hold:*

  (i) *Γ_1 is asymptotically stable relative to Γ_2,*

  (ii) *Γ_2 is locally attractive near Γ_1,*

  (iii) *there exists a neighbourhood $\mathcal{N}(\Gamma_1)$ such that, for all initial conditions in $\mathcal{N}(\Gamma_1)$, the associated solutions are bounded and such that the set $\mathrm{cl}(\phi(\mathbb{R}_+, \mathcal{N}(\Gamma_1))) \cap \Gamma_2$ is contained in the domain of attraction of Γ_1 relative to Γ_2.*

The set Γ_1 is globally attractive if:

  (i)' *Γ_1 is globally asymptotically stable relative to Γ_2,*

(ii)′ Γ_2 *is a global attractor,*

(iii)′ all trajectories in $\mathcal{X}$ *are bounded.*

Theorem 2.20 (Asymptotic stability)**.** *Let* Γ_1 *and* Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, *be two closed positively invariant sets, and assume* Γ_1 *is compact. Then,* Γ_1 *is [globally] asymptotically stable if the following conditions hold:*

  (i) Γ_1 *is [globally] asymptotically stable relative to* Γ_2,

  (ii) Γ_2 *is locally stable near* Γ_1,

  (iii) Γ_2 *is locally attractive near* Γ_1 *[*Γ_2 *is globally attractive],*

  (iv) *[all trajectories of* Σ *are bounded.]*

Similarly to before, we have the following corollary.

Corollary 2.21. *Let* Γ_1 *and* Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, *be two closed positively invariant sets, and assume that* Γ_1 *is compact. If* Γ_1 *is asymptotically stable relative to* Γ_2 *and* Γ_2 *is asymptotically stable, then* Γ_1 *is asymptotically stable. Moreover, if these assumptions hold globally and all trajectories of* Σ *are bounded, then* Γ_1 *is globally asymptotically stable.*

The reduction theorems for stability and asymptotic stability (Theorems 2.17, 2.20) and their corollaries presented above rely on the assumption that Γ_1 is compact. When Γ_1 is unbounded, the conditions of those theorems are no longer sufficient, an extra condition is needed.

Definition 2.22 (Local uniform boundedness (LUB))**.** System Σ is *locally uniformly bounded near* Γ *(LUB)* if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}_+, B_\lambda(x)) \subset B_m(x)$.

Theorem 2.23. *Let* Γ_1 *and* Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, *be two closed positively invariant sets. If* Γ_1 *is unbounded, the results of Theorems 2.17 and 2.20, as well as Corollaries 2.18 and 2.21, hold provided that* Σ *is LUB near* Γ_1.

2.4 CASCADE-CONNECTED SYSTEMS

Before using the reduction theorem for asymptotic stability to solve the hierarchical control problem, we consider a special case of interest: the stability/stabilization of cascade-connected systems. Cascade-connected systems have received much attention in nonlinear control theory ever since researchers discovered (see [Isi95]) that smooth single-input single-output control-affine systems with input $v \in \mathbb{R}$ and output $s \in \mathbb{R}$

$$\dot{z} = f_1(z) + f_2(z)v$$
$$s = h(z)$$

can be transformed, in a neighborhood of a point at which the system has well-defined relative degree, in the *normal form*

$$\dot{x} = f(x, y)$$
$$\dot{y} = Ay + Bu$$

where (A, B) is a controllable pair. Suppose we design a feedback $u = Ky$ stabilizing the linear subsystem $\dot{y} = Ay + Bu$, so that the closed-loop system

$$\dot{x} = f(x, y)$$
$$\dot{y} = (A + BK)y$$

is the cascade connection of a stable LTI system and a nonlinear system. Assuming that $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, the problem is to find conditions under which the equilibrium $(x, y) = (\mathbf{0}, \mathbf{0})$ is (globally) asymptotically stable for the above cascade.

More generally, consider the locally Lipschitz cascade-connected system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(y)$$

(2.3)

in which $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $g(\mathbf{0}) = \mathbf{0}$, and assume that the equilibrium $y = \mathbf{0}$ is (globally) asymptotically stable for the subsystem $\dot{y} = g(y)$. Under what conditions is it true that $(x, y) = (\mathbf{0}, \mathbf{0})$ is (globally) asymptotically stable for the overall system? This question is a special case of the reduction problem presented in Section 2.2. Indeed, if we define two nested sets $\Gamma_1 \subset \Gamma_2$ as

$$\Gamma_1 = \{(x, y) = (\mathbf{0}, \mathbf{0})\}$$
$$\Gamma_2 = \{(x, y) : y = \mathbf{0}\},$$

then the following is true:

(i) Assuming that $y = \mathbf{0}$ is an asymptotically stable equilibrium of $\dot{y} = g(y)$ corresponds to assuming that the set Γ_2 is asymptotically stable. (This is true provided the cascade-connected system has no finite escape times).

(ii) Assuming that $x = \mathbf{0}$ is an asymptotically stable equilibrium of the system $\dot{x} = f(x, \mathbf{0})$ corresponds to assuming that Γ_1 is asymptotically stable relative to Γ_2.

We therefore have the following corollary of Theorem 2.20.

Corollary 2.24. *Consider the locally Lipschitz cascade-connected system* (2.3) *and assume that* $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, $g(\mathbf{0}) = \mathbf{0}$. *If, and only if,* $x = \mathbf{0}$ *is asymptotically stable for* $\dot{x} = f(x, \mathbf{0})$ *and* $y = \mathbf{0}$ *is asymptotically stable for* $\dot{y} = g(y)$, *then* $(x, y) = (\mathbf{0}, \mathbf{0})$ *is asymptotically stable for* (2.3). *Furthermore, if the assumptions hold globally, then* $(x, y) = (\mathbf{0}, \mathbf{0})$ *is globally asymptotically stable for* (2.3) *provided that all solutions of* (2.3) *are bounded.*

2.5   SOLUTION OF HCP

In this section we solve the hierarchical control problem stated in Section 1.4. We consider again the locally Lipschitz control system

$$\dot{x} = f(x, u), \quad x \in \mathcal{X}. \tag{2.4}$$

Theorem 2.25. *Consider a hierarchy of feasible control specifications* SPEC 1 $\preceq \cdots \preceq$ SPEC n *for system* (2.4), *with corresponding nested sets* $\Gamma_1 \subset \cdots \subset \Gamma_n$, *and assume that* Γ_1 *is compact. Then, any locally hierarchical feedback asymptotically stabilizes* Γ_1. *Moreover, any globally hierarchical feedback globally asymptotically stabilizes* Γ_1 *provided that all solutions of the closed-loop system are bounded.*

When Γ_1 is unbounded, we have the following result.

Theorem 2.26. *If* Γ_1 *is a closed unbounded set, then the result of Theorem* 2.25 *holds provided that the closed-loop system is LUB near* Γ_1.

In the next chapter we will leverage these results to solve the three problems introduced in Chapter 1: the position control problem for thrust-propelled underactuated vehicles, the circular formation stabilization problem for kinematic unicycles, and the design of backstepping controllers without Lyapunov functions.

2.6   BIOGRAPHICAL NOTES

The notions of stability presented in Definition 2.3 are taken from [BS70]. If the set Γ is not compact, the notions presented in Definition 2.3 are weaker than notions of uniform set stability and uniform attractivity commonly used in the Lyapunov-based literature, e.g., [LSW96]. They are equivalent when Γ is compact.

The notions of relative stability of Definition 2.7 are taken from [SF95]. The notions of local stability near a set of Definition 2.10 are taken from [EHM13b]. The local stability definition used here and in [EHM13b] differs from the one used in [SF95], but yields equivalent assumptions in the reduction theorems for compact sets. The notion of local stability in [SF95] cannot be used for unbounded sets while the one used here can (Theorem 2.23).

The reduction problem of Section 2.2 was formulated by P. Seibert in [Sei69], [Sei70]. The reduction theorems for stability and asymptotic stability of compact sets (Theorems 2.17 and 2.20) were proved by Seibert and Florio in [SF95]. See also work by B.S. Kalitin [Kal99] and co-workers [IKO96]. Their extension to non-compact sets (Theorem 2.23) as well as the reduction theorem for attractivity (Theorem 2.19) were proved in [EHM13b]. This latter paper also presents the application to HCP and the stability of cascade-connected systems (Corollary 2.21 and Theorems 2.25, 2.26).

The literature on stability of equilibria of cascade-connected systems is vast. Vidyasagar [Vid80] showed that, for local asymptotic stability, it is necessary and sufficient that $x = \mathbf{0}$ be an asymptotically stable

equilibrium of the system $\dot{x} = f(x, \mathbf{0})$. As illustrated in Section 2.4, this result is actually a corollary of the reduction theorem 2.20. Vidyasagar's work was followed by research aimed at establishing global results, e.g., [SS90, SKS90, CTP95, JSK96, MP96, PL98, Isi99, PL01, Cha08]. The global asymptotic stability result of Corollary 2.24 relies on the assumption that all solutions of the cascaded system (2.3) are bounded. The above cited literature provides sufficient conditions guaranteeing that this assumption holds. Sontag, in [Son89], used a property of converging input bounded state (CIBS) stability. In the context of time-varying cascades, Panteley and Loria [PL98, PL01] proved global uniform stability of equilibria using Lyapunov-type conditions and growth rate conditions. In terms of control design, several results addressed the global stabilization problem for cascade systems, see [CTP95]. Several of these results present growth rate conditions, see for instance [SKS90], [JSK96], [MP96].

Chapter **3**

Applications

In this chapter we return to the design problems presented in Chapter 1 and solve them using the theory of Chapter 2.

3.1 Position Control of a Thrust-Propelled Vehicle

Recall the model of the thrust-propelled vehicle of Section 1.1:

$$
\begin{aligned}
\dot{x} &= v \\
m\dot{v} &= mg\mathbf{e}_3 - uR\mathbf{e}_3 \\
&= mg\mathbf{e}_3 + T.
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\dot{R} &= R\omega^\times \\
J\dot{\omega} + \omega \times J\omega &= \tau,
\end{aligned}
\tag{3.2}
$$

We will review the three-step design of Section 1.1 filling in some of the missing details. We will prove that the proposed design yields almost global asymptotic stability.

Position Control Design Following the design of Section 1.1, we consider first the translational subsystem

$$
\begin{aligned}
\dot{x} &= v \\
m\dot{v} &= mg\mathbf{e}_3 + T.
\end{aligned}
\tag{3.3}
$$

in which we view T as a control input. We seek a feedback $T_d(x,v)$ that globally asymptotically stabilizes the equilibrium $(x,v) = (x^\star, \mathbf{0})$ and is such that $\sup(\mathbf{e}_3^\top T_d(x,v)) < 0$. There are many possible choices, but here we will use the nested saturation feedback of A. Teel [Tee92, Tee96] (see also [RB01]):

$$
T_d(x,v) = -mge_3 - m\sigma\left(K_1 v + \lambda\sigma\left(\frac{K_2\tilde{x} + K_3 v}{\lambda}\right)\right)
\tag{3.4}
$$

where $\tilde{x} = x - x^\star$, K_1, K_2, K_3 are positive definite diagonal matrices, λ is a positive scalar, and $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth saturation function satisfying

(i) $\sigma(s) = (\sigma_1(s_1), \sigma_2(s_2), \sigma_3(s_3))$, $s = (s_1, s_2, s_3)$;

(ii) $s_i \sigma_i(s_i) > 0$ for all $s_i \neq 0$, $i = 1, 2, 3$;

(iii) $\dot{\sigma}(0) = I_3$;

(iv) $|\sigma_i(\cdot)| \leq M_i$ for some $M_i > 0$, $i = 1, 2, 3$, with $M_3 < g$.

In particular, we may choose $\sigma_i(s_i) = M_i \tanh\left(M_i^{-1} s_i\right)$. We have the following result for the feedback (3.4).

Proposition 3.1. *Consider the translational dynamics* (3.3) *with feedback* $T = T_d(x, v)$ *given in* (3.4). *The following are true:*

(i) There exist positive definite diagonal matrices K_1, K_2, K_3 *and a positive scalar* λ *such that the equilibrium* $(x, v) = (x^\star, \mathbf{0})$ *is globally asymptotically stable.*

(ii) There exists $\epsilon > 0$ *such that for any piecewise continuous function* $\rho : \mathbb{R} \to \mathbb{R}^3$ *such that* $\sup \|\rho\| < \epsilon$ *and* $\rho(t) \to \mathbf{0}$ *as* $t \to \infty$, *letting* $T = T_d(x, v) + \rho(t)$ *in* (3.1), *all solutions of* (3.1) *are bounded.*

(iii) The function T_d *satisfies* $\sup(\mathbf{e}_3^\top T_d(x, v)) < 0$ *and* $\sup \|T_d(x, v)\| < \infty$.

Proof. Properties (i) and (ii) follow from Theorem 5 in [Tee96]. Property (iii) is a direct consequence of the bounds chosen on the saturation functions. □

We remark that property (ii) of the nested saturation feedback is a form of converging input bounded state stability. It guarantees that small vanishing input disturbances do not produce unbounded solutions $(x(t), v(t))$.

Attitude extraction. As done in Section 1.1, we pick a unit vector v and then define

$$\mathbf{R}(T_d) := \begin{bmatrix} b_{1d} & b_{2d} & b_{3d} \end{bmatrix}, \text{ where}$$

$$b_{1d} = \frac{T_d}{\|T_d\|} \times v, \quad b_{2d} = -\frac{T_d}{\|T_d\|} \times b_{1d}, \quad b_{3d} = -\frac{T_d}{\|T_d\|}. \tag{3.5}$$

By definition, the columns of $\mathbf{R}$ have unit norm and are mutually orthogonal. Moreover, the third column is the vector product of the first and second columns. Therefore $\mathbf{R}$ maps into $SO(3)$. It is also clear that the map $\mathbf{R} : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to SO(3)$ in (3.5) is smooth and such that

$$\mathbf{R}(T_d)\mathbf{e}_3 = -\frac{T_d}{\|T_d\|}. \tag{3.6}$$

Attitude stabilization. Now we consider the attitude dynamics

$$\dot{R} = R\omega^\times$$
$$J\dot{\omega} + \omega \times J\omega = \tau, \tag{3.7}$$

and design a feedback that asymptotically stabilizes the equilibrium $(R, \omega) = (I_3, \mathbf{0})$.

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Proposition 3.2 (Theorem 1 of [CSM11])**.** *The feedback*

$$\tau = \tau_d(R, \omega) := -K_R \left(\sum_{i=1}^{3} a_i \mathbf{e}_i \times R\mathbf{e}_i \right) - K_\omega \omega \tag{3.8}$$

where K_R, K_ω are symmetric positive definite matrices, and a_1, a_2, a_3 are distinct positive numbers makes the equilibrium $(R, \omega) = (I_3, \mathbf{0})$ almost globally asymptotically stable for system (3.7).

Almost global position stabilizer.

The functions $T_d(x, v)$, $\mathbf{R}(T_d)$, and $\tau_d(R, \omega)$ are combined to form a position control feedback for system (3.1), (3.2) as follows:

$$
\begin{aligned}
u(\chi) &= \|T_d(x, v)\| \\
\tau(\chi) &= \tau_d(\tilde{\mathbf{R}}, \tilde{\omega}) - \tilde{\omega} \times J\tilde{\omega} + \omega \times J\omega - J\left(\tilde{\omega}^\times \tilde{\mathbf{R}}^{-1}\omega - \tilde{\mathbf{R}}^{-1}\dot{\omega}\right),
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
\tilde{\mathbf{R}}(\chi) &:= \mathbf{R}(T_d(x, v))^{-1} R \\
\omega(\chi) &:= \left[\mathbf{R}(T_d(x, v))^{-1} \dot{\mathbf{R}}(\chi) \right]_\times \\
\tilde{\omega}(\chi) &:= \omega - \tilde{\mathbf{R}}^{-1}(\chi)\omega(\chi).
\end{aligned}
$$

Theorem 3.3. *Consider system (3.1), (3.2) with the hierarchy of specifications $\Gamma_1 \subset \Gamma_2$,*

$$\Gamma_1 := \{\chi \in \Gamma_2 : x = x^\star, v = \mathbf{0}\}, \quad \Gamma_2 := \{\chi \in \mathcal{X} : R = \mathbf{R}(x, v), \, \omega = \tilde{\mathbf{R}}^{-1}(\chi(t))\omega(\chi)\}. \tag{3.10}$$

The controller (3.9) is an almost globally hierarchical feedback, and it almost globally asymptotically stabilizes the equilibrium $\chi = \chi^\star := (x^\star, \mathbf{0}, R^\star, \mathbf{0})$, where $R^\star = \mathbf{R}(-mg\mathbf{e}_3)$.

Proof. Consider the sets (3.10). We claim first that Γ_1 is the equilibrium $\chi^\star$. To this end, we need to show that, on Γ_1, $R = R^\star$ and $\omega = \mathbf{0}$. On Γ_1 we have $R = \mathbf{R}(x^\star, \mathbf{0}) = R^\star$, proving the first identity. For the second identity, we use the property of $\mathbf{R}$ in (3.6) to deduce that, for all $\chi \in \Gamma_1$,

$$m\dot{v} = mg\mathbf{e}_3 - \|T_d(x^\star, \mathbf{0})\|\mathbf{R}(T_d(x^\star, \mathbf{0}))\mathbf{e}_3 = mg\mathbf{e}_3 + T_d(x^\star, \mathbf{0}) = \mathbf{0}.$$

The latter identity follows from the fact that the feedback $T_d(x, v)$ renders $(x, v) = (x^\star, \mathbf{0})$ an equilibrium of the translational subsystem. Since $\dot{x} = \dot{v} = \mathbf{0}$ on Γ_1, using the chain rule we deduce that $\dot{\mathbf{R}}(\chi) = (d/dt)\mathbf{R}(x, v) = \mathbf{0}$, implying that $\omega(\chi) = \mathbf{0}$. In conclusion, on Γ_1 we have $\omega = \tilde{\mathbf{R}}^{-1}(\chi(t))\omega(\chi) = \mathbf{0}$. This concludes the proof that $\Gamma_1 = \{\chi^\star\}$.

In order to show that Γ_1 is almost globally asymptotically stable, we will show that Γ_2 is almost globally asymptotically stable, that Γ_1 is globally asymptotically stable relative to Γ_2, and then invoke Theorem 2.25.

One can check that the feedback $\tau(\chi)$ in (3.9) gives the following dynamics for the error variables $\tilde{\mathbf{R}}$ and $\tilde{\omega}$:

$$
\begin{aligned}
\dot{\tilde{\mathbf{R}}} &= \tilde{\mathbf{R}}\tilde{\omega}^\times \\
J\dot{\tilde{\omega}} + \tilde{\omega} \times J\tilde{\omega} &= \tau_d(\tilde{\mathbf{R}}, \tilde{\omega}).
\end{aligned}
$$

By Proposition 3.2, the equilibrium $(\tilde{\mathbf{R}}, \tilde{\omega}) = (I_3, \mathbf{0})$ is almost globally asymptotically stable, implying that Γ_2 is almost globally asymptotically stable *provided* that the closed-loop system given by (3.1), (3.2) and feedback (3.9) has no finite escape times. We will prove that this is the case, but for now let us assume this property holds. Then the domain of attraction of Γ_2 is a subset $\tilde{\mathcal{X}} \subset \mathcal{X}$ of full measure, so that Γ_2 is globally asymptotically stable relative to $\tilde{\mathcal{X}}$.

The set Γ_2 is invariant for the closed-loop system because $(\tilde{\mathbf{R}}, \tilde{\omega}) = (I_3, \mathbf{0})$ is an equilibrium. By the property (3.6) of the map $\mathbf{R}$, on Γ_2 we have $T = T_d(x, v)$. By Proposition 3.1, Γ_1 is globally asymptotically stable relative to Γ_2. This proves that the feedback (3.9) is almost globally hierarchical.

According to Theorem 2.25, we need to show that all solutions of the closed-loop system are bounded. We will first show that the closed-loop system has no finite escape times, so that Γ_2 is globally asymptotically stable relative to $\tilde{\mathcal{X}}$. Then we will show that all solutions are actually bounded.

Consider an arbitrary solution $\chi(t) \in \tilde{\mathcal{X}}$ of the closed-loop system. The dynamics of the translational subsystem (3.1) can be rewritten as

$$m\ddot{x} = mg\mathbf{e}_3 + T_d(x(t), \dot{x}(t)) + \left[-\|T_d(x(t), \dot{x}(t))\|R(t)\mathbf{e}_3 - T_d(x(t), \dot{x}(t)) \right].$$

The right-hand side of the above equation is bounded because T_d is bounded (property (iii) of Proposition 3.1) and $R(t)\mathbf{e}_3$ has unit norm. This implies that $(x(t), v(t))$ has no finite escape times. The smooth function $\omega(\chi(t))$ depends on $(x(t), v(t), R(t))$. Since $(x(t), v(t))$ and $R(t) \in \mathrm{SO}(3)$, a compact set, $\omega(\chi(t))$ is defined for all $t \geq 0$. Since $\tilde{\omega}(\chi(t)) = \omega(t) - \tilde{\mathbf{R}}^{-1}(\chi(t))\omega(\chi(t))$ is bounded, so is $\omega(t)$. In conclusion, the solution $\chi(t)$ has no finite escape times. Next we will show that $\chi(t)$ is actually bounded. To this end, consider again the $m\ddot{x}$ equation above. It is the perturbation of a system with a globally asymptotically stable equilibrium. The perturbation is the term $\left[-\|T_d(x(t), \dot{x}(t))\|R(t)\mathbf{e}_3 - T_d(x(t), \dot{x}(t)) \right]$ which is identically zero on Γ_2. Having proved that there are no finite escape times, we know that Γ_2 is globally asymptotically stable relative to $\tilde{\mathcal{X}}$, so that the foregoing perturbation vanishes asymptotically. By property (ii) of Proposition 3.1, the signal $(x(t), v(t))$ is bounded. This property implies that $\omega(\chi(t))$ is bounded. The boundedness of $\omega(\chi(t))$ and that of $\tilde{\omega}(\chi(t)) = \omega(t) - \tilde{\mathbf{R}}^{-1}(\chi(t))\omega(\chi(t))$ imply that $\omega(t)$ is bounded. Finally $R(t)$, being in $\mathrm{SO}(3)$, has unit norm columns. In conclusion, all solutions of the closed-loop system are bounded and, by Theorem 2.25, Γ_1 is globally asymptotically stable relative to $\tilde{\mathcal{X}}$ or, what is the same, almost globally asymptotically stable. $\square$

Remark 3.4. There are some practical benefits associated with the feedback (3.9). Consider again the

block diagram of the closed-loop system.
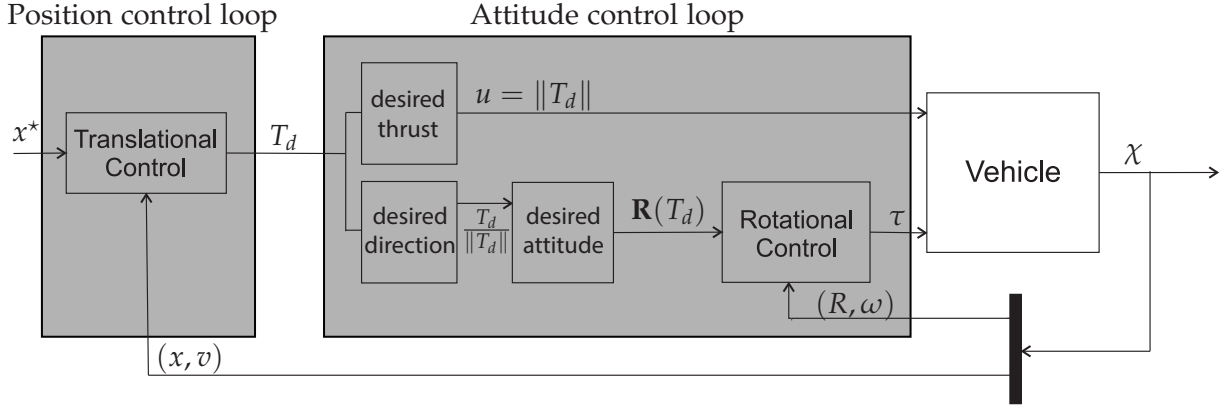
- The translational and rotational control modules (functions T_d and τ_d) are completely decoupled: the design of either one of them is independent of the design of the other.

- Being modular, the control design is intuitive and structured.

- One can replace either one of the control modules without having to redesign the remaining module. This allows one to easily test and compare different translational feedbacks: we could replace the nested saturation feedback by a wealth of available options.

- The combination of control modules in the final feedback (3.9) does not require the knowledge of Lyapunov functions, which in fact are not easy to obtain for the chosen controllers $T_d(x,v)$ and $\tau_d(R,\omega)$.

3.2 CIRCULAR FORMATION STABILIZATION FOR KINEMATIC VEHICLES

We return to the kinematic unicycles of Section 1.2,

$$
\begin{aligned}
\dot{x}_1^i &= u_1^i \cos x_3^i \\
\dot{x}_2^i &= u_1^i \sin x_3^i \qquad i \in \mathbf{n}, \\
\dot{x}_3^i &= u_2^i.
\end{aligned}
\tag{3.11}
$$

with state $\chi^i = (x_1^i, x_2^i, x_3^i)$, and collective state $\chi = (\chi^1, \ldots, \chi^n)$. Define, as before, a hierarchy of two specifications represented by sets $\Gamma_1 \subset \Gamma_2$,

$\Gamma_1 := \{\chi \in \Gamma_2 : x_3^{i+1} - x_3^i = \theta_i \bmod 2\pi, \ i = 1, \ldots, n-1\}$,　(a desired formation on a common circle)

$\Gamma_2 := \{\chi : c^1(\chi^1) = \cdots = c^n(\chi^n)\}$,　(vehicles on a common circle)

where

$$
c^i(\chi^i) = \begin{bmatrix} x_1^i - r \sin x_3^i \\ x_2^i + r \cos x_3^i \end{bmatrix}, \quad c(\chi) = [c^1(\chi^1)^\top \ \cdots \ c^n(\chi^n)^\top]^\top.
\tag{3.12}
$$

Our goal in this section is to design a hierarchical feedback consistent with the above specifications. We also put constraints on the information available for feedback to unicycle i.

Local and distributed feedback. As is customary in the literature, the information exchange between vehicles will be modeled by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ called the *sensor graph*. Here we will assume that $\mathcal{G}$ is *static and undirected*[1]. Each node of $v_i \in \mathcal{V}$ of $\mathcal{G}$ represents a vehicle. An edge in $\mathcal{E}$ from node i to node j of $\mathcal{G}$ signifies that vehicle i has access to the following variables:
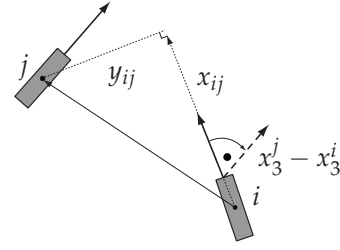
Sensor digraph $\mathcal{G}$

- The relative displacement of vehicle j with respect to vehicle i, measured in the local frame of vehicle i.

- The relative heading of vehicle j with respect to vehicle i.

Letting L denote the Laplacian of $\mathcal{G}$, we will use the notation L^i for the i-th row of L, and we will denote $L_{(2)} := L \otimes I_2$ (Kronecker product), $L^i_{(2)} := L^i \otimes I_2$.

Preliminary feedback transformation. Consider the feedback transformation

$$u^i_2 = \frac{u^i_1}{r} + \tilde{u}^i, \quad i \in \mathbf{n},$$

and denote $\tilde{u} = [\tilde{u}^1 \; \cdots \; \tilde{u}^n]^\top$. If $\tilde{u}^i = 0$ and $u^i_1 > 0$, the above control law makes the unicycle travel around a circle of radius r. The idea then is to use $\tilde{u}^i$, $i \in \mathbf{n}$, to stabilize a common circle (set Γ_2), and to use u^i_1 to make the unicycles assemble themselves in a formation around the circle (set Γ_1).

Stabilization of Γ_2. We wish to design $\tilde{u}^i$, $i \in \mathbf{n}$, to asymptotically stabilize the set

$$\Gamma_2 := \{\chi : c^1(\chi^1) = \cdots = c^n(\chi^n)\}.$$

We have that

$$\dot{c} = -rS(x_3)^\top \tilde{u},$$

where

$$S(x_3) = \text{blockdiag}\left\{[\cos x^1_3 \; \sin x^1_3], \cdots, [\cos x^n_3 \; \sin x^n_3]\right\}.$$

Since the graph $\mathcal{G}$ is undirected, its Laplacian matrix L is symmetric, and so is the matrix $L_{(2)}$. Consider the C^1 function

$$V(\chi) = \frac{1}{2}c(\chi)^\top L_{(2)} c(\chi),$$

and note that $V(\chi) = 0$ if and only if $c(\chi) \in \text{Ker}(L_{(2)})$. If the graph $\mathcal{G}$ is undirected, L has exactly one eigenvalue 0 with eigenvector $\mathbf{1}$. Therefore, $V(\chi) = 0$ if and only if $c^1(\chi) = \cdots = c^n(\chi)$, or $\chi \in \Gamma_2$. The time derivative of V is $\dot{V}(\chi) = -rc(\chi)^\top L_{(2)} S(x_3)^\top \tilde{u}$, which suggests defining $\tilde{u}(\chi) = KS(x_3)L_{(2)}c(\chi)$ to obtain $\dot{V} \leq 0$. We will do basically that, but we will use a saturation to avoid finite escape times and guarantee convergence.

---

[1]The assumption that $\mathcal{G}$ is undirected is made to simplify the proof that Γ_2 is globally asymptotically stable. The case of directed graphs is treated in [EHM13a]. As for the assumption that $\mathcal{G}$ is static, one may rightfully argue that it is nonsensical from a practical viewpoint. In practice, the graph $\mathcal{G}$ is *state-dependent* in that vehicles come in and out of each other's field of view as a function of their location. Even the most elementary coordination problems become formidably difficult when the sensor graph is state-dependent. We will therefore be content with the assumption that $\mathcal{G}$ is static.

Proposition 3.5. *Consider the kinematic unicycles in* (3.11) *and assume that the sensor graph $\mathcal{G}$ is connected. Let $u_1^i(\chi)$, $i \in \mathbf{n}$ be an arbitrary locally Lipschitz function such that $\sup_{\chi \in \mathcal{X}} |u_1^1(\chi)| < \infty$ and $\inf_{\chi \in \mathcal{X}} u_1^1(\chi) > 0$, and consider the feedback law*

$$
\begin{aligned}
u_1^i &= u_1^i(\chi) \\
u_2^i &= \frac{u_1^i(\chi)}{r} + K_i \phi_i(y_i) y_i, \text{ where } y = S(x_3) L_{(2)} c(\chi),
\end{aligned}
\qquad i \in \mathbf{n}, \qquad (3.13)
$$

where $\phi_i(y_i)$ is a locally Lipschitz function such that $\phi_i(y_i) > 0$ and $|\phi(y_i)y_i| < 1$. Then for any $K_i \in (0, (\inf u_1^i)/r)$, $i \in \mathbf{n}$, the set Γ_2 is globally asymptotically stable for the closed-loop system given by (3.11) *and* (3.13).

Proof. The assumption on $u_1^i(\chi)$ and the properties of the function $\phi_i(y_i)$ imply that there exist positive scalars μ_1, μ_2 such that $0 < \mu_1 < \dot{x}_3^i < \mu_2$ for all $i \in \mathbf{n}$. Moreover, the boundedness of u_1 and the fact that $x_3^i \in \mathbb{S}^1$, a compact set, imply that all solutions of (3.11) with feedback (3.13) are defined for all $t \geq 0$. This fact and the fact that $\dot{V} \leq 0$ imply that the set $\{\chi \in \mathcal{X} : V(\chi) = 0\}$ is stable for (3.11) with feedback (3.13). Let $\chi(0) \in \Gamma_1$ be arbitrary, and let $\chi(t)$ be the associated solution. One can easily see that the boundedness of $V(\chi(t))$ and the fact that $\dot{x}_3^i < \mu_2$ imply that $\ddot{V}(\chi(t))$ is bounded. By Barbalat's lemma, $y(t) = S(x_3(t)) L_{(2)} c(\chi(t)) \to 0$ as $t \to \infty$. In components, $y_i(t) = \begin{bmatrix} \cos x_3^i(t) & \sin x_3^i(t) \end{bmatrix} L_{(2)}^i c(\chi(t)) \to 0$. Thus, $L_{(2)}^i c(\chi(t)) \to a(t) \begin{bmatrix} -\sin x_3^i(t) & \cos x_3^i(t) \end{bmatrix}^\top$, where $a(t)$ is a continuous scalar function. Since the function $t \mapsto V(\chi(t))$ is continuous, bounded from below, and nonincreasing, it has a finite limit, implying that $L_{(2)} c^i(\chi(t))$ has a finite limit. Since $\dot{x}_3^i > \mu_1 > 0$, the only way that $a(t) \begin{bmatrix} -\sin x_3^i(t) & \cos x_3^i(t) \end{bmatrix}^\top$ may have a finite limit is that $a(t) \to 0$ as $t \to \infty$. This proves that for any $\chi(0) \in \mathcal{X}$, $L_{(2)} c(\chi(t)) \to \mathbf{0}$ as $t \to \infty$ or, what is the same, the set Γ_2 is globally attractive for (3.11), and hence also globally asymptotically stable. In conclusion, Γ_2 is globally asymptotically stable for (3.11) with feedback (3.13). $\qquad\square$

Stabilization of Γ_1 relative to Γ_2. Recall the definition of Γ_1,

$$
\Gamma_1 = \{\chi \in \Gamma_2 : x_3^{i+1} - x_3^i = \theta_i \bmod 2\pi, \ i = 1, \ldots, n-1\}.
$$

Define angles $\alpha_i \in \mathbb{S}^1$, $i \in \mathbf{n}$, as follows:

$$
\alpha_n = 0 \bmod 2\pi, \ \alpha_i = \sum_{j=i}^{n-1} \theta_j, \ i = 1, \ldots, n-1.
$$

Then, assuming the sensor graph is connected, Γ_1 can be equivalently expressed as

$$
\Gamma_1 = \{\chi \in \Gamma_2 : L(x_3 - \alpha) = \mathbf{0} \bmod 2\pi\},
$$

where $x_3 \in (\mathbb{S}^1)^n$ is the vector of unicycles' headings, $x_3 = [x_3^1 \ \cdots \ x_3^n]^\top$. To verify the equivalence of the two definitions of Γ_1, notice that $L(x_3 - \alpha) = \mathbf{0} \bmod 2\pi$ if and only if $x_3 - \alpha = \mathbf{1} \bmod 2\pi$, or component-wise, $x_3^1 - \alpha_1 = \cdots = x_3^n - \alpha_n \bmod 2\pi$, or $x_3^i - x_3^{i+1} = \alpha_i - \alpha_{i+1} \bmod 2\pi = \theta_i \bmod 2\pi$.

In order to write the dynamics of x_3^i on Γ_2, we notice that feedback u_2^i in (3.13) reduces to $u_2^i = u_1^i(\chi)/r$ on Γ_2. Therefore, denoting $u_1 := [u_1^1 \cdots u_1^n]^\top$ and letting $\tilde{x}_3 := x_3 - \alpha$, we have

$$
\dot{\tilde{x}}_3 = \frac{u_1^i}{r},
$$

and asymptotically stabilizing Γ_1 relative to Γ_2 is equivalent to designing u_1 stabilizing the set $\{\tilde{x}_3 \in (\mathbb{S}^1)^n : L\tilde{x}_3 = \mathbf{0} \bmod 2\pi\}$. A reader familiar with the distributed control literature will recognize this to be a problem of consensus on the n-torus, for which a well-known solution is the control law of the coupled Kuramoto oscillators with identical frequencies, $u_1^i = v - k_i \sum_{j \in N(i)} \sin(\tilde{x}_3^i - \tilde{x}_3^j)$. In the above, $N(i)$ denotes the set of nodes j in the sensor graph $\mathcal{G}$ such that there is an edge from node i to node j. In original coordinates, the feedback is given by

$$u_1^i = v - k_i \sum_{j \in N(i)} \sin(x_3^i - x_3^j + \alpha_j - \alpha_i). \tag{3.14}$$

Proposition 3.6 ([JMB04],[LFM07])**.** *Consider the rotational integrators $\dot{x}_3^i = u_1^i/r$, $i \in \mathbf{n}$, $x_3^i \in \mathbb{S}^1$, with feedback* (3.14)*. If the sensor graph is connected, for all $k_i > 0$, $i \in \mathbf{n}$, the set $\{x_3 \in (\mathbb{S}^1)^n : L(x_3 - \alpha) = \mathbf{0} \bmod 2\pi\}$ is asymptotically stable.*

We remark that if the controller parameters $k_i > 0$, $i \in \mathbf{n}$, are chosen sufficiently small, the feedback (3.14) meets the requirements of Proposition 3.5 (it is bounded away from zero and uniformly bounded from above), and it makes the set Γ_1 asymptotically stable relative to Γ_2.

Solution of the circular formation stabilization problem. We have designed the feedback

$$\begin{aligned} u_1^i(\chi) &= v - k_i \sum_{j \in N(i)} \sin(x_3^i - x_3^j + \alpha_j - \alpha_i) \\ u_2^i(\chi) &= \frac{u_1^i(\chi)}{r} + K_i \phi_i(y_i) y_i, \text{ where } y = S(x_3) L_{(2)} c(\chi), \end{aligned} \qquad i \in \mathbf{n}. \tag{3.15}$$

Theorem 3.7. *Consider the kinematic unicycles* (3.11) *with the hierarchy of specifications $\Gamma_1 \subset \Gamma_2$,*

$$\Gamma_1 := \{\chi \in \Gamma_2 : L(x_3 - \alpha) = \mathbf{0} \bmod 2\pi\}, \quad \Gamma_2 := \{\chi : c^1(\chi^1) = \cdots = c^n(\chi^n)\},$$

where $\alpha \in (\mathbb{S}^1)^n$ is a vector of angles specifying the ordering and spacing of the formation and $c^i(\chi^i)$ is defined in (3.12)*. If the sensor graph is undirected and connected, then for all $k_i \in (0, v/n)$ and $K_i \in (0, (\inf u_1^i)/r)$, $i \in \mathbf{n}$, the feedback* (3.15) *has the following properties*

   (i) *It is local and distributed.*

  (ii) *It is locally hierarchical.*

 (iii) *It globally asymptotically stabilizes Γ_2. In particular, for any initial condition all unicycles converge to a common stationary circle of radius r, whose centre depends on the initial condition, and travel counterclockwise along it.*

 (iv) *It asymptotically stabilizes Γ_1. In particular, if $L(x_3(0) - \alpha)$ is sufficiently close to $\mathbf{0}$, then for any initial position, the unicycles will converge to the desired formation on the circle.*

Some of the arguments in the proof below are technical and they are only outlined here. We refer the reader to [EHM13a] for the details.

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Proof. **Part (i).** The feedback $u_1^i(\chi)$ relies only on the relative heading angle of unicycle i with respect to its neighbors in the sensor graph. The feedback u_2^i relies also on the vector

$$
\begin{aligned}
y_i &= \begin{bmatrix} \cos x_3^i & \sin x_3^i \end{bmatrix} L_{(2)}^i c(\chi) \\
&= \begin{bmatrix} \cos x_3^i & \sin x_3^i \end{bmatrix} \sum_{j \in N(i)} c^i(\chi^i) - c^j(\chi^j) \\
&= \sum_{j \in N(i)} -x_{ij} + r \sin(x_3^j - x_3^i),
\end{aligned}
$$

where x_{ij} is the component along the heading axis of unicycle i of the relative displacement between unicycle i and unicycle j. This quantity is displayed in the figure of page 28. Thus, feedback (3.15) is both local and distributed.

Part (ii). By the choice of controller gains, the assumptions of Proposition 3.5 hold, so that Γ_2 is globally asymptotically stable. By Proposition 3.6, Γ_1 is asymptotically stable relative to Γ_2. According to Definition 1.3, the feedback (3.15) is locally hierarchical.

Part (iii). By Proposition 3.5, the feedback (3.15) globally asymptotically stabilizes Γ_1, and in particular $L_{(2)}c(\chi(t)) \to \mathbf{0}$ for any solution $\chi(t)$. One can show (this is done in Section V.B of [EHM13a]) that $L_{(2)}c(\chi(t))$ exponentially decreases to zero. We have $\|\dot{c}\| = \|rKc^\top L_{(2)}S(x_3)^\top \phi(y)S(x_3)L_{(2)}c\| \le C\|L_{(2)}c\|^2$ for a suitable constant $C > 0$. Since $L_{(2)}c(\chi(t)) \to 0$ exponentially, the above inequality implies that $c(\chi(t))$ has a finite limit. Combined with the fact that Γ_2 is globally asymptotically stable, we conclude that for any initial condition, there exists $\bar{c} \in \mathbb{R}^2$ such that $c^i(\chi(t)) \to \bar{c}$ for all $i \in \mathbf{n}$. Since $u_1^i > 0$ and since $u_2^i(\chi(t)) \to u_1^i/r$, each unicycle asymptotically moves counterclockwise around a circle of radius r of centre $c^i(\chi(t))$ and, as argued above, all centres converge to a fixed location $\bar{c}$ dependent upon the initial conditions.

Part (iv). The set Γ_1 is closed but unbounded because its definition doesn't pose any restriction on $c^i(\chi^i)$, and therefore on the positions of the unicycles. By Theorem 2.26, Γ_1 is asymptotically stable provided that the closed-loop system is LUB near Γ_1. The LUB property is proved in [EHM13a] using averaging. $\qquad\square$

3.3 BACKSTEPPING

We return to the class of lower-triangular control systems

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
&\vdots \\
\dot{x}_i &= f_i(x_1, \cdots, x_i) + g_i(x_1, \cdots, x_i)x_{i+1} \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \cdots, x_n) + g_n(x_1, \cdots, x_n)u,
\end{aligned}
\tag{3.16}
$$

In Section 1.3 we presented a backstepping control design resulting in the following feedback

$$\mu_1 = g_1^{-1}(-f_1 - K_1 x_1)$$
$$\mu_i = g_i^{-1}\Big(-f_i + \dot{\mu}_{i-1} - K_i(x_i - \mu_{i-1})\Big), \quad 2 = 1, \ldots, n-1 \qquad (3.17)$$
$$u = g_n^{-1}\Big(-f_n + \dot{\mu}_{n-1} - K_n(x_n - \mu_{n-1}).$$

Theorem 3.8. *Consider system (3.16), and assume that for each $i \in \mathbf{n}$, $f_i(\mathbf{0}) = 0$ and $g_i \neq 0$ is uniformly bounded. Consider the hierarchy of specifications $\Gamma_1 \subset \cdots \subset \Gamma_n$,*

$$\Gamma_1 = \{x = \mathbf{0}\}$$
$$\Gamma_i = \{(x_1, \ldots, x_n) : x_j = \mu_{j-1}(x_1, \ldots, x_{j-1}), j = i, \ldots, n\}, i \in \{2, \ldots, n\}.$$

For any $K_i > 0$, $i \in \mathbf{n}$, the feedback (3.17) is globally hierarchical and it globally asymptotically stabilizes Γ_1.

Proof. Assume for a moment that there are no finite escape times in the closed-loop system. Letting $e_i = x_{i+1} - \mu_i(x_1, \ldots, x_i)$, $i = 1, \ldots, n-1$, then $\Gamma_i = \{x : e_{i-1} = \cdots = e_{n-1} = 0\}$. The set Γ_n is globally asymptotically stable because $\dot{e}_{n-1} = -K_n e_{n-1}$. For $i = 1, \ldots, n-1$, Γ_i is globally asymptotically stable relative to Γ_{i+1} because, on it, $\dot{e}_{i-1} = -K_{i-1} e_{i-1}$. Therefore, feedback (3.17) is globally hierarchical. In light of Theorem 2.25, if all solutions are bounded then Γ_1 is globally asymptotically stable. Boundedness of all solutions in particular implies the absence of finite escape times, justifying our initial assumption. We prove the boundedness property inductively. Pick an arbitrary initial condition. First, $e_{n-1}(t)$ is bounded. Inductively, assume that $e_{i+1}(t)$ is bounded. It is readily checked that the dynamics of e_i is

$$\dot{e}_i = -K_{i+1} e_i + g_{i+1} e_{i+1}.$$

Since g_{i+1} is a uniformly bounded function, the boundedness of $e_{i+1}(t)$ implies that $e_i(t)$ is bounded. This shows that, for any initial conditions, the signals $e_1(t), \ldots, e_{n-1}(t)$ are bounded. Next, the dynamics of x_1 is $\dot{x}_1 = -K_1 x_1 + g_1 e_1$, and the boundedness of $e_1(t)$ implies that $x_1(t)$ is bounded. Finally, the boundedness of $(x_1(t), e_1(t), \ldots, e_{n-1}(t))$ implies that $x(t)$ is bounded. □

3.4 Biographical Notes

The material of Section 3.1 is taken from [RM14]. That paper leverages the reduction theorems to provided classes of translational and attitude controllers that can be combined to form almost global position stabilizers. In these notes we have presented one specific choice. Section 3.2 is deduced from [EHM13a], where the more general case of dynamic unicycles is considered, and the graph is allowed to be directed. Recently, M.I. El-Hawwary was able to extend the ideas presented in Section 3.2 to general rigid bodies in three-space, presenting results for fully actuated rigid bodies, as well as underactuated rigid bodies with degree of underactuation one or two, see [EH14]. Finally, Section 3.3 is taken from [EHM13b]. The results of this section are not particularly surprising, and certainly not groundbreaking. When compared with Lyapunov-based backstepping, the feedback in (3.17) is slightly less complex, see [EHM13b] for an illustration.

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