# Further Results on Transverse Feedback Linearization of Multi-Input Systems

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***Abstract*— In this paper we continue research on the transverse feedback linearization problem. In [1] we found sufficient conditions for this problem to be solvable. Here we present necessary and sufficient conditions for local transverse feedback linearization.**

## I. Introduction

The transverse feedback linearization problem (TFLP) was formulated in [2] by Banaszuk and Hauser. Given a controlled invariant manifold $\Gamma^\star$ embedded in the state space, this problem entails finding a coordinate and feedback transformation putting the dynamics transverse to $\Gamma^\star$ into linear controllable form. When feasible, transverse feedback linearization can simplify set stabilization problems.

Often, control objectives will dictate that the controller stabilize sets, rather than equilibria. In [3], for instance, the solution of a set stabilization problem is central to controlling bi-pedal locomotion. The "virtual constraints" technique in [4], used to stabilize oscillatory modes in Euler-Lagrange systems, relies on feedback linearization to stabilize an invariant set.

The work of Banaszuk and Hauser in [2] characterized the solution to TFLP for single-input systems when $\Gamma^\star$ is a diffeomorphic to the unit circle. In [5] we generalized Banaszuk and Hauser's results to the case when $\Gamma^\star$ is diffeomorphic to the generalized cylinder $\mathbb{R}^{n^\star-k}\times\mathbb{T}^k$, where $\mathbb{T}^k$ is the k-torus. In [1] we gave sufficient conditions to solve TFLP for multi-input systems. Theorem III.1 in [1], concerning the global solution to TFLP, contains a mistake. The theorem claims to give sufficient conditions for TFLP to be *globally* solvable under the assumption that $\Gamma^\star$ is a contractible set. Contractibility may not, in fact, be enough to guarantee the existence of a global solution to TFLP and hence Theorem III.1 in [1] must be considered a *local* result.

In this paper we further generalize the results of [1]. Our main result is Theorem V.1, which gives necessary and sufficient conditions for the existence of a local solution to TFLP. The global problem needs further investigation.

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## II. Notation and Problem Statement

Consider a control system modeled by equations of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i =: f(x) + g(x)u. \tag{1}$$

Here $x \in \mathbb{R}^n$ is the state, and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ is the control input. The vector fields $f, g_1, \ldots, g_m : \mathbb{R}^n \to T\mathbb{R}^n$ are smooth ($C^\infty$). We assume throughout this paper that $g_1, \ldots, g_m$ are linearly independent.

### A. Notation

If $k$ is a positive integer, $\mathbf{k}$ denotes the set of integers $\{0, 1, \ldots, k-1\}$. We let $\mathrm{col}(x_1, \ldots, x_k) := [x_1 \ \ldots \ x_n]^\top$ and, given two column vectors $a$ and $b$, we let $\mathrm{col}(a, b) := [a^\top \ b^\top]^\top$. Throughout this paper by a *manifold* is meant a *smooth manifold* and by a *submanifold* is meant an *embedded submanifold*. All objects are presumed to be smooth. On a manifold $M$, $V(M)$ will denote the set of all smooth vector fields on $M$ and $C^\infty(M)$ the ring of smooth real-valued functions on $M$. Given $v \in V(M)$, we denote by $\phi_t^v(x)$ an element of the local 1-parameter group of diffeomorphisms or flows generated by $v$ through the point $x \in M$ for sufficiently small $t$. Standard notations for Lie derivatives and Lie brackets are used which can be found in [6], [7]. Finally, we denote by $0_k$ the zero vector with $k$ elements and by $I_m$ the $m \times m$ identity matrix. Following [8], we denote the class of closed, connected embedded submanifolds of $\mathbb{R}^n$ which are controlled invariant for (1) by $\mathscr{I}(f, g, \mathbb{R}^n)$. If $N \in \mathscr{I}(f, g, \mathbb{R}^n)$, we write $\mathscr{F}(f, g, N)$ for the collection of "friends" of $N$, i.e. maps $\bar{u} : N \to \mathbb{R}^m$ such that $f + g\bar{u}$ is tangent to $N$, i.e.,

$$(f + g\bar{u})|_N : N \to TN.$$

Given a distribution $D$ on $\mathbb{R}^n$, we let $D^\perp$ be its orthogonal complement employing the natural orthogonal structure of $\mathbb{R}^n$. This stands in contrast to the notation $\mathrm{ann}(D)$ which we use to denote the annihilator of $D$. If $N \subset \mathbb{R}^n$ is a submanifold and $D$ a distribution on $\mathbb{R}^n$, by $TV + D$ is meant the vector bundle over $V$ defined fibre-wise by $T_pV + D(p)$; by $\mathrm{ann}(TV + D) \subset (T\mathbb{R}^n)^\star$ is meant the annihilator of $TV + D$ over $V$. The involutive closure of $D$ is denoted $\bar{D}$.

### B. Problem Statement

Suppose we are given a pair $(\Gamma^\star, u^\star)$, where $\Gamma^\star \in \mathscr{I}(f, g, \mathbb{R}^n)$, $\dim \Gamma^\star = n^\star$, and $u^\star \in \mathscr{F}(f, g, \Gamma^\star)$. In this paper we investigate the following.

**Local Transverse Feedback Linearization Problem (LTFLP):** *Given a point $p_0 \in \Gamma^\star$ and a pair $(\Gamma^\star, u^\star)$, find conditions which ensure that there exist a neighborhood $U$ of $p_0$ in $\mathbb{R}^n$, a local diffeomorphism $\Xi : U \to \Xi(U)$, and a regular static feedback $(\alpha, \beta)$ such that, letting $V := U \cap \Gamma^\star$, system* (1) *is feedback equivalent on $U$ to a system modeled by equations of the form*

$$\begin{aligned} \dot{z} &= f^0(z,\xi) + g^1(z,\xi)v_1 + g^2(z,\xi)v_2 \\ \dot{\xi} &= A\xi + Bv_1 \end{aligned} \tag{2}$$

*where $(z,\xi) \in \Xi(U) \subset \mathbb{R}^{n^\star} \times \mathbb{R}^{n-n^\star}$, $v = \mathrm{col}(v_1, v_2) \in \mathbb{R}^m$, $B$ is full rank, the pair $(A,B)$ is controllable and $\Xi|_V$ is the canonical immersion*

$$\begin{aligned} \Xi|_V : & V \to V \times \{0\} \\ & z \mapsto (z, 0). \end{aligned}$$

Under mild regularity conditions, necessary and sufficient conditions for the existence of a solution to LTFLP are presented in Theorem IV.3.

## III. Transverse Controllability Indices

In [1], we introduced the transverse controllability indices of (1) with respect to $\Gamma^\star$. For our present discussion we define, for $i = 0, 1, \ldots$, the following distributions associated with (1)

$$\begin{aligned} & G_i = \mathrm{span}\,\{ad_f^j g_k : 0 \le j \le i,\ 1 \le k \le m\}, \\ & \mathscr{G}_f = f + G_0 = \{f + g : g \in G_0\} \\ & [\Delta, \Lambda] = \{[X,Y] : X \in \Delta, Y \in \Lambda\}\ (\Delta, \Lambda \text{ distributions}), \\ & ad_f^i G_0 = \mathrm{span}\{ad_f^i X : X \in G_0\},\ i = 0, 1, \ldots, \\ & \mathscr{G}_i = \mathscr{G}_{i-1} + [\mathscr{G}_f, \mathscr{G}_{i-1}],\quad \mathscr{G}_0 = G_0,\ i = 1, 2, \ldots, \\ & \mathscr{S}_i = \bar{\mathscr{G}}_{i-1} + ad_f^i G_0,\quad \mathscr{S}_0 = G_0,\ i = 1, 2, \ldots. \end{aligned}$$

In [1] the distributions $G_i$ are assumed to be involutive in a neighborhood of $\Gamma^\star$. Here we omit this assumption and generalize our definition of transverse controllability indices.

For each $p \in \Gamma^\star$, $i \in \mathbb{N}$, define

$$\begin{aligned} \rho_0(p) &= \dim\left(T_p\Gamma^\star + G_0(p)\right) - n^\star \\ \rho_i(p) &= \dim\left(T_p\Gamma^\star + \bar{G}_{i-1}(p) + ad_f^i G_0(p)\right) \\ &\qquad - \dim\left(T_p\Gamma^\star + \bar{G}_{i-1}(p)\right). \end{aligned}$$

When the integers $\rho_i$ are constant over $\Gamma^\star$ (or an open subset of $\Gamma^\star$) we define a set of $\rho_0$ integers $k_1, \ldots, k_{\rho_0}$ which are the transverse controllability indices of (1) with respect to $\Gamma^\star$,

$$k_i := \mathrm{card}\,\{\rho_j \ge i, j \ge 0\},\quad i \in \boldsymbol{\rho_0}.$$

When the $G_i$'s are involutive, the definition of the integers $\rho_i$ agrees with that given in [1]. In the special case when $\Gamma^\star$ is an equilibrium point, it is useful to compare our definition of to that given by Marino in [9]. There, controllability indices are defined using the integers

$$\begin{aligned} r_0 &= \dim \mathscr{G}_0 \\ r_i &= \dim \mathscr{S}_i - \dim \bar{\mathscr{G}}_{i-1} \end{aligned}$$

in place of the integers $\rho_i$. The next result states that, when $T_p\Gamma^\star = \{0\}$, the integers $\rho_i$ and $r_i$ are identical.

**Lemma III.1** *For all non-negative integers $i$, $\bar{G}_i = \bar{\mathscr{G}}_i$. Thus, when $T_p\Gamma^\star = \{0\}$, $\rho_i = r_i$.*

The proof of Lemma III.1 is omitted for brevity. Inspired by [9], we conjecture that in cases when the dynamics of (1) transverse to $\Gamma^\star$ cannot be feedback linearized the transverse controllability indices characterize the largest transverse subsystem of (1) that can be feedback linearized.

## IV. Preliminary Results

Before proving the main results, we require some additional machinery which will generate a smooth feedback transformation which orders the vector fields of the distributions $G_i$ in a convenient way. We begin with a necessary condition for LTFLP to be solvable.

**Lemma IV.1** *Suppose that LTFLP is solvable at $p_0 \in \Gamma^\star$. Then there exists a neighborhood $U$ of $p_0$ in $\mathbb{R}^n$ such that, letting $V := \Gamma^\star \cap U$, we have that $(\forall p \in V)$ $(\forall i \in \mathbf{n} - \mathbf{n}^\star)$*

$$\dim(T_pV + \bar{G}_i(p)) = \dim(T_pV + G_i(p)) = \text{ const.},$$

*which implies that $\rho_0, \ldots, \rho_{n-n^\star-1}$ are constant on $V$.*

*Proof:* Since the stated condition is coordinate and feedback independent it suffices to show that the lemma holds in $(z,\xi)$ coordinates. By the properties of the normal form (2), for any $p \in V$, $\Xi(p) = (p, 0)$ so that in $(z,\xi)$ coordinates we have that for all $p \in V$

$$\begin{aligned} & T_pV + G_i(\mathrm{col}(p,0)) = \\ & \quad \mathrm{Im}\left(\begin{bmatrix} I_{n^\star} & \star & \star & \cdots & \star \\ 0_{n-n^\star \times n^\star} & B & AB & \cdots & A^iB \end{bmatrix}\right) \end{aligned} \tag{3}$$

for $i \in \mathbf{n} - \mathbf{n}^\star$. It is clear from (3) that for all $i \in \mathbf{n} - \mathbf{n}^\star$, the subspace $T_pV + G_i(\mathrm{col}(p,0))$ has constant dimension on $V$. We now show that, for all $p \in V$, $\bar{G}_i(\mathrm{col}(p,0)) \subseteq T_pV + G_i(\mathrm{col}(p,0))$. Once this is proven, the lemma follows from the fact that $T_pV + G_i(\mathrm{col}(p,0)) \subseteq T_pV + \bar{G}_i(\mathrm{col}(p,0))$.

In $(z,\xi)$ coordinates consider the constant distribution on $U$ given by

$$\Delta = \mathrm{Im}\begin{bmatrix} I_{n^\star} & 0 & 0 & \cdots & 0 \\ 0 & B & AB & \cdots & A^iB \end{bmatrix}$$

where each column is a (constant) vector. At each $p \in V$, $\Delta(p) = T_pV + G_i(\mathrm{col}(p,0))$. Since $\Delta$ is involutive and contains $G_i$, it follows that $\bar{G}_i(\mathrm{col}(p,0)) \subseteq \Delta(p) = T_pV + G_i(\mathrm{col}(p,0))$. ∎

The next result adapts [1, Lemma IV.1, Lemma IV.3] to the result in Lemma IV.1. In order to identify directions in the intersection $T_pV \cap \bar{G}_i(p)$ which are not contained in the intersection $T_pV \cap \bar{G}_{i-1}(p)$, define the integers

$$\begin{aligned} \mu_0(p) &:= \dim(T_pV \cap \bar{G}_0(p)) \\ \mu_i(p) &:= \dim(T_pV \cap \bar{G}_i(p)) - \dim(T_pV \cap \bar{G}_{i-1}(p)) \\ n_i(p) &:= \sum_{j=0}^{i} \mu_j. \end{aligned}$$

Thus, $n_i(p) = \dim(T_pV \cap \bar{G}_i(p))$. When the $\rho_i$'s and $\mu_i$'s are constant over an open subset of $\Gamma^\star$ we have the following result.

**Lemma IV.2** *Let $W \subseteq \mathbb{R}^n$ be an open set such that $W \cap \Gamma^\star \neq \emptyset$, and define $V := W \cap \Gamma^\star$. Assume that, for all $i \in \mathbf{n} - \mathbf{n}^\star$, and for all $p \in V$*

$$\begin{aligned} &\dim(T_pV + G_i(p)) = \dim(T_pV + \bar{G}_i(p)) = \text{ constant} \\ &(\forall p \in W) \ \dim(\bar{G}_i(p)) = \text{ constant}. \end{aligned}$$

*Then*

$$\begin{aligned} &\rho_0 \leq \rho_1 \leq \cdots \leq \rho_{n-n^\star-1} \\ &k_1 \geq k_2 \geq \cdots k_{\rho_0}. \end{aligned}$$

*Moreover there exists an open set $U \subseteq W$, a regular static feedback $(\alpha, \beta)$ defined on $U$ and $n_i$ vector fields $v_\ell^j : U \to T\mathbb{R}^n$, $1 \leq j \leq i$, $1 \leq \ell \leq \mu_i$, such that, letting $\tilde{V} := U \cap \Gamma^\star \neq \emptyset$, for all $i \in \mathbf{n} - \mathbf{n}^\star$,*

$$\begin{aligned} &(\forall p \in \tilde{V}) \ \ G_i^{\|}(p) := \operatorname{span}\{v_1^0, \dots, v_{\mu_i}^i\}(p) \subseteq T_p\tilde{V} \\ &(\forall p \in U) \\ &\qquad \bar{G}_i(p) = G_i^{\|}(p) \oplus \left( \bigoplus_{j=0}^{i} \operatorname{span}\{ad_{\tilde{f}}^j \tilde{g}_k : 1 \leq k \leq \rho_j\} \right) \end{aligned}$$

*where $\tilde{f} = f + g\alpha$ and $\tilde{g} = g\beta$.*

This lemma gives a basis of $\bar{G}_i$ by distinguishing between the vector fields in $\bar{G}_i$ which are tangent to $\Gamma^*$ and those which are transverse to it. Specifically, we have that

$$\begin{aligned} &(\forall p \in U) \ \ \bar{G}_i = G_i^{\|} \oplus G_i^{\pitchfork} \\ &= \left( G_0^{\|} + G_{1/0}^{\|} + \cdots G_{i/i-1}^{\|} \right) \oplus \left( G_0^{\pitchfork} + G_{1/0}^{\pitchfork} + \cdots G_{i/i-1}^{\pitchfork} \right) \end{aligned}$$

where $G_i^{\|}\Big|_V \subseteq TV$ and, for all $p \in U$,

$$\begin{aligned} G_{i/i-1}^{\|} &:= \operatorname{span}\{v_j^i : 1 \leq j \leq \mu_i\} \\ G_{i/i-1}^{\pitchfork} &:= \operatorname{span}\{ad_f^i g_j : 1 \leq j \leq \rho_i\} \subseteq G_i \end{aligned} \tag{4}$$

span, respectively, the tangential and transversal directions in $\bar{G}_i$ *not* contained in $\bar{G}_{i-1}$. The proof of Lemma IV.2 is omitted since it is conceptually the same as the proof of [1, Lemma IV.1]. An immediate consequence of Lemma IV.2 is that when $\sum k_i = n - n^\star$, i.e.,

$$(\forall p \in V) \ \dim(T_pV + \bar{G}_{k_1-1}(p)) = n,$$

then, after feedback transformation,

$$\begin{aligned} &T_pV \oplus \operatorname{span}\{ad_f^j g_k(p) : 0 \leq j \leq n - n^\star - 1, 1 \leq k \leq \rho_j\} \\ &\quad = T_p\mathbb{R}^n. \end{aligned} \tag{5}$$

As a result, Lemma IV.2 yields the following array of $n$ independent vector fields. In the array we use the symbols $G_{i/i-1}^{\|}$ and $G_{i/i-1}^{\pitchfork}$ to indicate a family of vector fields and not the span of vector fields.

$$\begin{array}{l|l} 1 & G_0^\tau, G_0^{\pitchfork}; \dots; G_{k_{\rho_0}-1/k_{\rho_0}-2}^\tau, G_{k_{\rho_0}-1/k_{\rho_0}-2}^{\pitchfork}; \\ 2 & G_{k_{\rho_0}/k_{\rho_0}-1}^\tau, G_{k_{\rho_0}/k_{\rho_0}-1}^{\pitchfork}; \dots\dots; \\ & \qquad G_{k_{\rho_0-1}-1/k_{\rho-1}-2}^\tau, G_{k_{\rho_0-1}-1/k_{\rho_0-1}-2}^{\pitchfork}; \\ & . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \rho_0 - 1 & G_{k_3/k_3-1}^\tau, G_{k_3/k_3-1}^{\pitchfork}; \dots; G_{k_2-1/k_2-2}^\tau, G_{k_2-1/k_2-2}^{\pitchfork}; \\ \rho_0 & G_{k_2/k_2-1}^\tau, G_{k_2/k_2-1}^{\pitchfork}; \dots; G_{k_1-1/k_1-2}^\tau, G_{k_1-1/k_1-2}^{\pitchfork}; \\ \rho_0 + 1 & v_1, , \dots, v_{n^\star - n_{k_1-1}}. \end{array} \tag{6}$$

All of the vector fields of (6) are defined on $U$ except for those in row $\rho_0 + 1$. Those vector fields are solely defined on $V \subset \Gamma^*$ and are not contained in any of the $\bar{G}_i$'s so that at each $p \in V$, $\operatorname{span}\{v_1, \dots, v_{n^\star - n_{k_1-1}}\}(p) \simeq \left(T_pV + \bar{G}_{k_1-1}(p)\right)/\bar{G}_{k_1-1}(p)$. They are chosen to complete the basis for $T_pV$, so that

$$T_pV = \operatorname{span}\{v_1, \dots, v_{n^\star - n_{k_1-1}}\}(p) \oplus G_{k_1-1}^{\|}(p).$$

We conclude this section with the local version of [1, Theorem VI.1] which is used to prove our main result in Section V.

**Theorem IV.3** *LTFLP is solvable at $p_0$ if and only if there exist $\rho_0$ smooth $\mathbb{R}$-valued functions $\alpha_1, \dots, \alpha_{\rho_0}$, defined on some open neighborhood $U$ of $p_0$ in $\mathbb{R}^n$, such that*

(1) $U \cap \Gamma^\star \subset \{x \in U : \alpha_i(x) = 0, \ i = 1, \dots, \rho_0\}$

(2) *The system*

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{\rho_0} g_i(x) u_i \\ y' &= \operatorname{col}(\alpha_1(x), \dots, \alpha_{\rho_0}(x)) \end{aligned} \tag{7}$$

*has vector relative degree $\{k_1, \dots, k_{\rho_0}\}$ at $p_0$.*

## V. Main Result

**Theorem V.1** *Assume that $\bar{G}_i$, $i \in \mathbf{k_1 - 2}$ are regular at $p_0$. Then LTFLP is solvable at $p_0$ if and only if*

(a) $\dim(T_{p_0}\Gamma^\star + G_{k_1-1}(p_0)) = n$

*and there exists an open neighborhood $O$ of $p_0$ in $\Gamma^\star$ such that*

(b) $(\forall i \in \mathbf{k_1 - 2})$ $(\forall p \in O)$ $\dim(T_p\Gamma^\star + G_i(p)) = \dim(T_p\Gamma^\star + \bar{G}_i(p)) =$ *constant.*

*Sketch of the Proof:* $(\Rightarrow)$ Assume LTFLP is solvable. Then condition (a) follows by [1, Lemma V.1]. Condition (b) follows by Lemma IV.1.

$(\Leftarrow)$ Conditions (a) and (b) along with the regularity of $\bar{G}_i$ imply that there exists a neighborhood $W$ of $p_0$ in $\mathbb{R}^n$ such that the assumptions of Lemma IV.2 hold, and thus after feedback transformation we obtain the $n$ independent vector fields of (6) defined on an open set $U \subseteq W$ with $V := U \cap O \neq \emptyset$. We now construct $\rho_0$ $\mathbb{R}$-valued functions $\alpha_1, \dots, \alpha_{\rho_0}$ satisfying Theorem IV.3. Let $p_0 \in V$ be the origin for $S$-coordinates [10]. These coordinates are generated by composing the flows of the vector fields in (6) in a special order and then using the $n$ flow times as coordinates. Scalar functions $\alpha_1, \ldots, \alpha_{\rho_0}$ satisfying Theorem IV.3 are chosen from among those times.

We compose the flows generated by the vector fields in (6) starting from the bottom row. Begin with the flows generated by the vector fields $v_1, \ldots, v_{n^\star - n_{k_1-1}}$. Consider the mapping $F_\emptyset : \Omega \subset \mathbb{R}^{n^\star - n_{k_1-1}} \to V \subset \mathbb{R}^n$.

$$\begin{aligned} F_\emptyset : S_\emptyset &:= (s_1^{\|}, \ldots, s^{\|}_{n^\star - n_{k_1-1}}) \\ &\mapsto \phi^{v_{n^\star - n_{k_1-1}}}_{s^{\|}_{n^\star - n_{k_1-1}}} \circ \cdots \circ \phi^{v_1}_{s_1^{\|}}(p_0). \end{aligned}$$

To each pair $(G^{\|}_{i/i-1}, G^{\pitchfork}_{i/i-1})$ in (6) we associate a set of times. For[1] $i \in \mathbf{k_1}$, let $S_{i/i-1} = (S^{\pitchfork}_{i/i-1}; S^{\|}_{i/i-1}) := (s^{\pitchfork}_{(i/i-1,1)}, \ldots, s^{\pitchfork}_{(i/i-1,\rho_i)}; s^{\|}_{(i/i-1,1)}, \ldots, s^{\|}_{(i/i-1,\mu_i)})$. Next we generate a collection of mappings $F_{i/i-1} : U_{i/i-1} \subset \mathbb{R}^{\mu_i + \rho_i} \to \mathbb{R}^n$, $(1 \le i \le k_1 - 1)$, given by

$$\begin{aligned} &F_{i/i-1} : S_{i/i-1} = \left(S^{\pitchfork}_{i/i-1}; S^{\|}_{i/i-1}\right) \mapsto \Phi^{\|}_{i/i-1} \circ \Phi^{\pitchfork}_{i/i-1}(p), \\ &\Phi^{\|}_{i/i-1} := \phi^{v^i_{\mu_i}}_{s^{\|}_{(i/i-1,\mu_i)}} \circ \cdots \circ \phi^{v^i_1}_{s^{\|}_{(i/i-1,1)}}, \\ &\Phi^{\pitchfork}_{i/i-1} := \phi^{ad^i_f g_1}_{s^{\pitchfork}_{(i/i-1,\rho_i)}} \circ \cdots \circ \phi^{ad^i_f g_{\rho_i}}_{s^{\pitchfork}_{(i/i-1,1)}}. \end{aligned}$$

The notation for $S_{i/i-1} = (S^{\pitchfork}_{i/i-1}; S^{\|}_{i/i-1})$ describes the fact that $S^{\pitchfork}_{i/i-1}$ is a collection of times associated with vector fields in $\bar{G}_i$, not in $\bar{G}_{i-1}$, which are transversal to $V$ on $V$. Meanwhile, $S^{\|}_{i/i-1}$ is a collection of times associated with vector fields in $\bar{G}_i$, not in $\bar{G}_{i-1}$, which are tangent to $V$ on $V$. Compose each of these maps together to obtain $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$,

$$F := F_0 \circ F_{1/0} \circ \cdots \circ F_{k_1-1/k_1-2} \circ F_\emptyset(p_0). \tag{8}$$

The fact that, for some sufficiently small neighborhood $U$ of $p_0$ in $\mathbb{R}^n$, (8) is a diffeomorphism onto its image is an obvious consequence of property (a). Globally, i.e., in a neighborhood of $\Gamma^\star$, it is not obvious. In the proof of Theorem III.1 in [1] we mistakenly claimed that (8) is a diffeomorphism from a neighborhood of $\Gamma^*$ onto its image. This mistake *does not affect* the conceptually similar, but significantly simpler, proof of Theorem 4.4 in [5], which provides sufficient conditions for the existence of a global solution to TFLP for single-input systems.

The final $S$-coordinates are given by

$$S = \mathrm{col}\left(S_\emptyset; S_{k_1-1/k_1-2}; \ldots; S_{1/0}; S_0\right).$$

As candidate output functions, let $\alpha_i$, $i \in \{1, \ldots, \rho_0\}$, be the time spent flowing along $ad_f^{k_i-1} g_i$, i.e. $\alpha_i(x) = s^{\pitchfork}_{(k_i-1/k_i-2, \rho_i - i+1)}(x)$. With this choice for $\alpha$, we must show that the conditions of Theorem IV.3 are satisfied.

Re-define $V$ as $V = F(U) \cap \Gamma^*$. In [1, Theorem III.1] it is shown that $V \subseteq \{x : \alpha(x) = 0\}$ and that for all $p \in F(U)$

$$L_{ad^\ell_f g_j} \alpha_i(p) = 0;\ 1 \le i \le \rho_0,\ 1 \le j \le m,\ 0 \le \ell \le k_i - 2. \tag{9}$$

[1]We define i/(i-1) := 0 for $i = 0$ to be consistent with the array (6).

Since the proof of the above facts remains the same in the more general setting of this theorem, we focus on showing that the $\rho_0 \times m$ decoupling matrix

$$\begin{pmatrix} L_{g_1} L_f^{k_1-1} \alpha_1(p) & \cdots & L_{g_m} L_f^{k_1-1} \alpha_1(p) \\ L_{g_1} L_f^{k_2-1} \alpha_2(p) & \cdots & L_{g_m} L_f^{k_2-1} \alpha_2(p) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{k_{\rho_0}-1} \alpha_{\rho_0}(p) & \cdots & L_{g_m} L_f^{k_{\rho_0}-1} \alpha_{\rho_0}(p) \end{pmatrix} \tag{10}$$

is full rank for any $p \in V$. Notice that for any point on $V$ the last $m - \rho_0$ columns of (10) are zero. To see this, recall that the preliminary feedback transformation of Lemma IV.2 is such that,

$$\begin{aligned} &(\forall i \in \{0, 1, \ldots\})\ (\forall p \in V)\ (\forall k \in \{\rho_i + 1, \ldots, m\}) \\ &\qquad ad^i_f g_k(p) \in T_p V + \bar{G}_{i-1}(p). \end{aligned}$$

This implies, by the fact that $V \subseteq \{x : \alpha(x) = 0\}$ and by (9), that

$$L_{ad_f^{k_i-1} g_j} \alpha_i(p) = 0;\ 1 \le i \le \rho_0,\ i < j \le m.$$

Using [6, Lemma 4.1.2] allows us to set the final $m - \rho_0$ columns of (10) to zero. Thus we concentrate on the $\rho_0 \times \rho_0$ sub-matrix of (10) consisting of the first $\rho_0$ columns. By [6, Lemma 4.1.2] and by (9) we have that the $\rho_0 \times \rho_0$ sub-matrix of (10) consisting of the first $\rho_0$ columns is nonsingular if and only if

$$\begin{pmatrix} L_{ad_f^{k_1-1} g_1} \alpha_1(p) & \cdots & L_{ad_f^{k_1-1} g_{\rho_0}} \alpha_1(p) \\ L_{ad_f^{k_2-1} g_1} \alpha_2(p) & \cdots & L_{ad_f^{k_2-1} g_{\rho_0}} \alpha_2(p) \\ \cdots & \cdots & \cdots \\ L_{ad_f^{k_{\rho_0}-1} g_1} \alpha_{\rho_0}(p) & \cdots & L_{ad_f^{k_{\rho_0}-1} g_{\rho_0}} \alpha_{\rho_0}(p) \end{pmatrix} \tag{11}$$

is non-singular. Now suppose $k_1 = k_2 = \cdots k_{m_1} > k_{m_1+1}$, $1 \le m_1 \le \rho_0$. Intuitively this means that the functions $\alpha_1, \ldots, \alpha_{m_1}$ correspond to the times flowing along vector fields in $\bar{G}_{k_1-1}$, but not in $\bar{G}_{k_1-2}$. In terms of the $\rho_i$, this means $\rho_{k_1-1} = \rho_{k_1-2} = \cdots = \rho_{k_{m_1+1}} = m_1$. We now show that the first $m_1$ rows of (11) are full rank. Suppose that there exist $m_1$ scalars $c_i$ such that

$$\sum_{i=1}^{m_1} c_i \left\langle d\alpha_i, ad_f^{k_1-1} g_j \right\rangle (p) = 0; \quad 1 \le j \le \rho_0.$$

This implies that $\sum c_i d\alpha_i \in \mathrm{ann}\,(\bar{G}_{k_1-1})$. However, $\sum c_i d\alpha_i \in \mathrm{ann}\,(TV)$. Therefore

$$\begin{aligned} \sum_{i=1}^{m_1} c_i d\alpha_i &\in \mathrm{ann}\,(TV) \cap \mathrm{ann}\,(\bar{G}_{k_1-1}) \\ &= \mathrm{ann}\left(TV + \bar{G}_{k_1-1}\right) = 0. \end{aligned}$$

Since $\{d\alpha_1, \ldots, d\alpha_{m_1}\}$ are linearly independent, (a fact easily seen in $S$-coordinates), we conclude that $c_1 = \cdots = c_{m_1} = 0$ and the first $m_1$ rows of (11) are full rank as claimed.

Now suppose that $k_{m_1+1} = \cdots = k_{m_1+m_2} > k_{m_1+m_2+1}$, $0 \le m_2 \le \rho_0 - m_1$. We want to show that the first $m_1 + m_2$

rows of (11) are full rank. In order to do this we first show that the exact one-forms

$$dL_f^i\alpha_j, \quad 1 \le j \le m_1, \quad 0 \le i \le k_1 - k_{m_1+1} - 1$$

are

(i) linearly independent on $V$.
(ii) Contained in $\operatorname{ann}\left(TV + \bar{G}_{k_{m_1+1}-2}\right)$.

Fact (ii) follows directly from (9) and the fact that $V \subseteq \{x : \alpha(x) = 0\}$. To prove (i), consider the linear combination

$$\sum_{i=1}^{m_1} a_i^0 d\alpha_i + \cdots + a_i^{k_1-k_{m_1+1}} dL_f^{k_1-k_{m_1+1}}\alpha_i = 0. \tag{12}$$

Next take the inner product of (12) with $ad_f^{k_{m_1+1}-1}g_j$, $1 \le j \le m$. Using (9) and [6, Lemma 4.1.2] we have

$$\left\langle \sum_{i=1}^{m_1} a_i^{k_1-k_{m_1+1}} dL_f^{k_1-k_{m_1+1}}\alpha_i, ad_f^{k_1-k_{m_1+1}-1}g_j \right\rangle = 0$$
$$\Leftrightarrow \left\langle \sum_{i=1}^{m_1} a_i^{k_1-k_{m_1+1}} d\alpha_i, ad_f^{k_1-1}g_j \right\rangle = 0.$$

Since the first $m_1$ rows of (11) are linearly independent, we conclude that $a_i^{k_1-k_{m_1+1}} = 0$ for $1 \le i \le m_1$. Following this same procedure one can recursively show that all the coefficients in (12) are identically zero and (i) is proven. An important consequence of this fact is that

$$\operatorname{ann}(TV + \bar{G}_{k_{m_1+1}-1}) = \bigoplus_{i=0}^{k_1-k_{m_1+1}-1} \operatorname{span}\{dL_f^i\alpha_1, \ldots, dL_f^i\alpha_{m_1}\}. \tag{13}$$

Returning our attention to the first $m_1 + m_2$ rows of (11), suppose there exist $m_1 + m_2$ scalars such that for $1 \le j \le \rho_0$

$$\sum_{i=1}^{m_1} a_i \left\langle d\alpha_i, ad_f^{k_1-1}g_j \right\rangle + \sum_{i=1}^{m_2} b_i \left\langle d\alpha_{m_1+i}, ad_f^{k_{m_1+1}-1}g_j \right\rangle = 0.$$

Using, once again [6, Lemma 4.1.2] and (9) this can be written as

$$\left\langle \sum_{i=1}^{m_1} a_i dL_f^{k_1-k_{m_1+1}}\alpha_i + \sum_{i=1}^{m_2} b_i d\alpha_{m_1+i}, ad_f^{k_{m_1+1}-1}g_j \right\rangle = 0.$$

which implies that at each $p \in V$

$$\sum_{i=1}^{m_1} a_i(p) dL_f^{k_1-k_{m_1+1}}\alpha_i(p) + \sum_{i=1}^{m_2} b_i(p) d\alpha_{m_1+i}(p) \tag{14}$$

belongs to $\operatorname{ann}\left(TV + \bar{G}_{k_{m_1+1}-1}\right)$. We now show that this contradicts (13). Consider the $S$-coordinates representation of the term $\sum_{i=1}^{m_2} b_i d\alpha_{m_1+i}$ in (14). Since the one-forms $\{d\alpha_{m_1+1}, \ldots, d\alpha_{m_1+m_2}\}$ are part of the dual basis in $S$-coordinates, it has a particularly simple vector notation given by

$$\left[\begin{array}{c|ccc|c|c} 0_h & b_1 & \cdots & b_{m_2} & 0_{m_1} & 0_k \end{array}\right]$$

where $h = m_1(k_1 - k_{m_1+1}) + n_{k_{m_1+1}-1}$ and $k = n - m_1(k_1 - k_{m_1+1} + 1) - m_2 - n_{k_{m_1+1}-1}$. In light of this the term $\sum_{i=1}^{m_1} a_i dL_f^{k_1-k_{m_1+1}}\alpha_i$ in (14) must have, in $S$-coordinates, the form

$$\left[\begin{array}{c|ccc|c|c} \star_h & -b_1 & \cdots & -b_{m_2} & 0_{m_1} & 0_k \end{array}\right].$$

However, in $S$-coordinates, vector fields in $\bar{G}_{k_{m_1+1}-1}$ have the form $\operatorname{col}(0_h, \star)$ with zeros corresponding precisely with the term $\star_h$ above. In fact it is possible to find a $v \in \bar{G}_{k_{m_1+1}-1}$, $v \notin \bar{G}_{k_{m_1+1}-2}$, given by

$$v = \sum_{i=0}^{k_{m_1+1}-1} c_i^0 g_i + \cdots + c_i^{k_{m_1+1}-1} ad_f^{k_{m_1+1}-1} g_i$$

such that in $S$-coordinates

$$v = \operatorname{col}\left(\begin{array}{c|ccc|c|c} 0_h & 0 & \cdots & 0 & \star_{m_1} & 0_k \end{array}\right).$$

This means that

$$\left\langle \sum_{i=1}^{m_1} a_i dL_f^{k_1-k_{m_1+1}}\alpha_i, v \right\rangle = 0$$

and hence

$$\left\langle \sum_{i=1}^{m_1} a_i dL_f^{k_1-k_{m_1+1}}\alpha_i, \sum_{i=0}^{k_{m_1+1}-1} c_i^{k_{m_1+1}-1} ad_f^{k_{m_1+1}-1} g_i \right\rangle = 0.$$

Thus $\sum_{i=1}^{m_1} a_i dL_f^{k_1-k_{m_1+1}}\alpha_i \in \operatorname{ann}\left(TV + \bar{G}_{k_{m_1+1}-1}\right)$ which by the fact (i) shown earlier, implies that $a_i \equiv 0$. We are left to show that the $b_i$ in (14) are zero. This can be done directly using (9) and [6, Lemma 4.1.2] and considering the expression

$$\sum_{i=1}^{m_1} a_i^0 d\alpha_i + \cdots + a_i^{k_1-k_{m_1+1}-1} dL_f^{k_1-k_{m_1+1}-1}\alpha_i + \sum_{i=1}^{m_2} b_i d\alpha_{m_1+i} = 0.$$

One now proceeds in exactly the same way as was used to show that the coefficients in (12) are all zero.

At this point the proof technique can be repeated until all the rows of (11) are accounted for. Specifically, the next step in the proof is to assume that $k_{m_1+m_2+1} = \cdots = k_{m_1+m_2+m_3} > k_{m_1+m_2+m_3+1}$. Now take a linear combination of the first $m_1 + m_2 + m_3$ rows of (11) and assume there exists $m_1 + m_2 + m_3$ scalars such that, for $1 \le j \le \rho_0$,

$$\sum_{i=1}^{m_1} a_i \left\langle d\alpha_i, ad_f^{k_1-1}g_j \right\rangle + \sum_{i=1}^{m_2} b_i \left\langle d\alpha_{m_1+i}, ad_f^{k_{m_1+1}-1}g_j \right\rangle + \sum_{i=1}^{m_3} c_i \left\langle d\alpha_{m_1+m_2+i}, ad_f^{k_{m_1+m_2+1}-1}g_j \right\rangle = 0.$$

Arguing in the same way as above, one shows that the integers $a_i$, $b_i$, and $c_i$ must be identically zero. In this way one shows that (11) is full rank.

In conclusion, the function $(\alpha_1, \ldots, \alpha_{\rho_0})$ constructed using $S$-coordinates satisfy both conditions in Theorem IV.3. ∎

## VI. Example

To illustrate these ideas we present an example of centralized control of two Lorenz oscillators. The equations of motion are

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) + u_1 & \dot{y}_1 &= \sigma(y_2 - y_1) + u_2 \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3 & \dot{y}_2 &= r y_1 - y_2 - y_1 y_3 \\ \dot{x}_3 &= -b x_3 + x_1 x_2 + u_1 & \dot{y}_3 &= -b y_3 + y_1 y_2 + u_2. \end{aligned} \quad (15)$$

For simplicity, we assume that $\sigma = r = b = 1$. We consider two separate problems: (a) the problem of full state synchronization and (b) a partial synchronization problem. We will show that the latter is solvable while the former is not by using transverse feedback linearization. These types of problems are common and have appeared in the literature [11]. We begin with the partial synchronization problem.

Suppose we are interested in forcing the the variables $x_1$ and $y_1$ to lie on a unit circle $\Gamma^\star = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_1^2 + y_1^2 = 1\}$. In this case it is clear that $u^\star = \mathrm{col}(-\sigma(x_2 - x_1) + y_1, -\sigma(y_2 - y_1) - x_1)$ is a suitable, though not unique, friend. The constraint $h = x_1^2 + y_1^2 - 1$ defining $\Gamma^\star$ satisfies condition $(1)$ of Theorem IV.3. It turns out that condition $(2)$ holds as well signifying that the constraint can be used as the virtual output $y'$ in (7). It is instructive to also check the conditions of Theorem V.1. In this example we have that for all $x \in \Gamma^\star$,

$$T_x\Gamma^\star = \mathrm{Im} \begin{bmatrix} 0 & 0 & 0 & 0 & y_1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Simple calculations reveal that for all $x \in \Gamma^\star$, $\dim(T_x\Gamma^\star + G_0(x)) = 6$, i.e. condition (a) of Theorem V.1 is satisfied. In the special case when $n^\star = n - 1$, the conditions of Theorem V.1 simplify and condition (a) becomes both necessary and sufficient.Following the procedure in the proof of Theorem IV.3 one obtains the output function

$$\alpha(x, y) = \ln\left(\sqrt{x_1^2 + y_1^2}\right).$$

Next we pursue the question of whether or not transverse feedback linearization can be used to synchronize system (15), i.e we want to know if the diagonal

$$\Gamma^\star = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_1 = y_1, x_2 = y_2, x_3 = y_3\}$$

can be stabilized using transverse feedback linearization. The set $\Gamma^\star$ is invariant for any choice of $u^\star$ so long as $u_1^\star = u_2^\star$. In particular $u^\star = 0$ is a suitable candidate. Thus $n^\star = 3$ and for any $p \in \Gamma^\star$ we have that

$$T_p\Gamma^\star = \mathrm{Im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To check that conditions of theorem V.1 we note that $G_0 = \bar{G}_0$ and that for all $p \in \Gamma^\star$ $\dim(T_p\Gamma^\star + G_0(p)) = 4$ which means that $\rho_0 = 1$. Next we find the distribution $G_1$ by noting that $ad_f g_1(x, y) = \mathrm{col}(1, x_1 + x_3 - 1, 1 - x_2, 0, 0, 0)$ and $ad_f g_2(x, y) = \mathrm{col}(0, 0, 0, 1, y_1 + y_3 - 1, 1 - y_2)$. Simple calculations give that $\rho_1 = 1$ and so for condition (b) of Theorem V.1 to hold we require that for all $p \in \Gamma^\star$, $\dim(T_p\Gamma^\star + G_1(p)) = \dim(T_p\Gamma^\star + \bar{G}_1(p))$. One can easily check that this condition fails and hence the conditions of Theorem V.1 do not hold. We conclude that transverse feedback linearization cannot be used to synchronize the Lorenz oscillators (15).

## VII. Conclusions

Together with earlier results in [5] and [1], this paper completes the characterization of the local transverse feedback linearization problem. The main contributions of this paper are: a generalization of the definition of controllability indices which can be used when the distributions $G_i$ are not involutive; the comparison between our definition of transverse controllability indices in the special case when $T_p\Gamma^\star = \{0\}$ and Marino's definition (Lemma III.1); a new necessary condition (Lemma IV.1) and checkable necessary and sufficient conditions (Theorem V.1) for the existence of a solution to LTFLP. Future research includes solving the global version of TFLP.

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