ON LOCAL TRANSVERSE FEEDBACK LINEARIZATION[*†]

CHRISTOPHER NIELSEN[‡] AND MANFREDI MAGGIORE[‡]

PUBLISHED IN *SIAM JOURNAL ON CONTROL AND OPTIMIZATION*, VOL. 47, NO. 5, 2008, PP. 2227–2250

Abstract.
Given a control-affine system and a controlled invariant submanifold, we present necessary and sufficient conditions for local feedback equivalence to a system whose dynamics transversal to the submanifold are linear and controllable. A key ingredient used in the analysis is the new notion of transverse controllability indices of a control system with respect to a set.

Key words. set stabilization, feedback linearization, controlled invariant sets, zero dynamics, multi-input systems, nonlinear geometric control

AMS subject classifications. 37N35, 93B10, 93B29, 93B27,

1. Introduction.

Ever since Poincaré's seminal work [22], the problem of equivalence of vector fields has been a central question in the field of dynamics. In his 1879 work, Poincaré found sufficient conditions for an analytic vector field to be locally equivalent to a linear one by means of an analytic transformation. Poincaré's key insight in formulating this problem was that, rather than trying to solve a differential equation, it is convenient to seek a coordinate transformation reducing the associated vector field to its "simplest" form, the normal form. In control theory, the problem of equivalence of a control system to a linear controllable system by means of smooth coordinate transformations was first formulated by Krener in 1973, [13]. In 1978, Brockett [3] formulated and solved the feedback linearization problem for single-input, single-output systems, whereby the equivalence to a linear controllable system is established by means of a smooth coordinate transformation and a regular feedback transformation; this is referred to as feedback equivalence. The multi-input multi-output extension of Brockett's work was carried out by Jakubczyk and Respondek in [12] and, independently, by Hunt, Su, and Meyer in [8]; see also [26]. When a control system is not feedback linearizable, it is natural to ask whether it admits a feedback linearizable subsystem. This problem, first posed by Isidori and Krener in [11], is referred to as partial feedback linearization. For single-input systems, Krener, Isidori, and Respondek [14] investigated partial feedback linearization yielding a linear subsystem of maximal dimension. This result was extended by Marino in [15] to the multi-input case; see also [16], [23]. For systems with outputs, Xu and Hunt [30], [31], consider a similar problem.

In [2], Banaszuk and Hauser formulated and solved the transverse feedback linearization problem (TFLP) for periodic orbits of single-input control-affine systems. If $\Gamma^\star$ is a periodic orbit of the open loop system, the problem entails finding conditions for feedback equivalence to a control system whose dynamics transversal to $\Gamma^\star$ are linear, and controllable. In [19], we generalized Banaszuk and Hauser's results to

*Received by the editors February 7, 2007; accepted for publication (in revised form) March 21, 2008.

†Both authors were supported by the National Science and Engineering Research Council (NSERC) of Canada. Christopher Nielsen was partially supported by the Ontario Graduate Scholarship (OGS).

‡Department of Electrical and Computer Engineering, University of Toronto, 10 King's College Road, Toronto, ON, M5S 3G4, Canada ({nielsen,maggiore}@control.utoronto.ca).

the case when $\Gamma^\star$ is an arbitrary controlled invariant embedded submanifold of the state space. In this paper we present the complete solution to the local TFLP for multi-input systems, relying on a mild regularity assumption. A key ingredient used in the analysis of the problem is the new notion of transverse controllability indices of a control system with respect to a set. The transverse controllability indices are an adaptation of those introduced by Marino [15].

When the set $\Gamma^\star$ is an equilibrium point, the problem considered in this paper (see Section 3) reduces to the classical state-space exact linearization problem. In this special case our conditions coincide with those of the classical results on feedback equivalence to linear, time-invariant, controllable systems [8, 12], and the transverse controllability indices coincide with the controllability indices introduced by Marino [15].

We now discuss some of the applications of transverse feedback linearization (TFL). While classical feedback linearization is used to stabilize equilibria of nonlinear systems, TFL is applicable to the more general set stabilization problem. Indeed, if a system is transversely feedback linearizable with respect to a controlled invariant manifold $\Gamma^\star$, then designing a controller that locally stabilizes $\Gamma^\star$ amounts to designing a stabilizer for the origin of a linear time-invariant system and so the set stabilization problem is greatly simplified (see Section 3 for a more precise discussion). In light of the above, TFL is relevant to all those problems where the control objective is the stabilization of a manifold, rather than an equilibrium. Consider, for instance, the simplest synchronization (or state agreement) problem: make the states of two coupled dynamical systems converge to one another. This is equivalent to stabilizing the diagonal subspace. In the more general case when one wants to make the outputs of several coupled dynamical systems converge to one another then, generally, the set to be stabilized is a manifold. As other relevant applications of TFL we mention path following (make the output of a dynamical system approach and follow a path) [2, 18, 19], and the stabilization of virtual constraints in mechanical systems [24].

Another important application of our main result is the solution of the following *zero dynamics assignment problem with relative degree.* Given a control-affine system and a controlled invariant manifold $\Gamma^\star$, does there exist an output function yielding a well-defined vector relative degree whose associated zero dynamics manifold locally coincides with $\Gamma^\star$? Our main result in Theorem 3.2 gives checkable necessary and sufficient conditions that completely answer this question.

This paper is organized as follows: Section 2 contains mathematical preliminaries. Section 3 presents the formal problem statement, the statement of our main result, Theorem 3.2, and a comparison of our result to the solution of the classical state-space exact linearization problem [8, 12]. A relationship to the partial feedback linearization problem is established in Theorem 3.5. In Section 4 we introduce the notion of transverse indices, compare them to the controllability indices of Marino [15] in Lemma 4.1 and establish their feedback invariance. The proof of the main result is presented in Section 5 and Section 6 contains concluding remarks.

2. Preliminaries.

Consider a control system Σ modeled by equations of the form

$$\Sigma: \quad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i =: f(x) + g(x)u. \tag{2.1}$$

Here $x \in \mathbb{R}^n$ is the state, and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ is the control input. The vector fields $f, g_1, \ldots, g_m : \mathbb{R}^n \longrightarrow T\mathbb{R}^n$ are smooth (C^∞). We assume throughout this paper that $g_1, \ldots, g_m$ are linearly independent.

When we talk of a manifold M, we mean a smooth manifold without boundary. By a submanifold is meant an embedded submanifold. All objects are presumed to be smooth. In this paper we consider submanifolds of $\mathbb{R}^n$, where $\mathbb{R}^n$ is identified with Euclidean n-space.

2.1. Notation. If k is a positive integer, $\mathbf{k}$ denotes the set of integers $\{0, 1, \ldots, k-1\}$. We let $\mathrm{col}(x_1, \ldots, x_k) := [x_1 \ \ldots \ x_n]^\top$ and, given two column vectors a and b, we let $\mathrm{col}(a, b) := [a^\top \ b^\top]^\top$. If U is an open set of $\mathbb{R}^n$, let $\mathrm{Diff}(U)$ denote the collection of diffeomorphisms from U to some open set $\tilde{U} \subset \mathbb{R}^n$. If $F : M \to N$ is a map between manifolds then $dF_x : T_xM \to T_{F(x)}N$ denotes its differential. If M and N are vector spaces, then use dF_x to denote the Jacobian matrix of F at x. If $F : M \to N$ is a diffeomorphism between two manifolds, and if v is a vector field on M, then the differential of F can be used to define a vector field on N by means of the push-forward map $F_\star$, defined as $F_\star v(q) = (dF_p v(p))\big|_{p=F^{-1}(q)}$. This corresponds to the usual change of coordinates in a differential equation. We denote by I_m the $m \times m$ identity matrix. The direct sum of two matrices A and B is the block diagonal matrix

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where the zeros denote matrices of suitable size. Given two subspaces V and W of the same vector space, the notation $V \oplus W$ (internal direct sum) is used to represent the subspace $V + W$ when V and W are linearly independent.

Definition 2.1. *Given an open set $U \subseteq \mathbb{R}^n$, a* regular static feedback*, denoted (α, β), on U for the control system* (2.1) *is a relation*

$$u = \alpha(x) + \beta(x)v$$

where $u = (u_1, \ldots, u_m)$ and $\alpha : U \to \mathbb{R}^m$, $\beta : U \longrightarrow \mathrm{GL}(m, \mathbb{R})$ are smooth mappings. We denote by $\tilde{f} := f + g\alpha$ and $\tilde{g} := g\beta$ the vector fields obtained after the application of (α, β).

Definition 2.2. *Two control systems $\Sigma : \dot{x} = f(x) + g(x)u$ and $\hat{\Sigma} : \dot{\hat{x}} = \hat{f} + \hat{g}\hat{u}$, are* feedback equivalent *on an open set $U \subseteq \mathbb{R}^n$ if there exist a regular static feedback (α, β) defined on U and a diffeomorphism $\Xi \in \mathrm{Diff}(U)$ such that*

$$\hat{f} = \Xi_\star(f + g\alpha), \quad \hat{g} = \Xi_\star(g\beta)$$

on U.

On a manifold M, $\mathsf{V}(M)$ will denote the set of all smooth vector fields on M and $C^\infty(M)$ the ring of smooth real-valued functions on M. Given $v \in \mathsf{V}(M)$, $\phi^v_t(p)$ denotes the solution of $\dot{x} = v(x)$ with initial condition $x(0) = p$ at time t. A closed connected set $N \subset M$ is said to be invariant for $f \in \mathsf{V}(M)$ if

$$(p_0 \in N) \Rightarrow (\forall t \in \mathbb{R})(\phi^f_t(p_0) \in N).$$

A closed connected set $N \subset \mathbb{R}^n$ is called controlled invariant for (2.1) if there exists a smooth feedback smooth $\overline{u} : N \to \mathbb{R}^m$ making N an invariant set for the closed-loop system. Following [28], we denote the class of closed, connected, embedded submanifolds of $\mathbb{R}^n$ which are controlled invariant for (2.1) by $\mathscr{I}(f, g, \mathbb{R}^n)$. If $N \in \mathscr{I}(f, g, \mathbb{R}^n)$, we write $\mathscr{F}(f, g, N)$ for the collection of maps that render N controlled invariant, i.e. maps $\overline{u} : N \to \mathbb{R}^m$ such that $f + g\overline{u}$ is tangent to N, i.e.,

$$(f + g\overline{u})|_N : N \to TN.$$

If $f \in \mathsf{V}(M)$ and $\lambda \in C^\infty(M)$ then

$$L_f\lambda(p) = \lim_{h\to 0} \frac{1}{h}\left[\lambda(\phi_h^f(p)) - \lambda(p)\right]$$

is the Lie derivative of λ with respect to f at p, it is also an element of $C^\infty(M)$. If $f, g \in \mathsf{V}(M)$ then the Lie bracket of f and g is defined by the following relation

$$L_{[f,g]}\lambda = L_f(L_g\lambda) - L_g(L_f\lambda), \quad \forall\lambda \in C^\infty(M).$$

We will use the standard notation for iterated Lie derivatives and Lie brackets

$$\begin{aligned}
&L_gL_f\lambda := L_g(L_f\lambda),\\
&L_g^0\lambda := \lambda, \quad L_g^k\lambda := L_g(L_g^{k-1}\lambda)\\
&ad_f^0 g := g, \quad ad_f^k g := \left[f, ad_f^{k-1} g\right], \quad k \geq 1.
\end{aligned}$$

The set $\mathsf{V}(M)$ can be equipped with different algebraic structures. For our purposes it suffices to consider $\mathsf{V}(M)$ as either (i) a vector space (infinite dimensional) over $\mathbb{R}$ which, when endowed with the Lie bracket $[\ ,\ ] : \mathsf{V}(M) \times \mathsf{V}(M) \to \mathsf{V}(M)$, becomes a Lie algebra, or (ii) a module over the ring $C^\infty(M)$. Given a Lie algebra $\mathfrak{g}$, a subset $\mathfrak{h} \subset \mathfrak{g}$ is called a subalgebra if $h_1, h_2 \in \mathfrak{h}$ implies $[h_1, h_2] \in \mathfrak{h}$.

Finally, if $A \subset M$ is any subset, then a smooth map $r : M \to A$ such that $r|_A = 1_A$, where 1_A is the identity map on A, is called a smooth retraction of M onto A. The following lemma regarding retractions is a simpler, local version, of the Tubular Neighborhood Theorem [5].

LEMMA 2.3. *Let $N \subset \mathbb{R}^n$ be an $n^\star$-dimensional submanifold of $\mathbb{R}^n$. Then, for every $p \in N$ there exist a neighborhood U of p in $\mathbb{R}^n$ and a smooth retraction $r : U \to N \cap U$.*

2.2. Vector Bundles. The discussion, terminology and notation of this section is standard and can be found in [6] or [25]. A smooth n-dimensional vector bundle (or n-plane bundle) $\xi = (\pi, E, B)$ can be thought of as[1] a family $\{E_p\}_{p\in B}$ of disjoint n-dimensional vector spaces parameterized by a space B. The union of these vector spaces is the space E and B is called the base space. The map $\pi : E \to B$, $E_p \mapsto p$ is a smooth surjective submersion and is called the vector bundle projection. Moreover, ξ is "locally trivial" in the sense that, locally (with respect to B), E looks like a product with $\mathbb{R}^n$: for each $p \in B$, there is a neighborhood U of p and a diffeomorphism $t : \pi^{-1}(U) \to U \times \mathbb{R}^n$, $v_q \mapsto (t_1(q), t_2(q)v)$, which is an isomorphism from each fibre $\pi^{-1}(q)$ onto $q \times \mathbb{R}^n$ for each $q \in U$. This property is exhibited by the commutative diagram below where $\tilde{\pi}(q, v) = q$.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "t"] \arrow[d, "\pi"'] & U \times \mathbb{R}^n \arrow[dl, "\tilde{\pi}"] \\
U &
\end{tikzcd}$$

[1]The discussion here is informal. In particular we define a vector bundle as a triple (π, E, B) where more formally one defines a vector bundle as a 5-tuple by augmenting the above triple with two operations $\oplus$ and $\otimes$. See [6], [25] for more details.

The pair $\big(\pi^{-1}(U), t\big)$ is called a vector bundle chart with domain U and dimension n. The collection of all vector bundle charts of ξ is a vector bundle atlas. A vector bundle is a manifold in its own right with an atlas of compatible vector bundle charts. We will usually refer to a vector bundle as simply ξ, or $\pi : E \to B$ or even denote the bundle by E alone.

If $A \subset B$ is any subset and $\xi = (\pi, E, B)$, then we denote $\pi^{-1}(A)$ by $E|_A$ the restriction of ξ to A. It is a well-defined vector bundle. A subbundle of the bundle $\xi = (\pi, E, B)$ is a bundle $\xi_0 = (\pi_0, E_0, B)$, over the same base space B such that $E_0 \subset E$, and $\pi|_{E_0} = \pi_0$. Additionally, there must exist a vector bundle atlas $\mathbf{\Phi}$ for ξ such that if $\big(\pi^{-1}(U), t\big) \in \mathbf{\Phi}$, the following diagrams commute

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "t"] \arrow[d, "\pi"'] & U \times \mathbb{R}^k \times \mathbb{R}^{n-k} \arrow[dl, "\tilde{\pi}"] \\ U & \end{tikzcd} \qquad \begin{tikzcd} \pi_0^{-1}(U) \arrow[r, "t"] \arrow[d, "\pi_0"'] & U \times \mathbb{R}^k \times \{0\} \arrow[dl, "\tilde{\pi}"] \\ U & \end{tikzcd}$$

The prime example of a vector bundle is the tangent bundle $\pi : TM \to M$ of a manifold M. If $N \subset M$ is a submanifold of M, then TN is a subbundle of $TM|_N$

$$TN = \{v_x \in TM|_N : v(x) \in T_xN\}.$$

The algebraic normal bundle of N in M is the subbundle over N whose fibres are the quotient spaces T_xM/T_xN. It is denoted $TM|_N /TN$.

Let $\xi = (\pi, E, B)$ be a smooth vector bundle. A C^∞ inner product or orthogonal structure on ξ is a family $\{\alpha_p\}_{p\in B}$ where each α_p is an inner product on E_p and the map $(p, y, z) \mapsto \alpha_p(y, z)$ defined on $\{(p, y, z) \in B \times E \times E : p = \pi(y) = \pi(z)\}$ is C^∞. The pair (ξ, α) is called an orthogonal vector bundle. If M is a manifold, a C^∞ orthogonal structure on TM is called a Riemannian metric. In this paper, orthogonal structures will always arise in subbundles of $T\mathbb{R}^n|_V$, where V is a submanifold of $\mathbb{R}^n$, whereby the standard inner product on $\mathbb{R}^n$ is used. Suppose (ξ, α) is an orthogonal vector bundle. If y, z are in the same fibre E_p, we write $\langle y, z\rangle$ or $\langle y, z\rangle_p$ for $\alpha_p(y, z)$. If $\xi = (\pi^0, E^0, B) \subset \eta = (\pi, E, B)$ is a subbundle, the orthogonal complement $\xi^\perp \subset \eta$ is the subbundle defined fibre-wise as

$$(\xi^\perp)_p = (\xi_p)^\perp = \{y \in E_p : \langle y, z\rangle = 0,\ z \in E_p^0\}.$$

Note that $\xi^\perp$ is isomorphic to η/ξ. Of particular interest to us will be the case when $N \subset M$ is a submanifold and M has a Riemannian metric. In this case $TN^\perp \subset TM|_N$ is called the geometric normal bundle of N in M.

2.3. Distributions. A smooth distribution D on a manifold M is an assignment to each $p \in M$ of a subspace $D(p) \subseteq T_pM$ which varies smoothly as p varies. Locally, a smooth distribution is spanned by a collection of smooth vector fields which are called local generators. A point $p \in M$ is a regular point of D if there exists a neighborhood U containing p for which $\dim(D(q))$ is constant for all $q \in U$. In this case D is said to be non-singular on U. If p is a regular point of a distribution D with $\dim D(p) = d$, then there exist an open neighborhood U^0 of p and d smooth local generators $f_1, \ldots, f_d$ defined on U^0 such that for each $q \in U^0$, $D(q) = \mathrm{span}\{f_1(q), \ldots, f_d(q)\}$. We will write $D = \mathrm{span}\{f_1, \ldots, f_d\}$ when such a finite set of local generators exist.

A non-singular distribution can be viewed as a subbundle of TM. As such, when TM has an orthogonal structure, we will use the notation $D^\perp$ to indicate the orthogonal complement of D in TM. This stands in contrast to the notation $\operatorname{ann}(D)$ which we use to denote the annihilator of D contained in $TM^\star$, the cotangent bundle. If D is a distribution defined on a manifold M and $N \subset M$ is a submanifold we will at times consider the subbundles $TN + D$ and $TN \cap D$ of $TM|_N$ defined fibre-wise, for each $p \in N$, by $T_pN + D(p)$ and $T_pN \cap D(p)$, respectively. The following fact is used in the sequel.

LEMMA 2.4. *Let $N \subset M$ be a C^∞-submanifold of the C^∞-manifold M. Let $p \in N$ be a regular point of a C^∞-distribution D on M. Suppose there exists an open neighborhood V in N such that $\dim(T_pN \cap D(p))$ is constant for all $p \in V$. Then there exists a neighborhood U of p in V such that $TN \cap D$ and $(TN \cap D)^\perp$ are smooth in U.*

Given a smooth distribution D on M, we denote by $\bar{D}$ the involutive closure of D, i.e., the intersection of all involutive distributions containing D. We denote by $\operatorname{Lie}(D)$ the smallest subalgebra of $\mathsf{V}(M)$ containing the vector fields in D. We will use $\operatorname{Lie}_{C^\infty(M)}(D)$ to denote the smooth distribution spanned by vector fields in $\operatorname{Lie}(D)$. Then, $\operatorname{Lie}_{C^\infty(M)}(D) \subseteq \bar{D}$ and generally the inclusion is proper.

If Δ and Λ are distributions and f is a vector field, then we use the following notation

$$\begin{aligned} [\Delta, \Lambda] &= \operatorname{span}\{[X, Y] : X \in \Delta, Y \in \Lambda\} \\ [f, \Delta] &= \operatorname{span}\{[f, \tau] : \tau \in \Delta\}. \end{aligned}$$

A smooth distribution Δ is invariant under a vector field f if $[f, \Delta] \subseteq \Delta$. A distribution Δ defined on an open set U is called locally controlled invariant for the dynamics (2.1) if for each $x_0 \in U$ there exist a neighborhood U^0 of x_0 and a regular static feedback (α, β) on U^0 such that

$$\begin{aligned} &\left[\tilde{f}, \Delta\right] \subseteq \Delta \\ &[\tilde{g}_i, \Delta] \subseteq \Delta, \quad i \in \{1, \ldots, m\}, \end{aligned}$$

where $\tilde{f} = f + g\alpha$ and $\tilde{g} = g\beta$.

THEOREM 2.5 ([7], [9], [10], [20], [21]). *Let Δ be an involutive distribution. Suppose Δ, and $\Delta + \operatorname{span}\{g_1, \ldots, g_m\}$ are nonsingular on U. Then Δ is locally controlled invariant for the dynamics* (2.1) *if and only if*

$$\begin{aligned} [f, \Delta] &\subseteq \Delta + \operatorname{span}\{g_1, \ldots, g_m\} \\ [g_i, \Delta] &\subseteq \Delta + \operatorname{span}\{g_1, \ldots, g_m\}, \quad i \in \{1, \ldots, m\}. \end{aligned}$$

3. Local Transverse Feedback Linearization Problem and Solution. In this section we present the main problem studied in this paper. Suppose we are given a pair $(\Gamma^\star, u^\star)$, where $\Gamma^\star \in \mathscr{I}(f, g, \mathbb{R}^n)$, $\dim \Gamma^\star = n^\star$, and $u^\star \in \mathscr{F}(f, g, \Gamma^\star)$. In this presentation we consider the controlled invariant set $\Gamma^\star$ as a given data. For example in a mechanical system this might correspond to a motion planning task being solved in order to obtain the shortest path between two given points. In most situations, however, one is given a set Γ, perhaps defined by virtual constraints or design goals, and then one must pare away pieces of Γ until all which remains is the largest controlled invariant submanifold $\Gamma^\star$ *contained* in Γ. Various tools in the literature exist for this purpose (see for instance the zero dynamics algorithm [9] or the constrained dynamics algorithm [21]). Generally, the existing tools generate a local characterization of $\Gamma^\star$ about the initialization point of the algorithms. In some cases, viability theory [1] can be used to obtain global characterizations of invariant sets for dynamical systems. The main problem investigated in this paper, stated next, concerns the decomposition of the system dynamics into a subsystem describing the motion on $\Gamma^\star$ and one describing the motion transversal to $\Gamma^\star$, with the essential requirement that the transversal subsystem be feedback linearizable.

Local Transverse Feedback Linearization Problem (LTFLP): Given a pair $(\Gamma^\star, u^\star) \in \mathscr{I}(f, g, \mathbb{R}^n) \times \mathscr{F}(f, g, \mathbb{R}^n)$ and a point $p_0 \in \Gamma^\star$ find, if possible, a neighborhood U of p_0 in $\mathbb{R}^n$, a transformation $\Xi \in \mathrm{Diff}(U)$, $\Xi : U \to \mathbb{R}^{n^\star} \times \mathbb{R}^{n-n^\star}$, $x \mapsto (z, \xi)$, and a feedback transformation (α, β), such that (2.1) is feedback equivalent on U to

$$\begin{aligned} \dot{z} &= f^0(z,\xi) + g^1(z,\xi)v_1 + g^2(z,\xi)v_2 \\ \dot{\xi} &= A\xi + Bv_1 \end{aligned} \tag{3.1}$$

where $v = \mathrm{col}(v_1, v_2) \in \mathbb{R}^m$, B is full rank, the pair (A, B) is controllable, and $\Xi(\Gamma^\star \cap U) = \{(z, \xi) \in \mathbb{R}^{n^\star} \times \mathbb{R}^{n-n^\star} : \xi = 0\}$.

In words, we seek to characterize conditions under which (2.1) is feedback equivalent to a system whose dynamics transversal to the set $\Gamma^\star$ are linear, time-invariant and controllable. LTFLP asks for a coordinate and feedback transformation valid on U which generates a normal form with two types of decompositions. On the one hand, system dynamics near $\Gamma^\star \cap U$ are decomposed into a tangential subsystem, the z-dynamics and a transversal subsystem, the ξ-dynamics. On the other hand, the original m control inputs are decomposed into transversal and tangential components v_1 and v_2, respectively.

The terminology "transverse feedback linearization" should not be mistaken for "transverse linearization," a technique consisting in the Jacobian linearization of the dynamics transversal to a periodic orbit. In the special case when $\Gamma^\star$ is a periodic orbit, the notions of "transverse feedback linearization" and "transverse linearization" differ similarly to the way that "feedback linearization" around an equilibrium differs from "linearization" around the equilibrium.

Transverse feedback linearization finds application in the stabilization of $\Gamma^\star$. For, if a transversal controller v_1 is designed that stabilizes $\xi = 0$, and the trajectories of the closed loop system are bounded, then the controller stabilizes $\Gamma^\star$ in original coordinates. If, on the other hand, the trajectories of the closed-loop system are not all bounded, then stabilization of $\xi = 0$ implies the stabilization of $\Gamma^\star$ in original coordinates if there exists a class-$\mathcal{K}$ function α such that $\|\xi(x)\| \geq \alpha(\|x\|_{\Gamma^\star})$, where $\|x\|_{\Gamma^\star}$ is the point-to-set distance of a point x to the set $\Gamma^\star$, defined as $\|x\|_{\Gamma^\star} := \inf_{p \in \Gamma^\star} \|x - p\|$. Hereafter, we assume that the preliminary regular feedback $(u^\star, I_m)$ is applied to (2.1) so that $f|_{\Gamma^\star}$ is tangent to $\Gamma^\star$. Next, we present a technical result which is useful in proving the main theorem.

THEOREM 3.1. *LTFLP is solvable if and only if there exist ρ_0 smooth $\mathbb{R}$-valued functions $\alpha_1, \ldots, \alpha_{\rho_0}$, defined on an open neighborhood U of p_0 in $\mathbb{R}^n$, such that*

(1) $U \cap \Gamma^\star \subset \{x \in U : \alpha_i(x) = 0,\ i = 1, \ldots, \rho_0\}$

(2) The system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ y' &= \mathrm{col}(\alpha_1(x), \ldots, \alpha_{\rho_0}(x)) \end{aligned} \tag{3.2}$$

has vector relative degree $\{k_1, \ldots, k_{\rho_0}\}$ with $k_1 + \cdots + k_{\rho_0} = n - n^\star$ at p_0.

Moreover, the zero dynamics $\mathcal{Z}^\star$ *of* (3.2) *coincide with* $\Gamma^\star$ *on* U*:* $\mathcal{Z}^\star \cap U = \Gamma^\star \cap U$.
We omit the proof of Theorem 3.1 and instead refer the reader to [19, Theorem 4.1] whose proof is identical. System (3.2) has m inputs and ρ_0 outputs and hence is not square. The notion of vector relative degree of a non-square system is the same as that of a square system given in Section 5.1 of [9], with the difference that the $\rho_0 \times m$ decoupling matrix $A(x)$ with components $a_{ij}(x) = L_{g_j} L_f^{k_i - 1} \alpha_i(x)$ is assumed to be full-rank, rather than non-singular, at p_0.

In Section 4 we give the coordinate-free definition of transverse controllability indices. It turns out (see Lemma 4.3) that $\{k_1, \dots, k_{\rho_0}\}$ in Theorem 3.1 are precisely the transverse controllability indices of (2.1) with respect to $\Gamma^\star$.

Theorem 3.1 characterizes the solvability of LTFLP in terms of the existence of a virtual output function $\alpha : U \to \mathbb{R}^{\rho_0}$ satisfying (1) and (2). Once the output function is known, a coordinate and feedback transformation yielding (3.1) is found constructively using [9, Proposition 5.1.2] and additional elementary manipulations. The theorem also shows that LTFLP is equivalent to the *zero dynamics assignment problem with relative degree* mentioned in the introduction. For, the theorem states that LTFLP is solvable if and only if $\Gamma^\star$ can be made into the zero dynamics manifold of (2.1) induced by a suitable output yielding a well-defined vector relative degree. On the other hand, Theorem 3.1 does not give any way of finding the output function or even to determine whether it exists. Hence, it has limited value for constructing the coordinate and feedback transformation.

Consider the distributions

$$G_i := \operatorname{span}\{ad_f^j g_k : 0 \le j \le i,\ 1 \le k \le m\}, \tag{3.3}$$

and recall from Section 2 that $\bar{G}_i$ denotes the involutive closure of G_i. Now the main result of this paper.

THEOREM 3.2 (Main Result). *Suppose that* $\bar{G}_i$, $i \in \mathbf{n} - \mathbf{n}^\star - \mathbf{1}$ *are regular at* $p_0 \in \Gamma^\star$. *Then, LTFLP is solvable at* p_0 *if and only if*

(a) $\dim\left(T_{p_0}\Gamma^\star + G_{n-n^\star-1}(p_0)\right) = n$

and there exists an open neighborhood U *of* p_0 *in* $\mathbb{R}^n$ *such that for all* $i \in \mathbf{n} - \mathbf{n}^\star - \mathbf{1}$

(b) $(\forall p \in \Gamma^\star \cap U)\ \dim(T_p\Gamma^\star + G_i(p)) = \dim(T_p\Gamma^\star + \bar{G}_i(p)) =$ *constant.*

It is useful to specialize Theorem 3.2 to the case when $\Gamma^\star$ is an equilibrium, because in this special case LTFLP coincides with the state-space exact linearization problem whose solution was given in [8, 12].

COROLLARY 3.3. *Assume that* $\Gamma^\star = \{p_0\}$ *is an equilibrium point of the open-loop system* $\dot{x} = f(x)$ *and that* $\bar{G}_i$, $i \in \mathbf{n} - \mathbf{1}$, *are regular at* p_0. *Then, LTFLP is solvable at* p_0 *if and only if*

(a) $\dim G_{n-1}(p_0) = n$

(b′) G_i, $i \in \mathbf{n} - \mathbf{1}$, *are involutive and regular at* p_0.

Proof. It suffices to show that, under the assumption that the distributions $\bar{G}_i$ are regular, (b') is equivalent to condition (b) in Theorem 3.2. Assume that (b) holds, i.e., $G_i(p_0) = \bar{G}_i(p_0)$. For all p in a neighborhood of p_0, one has

$$\dim(G_i(p_0)) \le \dim(G_i(p)) \le \dim(\bar{G}_i(p)) = \dim(\bar{G}_i(p_0)) = \dim(G_i(p_0)),$$

and so all inequalities above are equalities. Therefore, $\dim(G_i(p)) = \dim(G_i(p_0))$ and $\dim(G_i(p)) = \dim(\bar{G}_i(p))$, proving that $(b) \implies (b')$. The converse implication is obvious. ∎

We recall the classical result

THEOREM 3.4 (State-Space Exact Linearization [8, 12]). *Assume that $\Gamma^\star = \{p_0\}$ is an equilibrium point of the open-loop system $\dot{x} = f(x)$. Then, LTFLP is solvable at p_0 if and only if conditions (a) and (b') in Corollary 3.3 hold.*

Note that conditions (a) and (b') in Corollary 3.3 imply that the distributions $\bar{G}_i$, $i \in \mathbf{n-1}$, are regular at p_0. Thus, in the special case when $\Gamma^\star$ is an equilibrium point, the conditions of our main result coincide with those of the state-space exact linearization problem. The difference between Corollary 3.3 and Theorem 3.4 is that the former relies on the *preliminary assumption* that the distributions $\bar{G}_i$ are regular at p_0, while the latter shows that regularity of $\bar{G}_i$ at p_0 is actually necessary for the solvability of LTFLP, and hence there is no need to impose it as preliminary requirement.

The assumptions of Theorem 3.2 are checkable, however its proof does not provide a constructive procedure for finding the virtual outputs described in Theorem 3.1. The next result sheds additional light on LTFLP by relating it to the partial feedback linearization problem. The result isn't a viable solution to LTFLP because its assumptions are not checkable. On the other hand, the theorem provides guidelines for finding the output function in Theorem 3.1, as discussed below.

THEOREM 3.5. *Suppose that $\bar{G}_i$, $i \in \mathbf{n} - \mathbf{n}^\star - \mathbf{1}$ are regular at $p_0 \in \Gamma^\star$. Then, LTFLP is solvable at p_0 if and only if there exist a neighborhood U of p_0 and a smooth, involutive, and regular distribution Δ on U such that*

(i) $\Delta|_{\Gamma^\star} = T\Gamma^\star$.

(ii) *Δ is locally controlled invariant under (2.1).*

(iii) $(\forall\ p \in \Gamma^\star \cap U)\ \dim\left(T_p\Gamma^\star + G_{n-n^\star-1}(p)\right) = n$.

(iv) *$(\forall\ i \in \mathbf{n} - \mathbf{n}^\star - \mathbf{1})$ $\Delta + G_i$ is regular and involutive on U.*

Proof. Suppose that LTFLP is solvable at p_0. The necessity of conditions (i) - (iv) can be easily shown by considering the normal form (3.1) and taking

$$\Delta = \operatorname{span}\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n^\star}}\right\}. \tag{3.4}$$

Conversely, suppose conditions (i)-(iv) hold. These conditions imply the conditions of [11, Theorem 2.1]. In particular, condition (iv) implies conditions (a) and (b) of [11, Theorem 2.1]. Therefore, by [11, Theorem 2.1] we obtain a system whose dynamics in transformed coordinates is given by (3.1) and where Δ is given by (3.4). The integral submanifolds of Δ foliate a neighborhood U of p_0 and are locally given by the sets $\{(z,\xi) : \xi = \xi^0 = \text{constant}\}$. Condition (i) means that one of the leaves of the foliation is precisely $\Gamma^\star \cap U$. Without loss of generality this leaf is taken as the zero level set $\{(z,\xi) : \xi = 0\}$. ◻

Note that the distribution Δ in Theorem 3.5 is not unique. Also note that this theorem involves an interaction between the concepts of controlled invariant distributions and controlled invariant manifolds. Together, Theorems 3.1, 3.2 and 3.5 can be used to find solutions to LTFLP. The following steps outline the typical procedure one may follow in searching for the output function.

1. Represent $\Gamma^\star$ in a neighborhood of p_0 as the zero level set of $n - n^\star$ $\mathbb{R}$-valued functions. Using Theorem 3.1, check if there exists a subset of ρ_0 of these functions, with ρ_0 defined in (4.1), yielding the correct vector relative degree.
2. If the above step fails, check the conditions of Theorem 3.2 to verify whether or not the problem is solvable. In simple cases, the procedure described in the proof of Theorem 3.2 may yield the desired output functions.
3. If Theorem 3.2 establishes that the problem is solvable, then there exists a

distribution Δ satisfying the conditions in Theorem 3.5. If Δ is found then, after computing the controllability indices defined in (4.2), the output functions are obtained by finding those exact one-forms that span the codistributions $\operatorname{ann}(\Delta + G_i)$, $i = 0, \ldots, k_1 - 2$, and arranging them in the order illustrated below, with the integers ρ_i defined in (4.1).

$$
\begin{array}{c|lllll}
\operatorname{ann}(\Delta + G_{k_1-2}) & d\alpha_1 & & & & \\
\vdots & \vdots & & & & \\
\operatorname{ann}(\Delta + G_{k_1-2-j}) & dL_f^j\alpha_1 & & & & \\
\vdots & \quad\vdots & & & & \\
\operatorname{ann}(\Delta + G_{k_2-2}) & dL_f^{k_1-k_2}\alpha_1 & d\alpha_2 & & & \\
\vdots & \quad\vdots & \vdots & & & \\
\operatorname{ann}(\Delta + G_{k_2-2-j}) & dL_f^{k_1-k_2+j}\alpha_1 & dL_f^j\alpha_2 & & & \\
\vdots & \quad\vdots & \quad\vdots & & & \\
\operatorname{ann}(\Delta + G_{k_3-2}) & dL_f^{k_1-k_3}\alpha_1 & dL_f^{k_2-k_3}\alpha_2 & d\alpha_3 & & \\
\vdots & \quad\vdots & \quad\vdots & \vdots & & \\
\operatorname{ann}\left(\Delta + G_{k_{\rho_0}-2}\right) & dL_f^{k_1-k_{\rho_0}}\alpha_1 & dL_f^{k_2-k_{\rho_0}}\alpha_2 & \cdots & d\alpha_{\rho_0} \\
\vdots & \quad\vdots & \quad\vdots & & \quad\vdots \\
\operatorname{ann}(\Delta + G_0) & dL_f^{k_1-2}\alpha_1 & dL_f^{k_2-2}\alpha_2 & \cdots & dL_f^{k_{\rho_0}-2}\alpha_{\rho_0}.
\end{array}
$$

In each row of the above table, the codistribution in the left column is locally spanned by all of the differentials in that row *plus* all the differentials in the rows above. For example, locally we have that

$$\operatorname{ann}(\Delta + G_{k_2-2}) = \operatorname{span}\{d\alpha_1, dL_f\alpha_1, \ldots, dL_f^{k_1-k_2}\alpha_1, d\alpha_2\}.$$

The functions $\alpha_1, \ldots, \alpha_{\rho_0}$, resulting from the integration of the exact one forms $d\alpha_1, \ldots, d\alpha_{\rho_0}$ along the diagonal of the table, are the required outputs.

Next we present an example to illustrate the use of Theorems 3.1, 3.2 and 3.5.

Example. Consider the system

$$\dot{x} = \begin{bmatrix} -x_2 \\ x_1 \\ x_3x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{bmatrix} u_2$$

along with the pair $(\Gamma^\star, u^\star) \in \mathscr{I}(f, g, \mathbb{R}^4) \times \mathscr{F}(f, g, \mathbb{R}^4)$

$$\begin{aligned}
\Gamma^\star &= \left\{x \in \mathbb{R}^4 : x_1^2 + x_2^2 - x_3 = x_4 = 0\right\} \\
u^\star &= \operatorname{col}(0, 0).
\end{aligned}$$

The set $\Gamma^\star$ is an elliptic paraboloid embedded in the subspace $\{x \in \mathbb{R}^4 : x_4 = 0\}$. We want to perform transverse feedback linearization of (3) with respect to $\Gamma^\star$ near $p_0 = \operatorname{col}(4, 0, 2, 0)$. In this example $n = 4$ and $n^\star = 2$ so we seek to feedback linearize a subsystem of dimension $n - n^\star = 2$. The natural approach to solving this problem is to check if one of the two constraints which define $\Gamma^\star$ satisfy the conditions of Theorem 3.1. In this case, both constraints, taken individually as scalar outputs, yield a well-defined relative degree near p_0 of 1 which does not equal $n-n^\star$. Taken together, as a vector output, the constraints do not yield a well-defined vector relative degree. In both cases, the conditions of Theorem 3.1 are not satisfied by these constraints. Next, we check whether or not LTFLP is solvable for (3) using Theorem 3.2. Checking condition (a) one finds

$$\dim\left(T_{p_0}\Gamma^\star + G_1(p_0)\right) = 4.$$

Also, since $[g_1, g_2] = 0$, it follows that $G_0 = \bar{G}_0$ everywhere. It is then an easy matter to check that for any $p \in \Gamma^\star$, $\dim\left(T_p\Gamma^\star + G_0(p)\right) = 3$. Thus condition (b) holds and LTFLP is solvable despite the fact that the constraints which locally define $\Gamma^\star$ do not satisfy Theorem 3.1.

The fact that Theorem 3.2 holds for this system implies that there exists a distribution Δ satisfying Theorem 3.5. After some trial and error, one finds that the distribution

$$\Delta = \operatorname{span}\left\{ \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ 2x_3 \\ 0 \end{bmatrix} \right\}$$

satisfies the conditions of Theorem 3.5. Then, by Frobenius theorem, there exists an exact one-form $d\alpha$ which spans $\operatorname{ann}(\Delta + G_0)$. The corresponding function α is given by

$$\alpha(x) = \ln\left(\frac{x_3}{x_1^2 + x_2^2}\right) - x_4$$

which yields a well-defined relative degree of 2 at p_0 and satisfies Theorem 3.1. As a result, the coordinate transformation $\Xi : x \mapsto \operatorname{col}(z_1, z_2, \xi_1, \xi_2)$ defined as $z_1 = x_1$, $z_2 = x_2$, $\xi_1 = \alpha$, $\xi_2 = L_f\alpha = x_4$, directly yields the normal form (3.1), with $v_1 = u_1$ and $v_2 = u_2$. The coordinate transformation $\Xi(x)$ is valid on $\mathbb{R}^4 \setminus \big(\{x \in \mathbb{R}^4 : x_1 = x_2 = 0\} \cup \{x \in \mathbb{R}^4 : x_3 = 0\}\big)$.

4. Transverse Controllability Indices and Preliminary Results. In linear systems theory, controllability indices, see [4] and [29], describe certain properties which are invariant under coordinate and nonsingular feedback transformations and serve to categorize controllable linear systems. Controllability indices have been ported to the nonlinear setting. They have been used to characterize the largest feedback linearizable subsystem of a nonlinear system [15], and conditions under which a system is feedback linearizable, see [17], [21]. Here, we adapt these ideas to the framework of transverse feedback linearization. Let V be an open subset of $\Gamma^\star$. For each $p \in V$, let

$$\begin{aligned} \rho_0(p) &:= \dim(T_p\Gamma^\star + G_0(p)) - n^\star \\ \rho_i(p) &:= \dim(T_p\Gamma^\star + \bar{G}_{i-1}(p) + ad_f^i G_0(p)) - \dim(T_p\Gamma^\star + \bar{G}_{i-1}(p)), \end{aligned} \tag{4.1}$$

$i = 1, 2, \ldots$, where G_i are defined in (3.3), and

$$ad_f^i G_0 = \operatorname{span}\{ad_f^i X : X \in G_0\},\ i = 0, 1, \ldots.$$

Geometrically, at each $p \in \Gamma^\star$, the integers $\rho_i(p)$ represent the number of linearly independent vectors in $ad_f^i G_0(p)$ which are not in $T_p\Gamma^\star + \bar{G}_{i-1}(p)$. Associated to the list $\{\rho_0(p), \dots, \rho_i(p), \dots\}$ is a set of $\rho_0(p)$ integers, $\{k_1(p), \dots, k_{\rho_0}(p)\}$, which we refer to as the *transverse controllability indices of* (2.1) *with respect to* $\Gamma^\star$, defined as (we omit the argument p)

$$k_i := \operatorname{card}\{\rho_j \geq i,\ j \geq 0\}, \quad i \in \{1, \dots, \rho_0\}. \tag{4.2}$$

Note that $k_1 \geq k_2 \geq \cdots \geq k_{\rho_0}$. We show in Corollary 4.2 that the transverse controllability indices are invariant under coordinate and feedback transformations.

Condition (b) of Theorem 3.2 implies that $\rho_0, \rho_1, \dots, \rho_{n-n^\star-2} =$ constant while condition (a) implies that $\sum_i \rho_i = n - n^\star$. In the special case when $\Gamma^\star$ is an equilibrium point, it is useful to compare our definition of controllability indices with the definition by Marino in [15]. Marino's definition relies on the distributions

$$\begin{aligned}
&\mathscr{G}_f = f + G_0 = \{f + g : g \in G_0\} \\
&\mathscr{G}_i = \mathscr{G}_{i-1} + [\mathscr{G}_f, \mathscr{G}_{i-1}], \quad \mathscr{G}_0 = G_0,\ i = 1, 2, \dots, \\
&\mathscr{S}_i = \bar{\mathscr{G}}_{i-1} + ad_f^i G_0, \quad \mathscr{S}_0 = G_0,\ i = 1, 2, \dots.
\end{aligned}$$

and uses the integers

$$\begin{aligned}
r_0 &= \dim \mathscr{G}_0 \\
r_i &= \dim \mathscr{S}_i - \dim \bar{\mathscr{G}}_{i-1}
\end{aligned}$$

in place of the integers ρ_i in the definition of controllability indices. We now show that, when $\Gamma^\star$ is an equilibrium point, the integers ρ_i and r_i are identical and thus the notion of transverse controllability indices reduces to the classical notion of controllability indices.

LEMMA 4.1. *For all non-negative integers i, $\bar{G}_i = \bar{\mathscr{G}}_i$. Thus, when $\Gamma^\star = \{p_0\}$ is an equilibrium point, $\rho_i = r_i$.*

Proof. By definition, $G_0 = \mathscr{G}_0$ so the lemma trivially holds for $i = 0$. We now show that $\bar{G}_i \subseteq \bar{\mathscr{G}}_i$ for all $i \in \mathbb{N}$. By definition,

$$\mathscr{G}_i = \mathscr{G}_{i-1} + [\mathscr{G}_f, \mathscr{G}_{i-1}].$$

Since $f \in \mathscr{G}_f$, it follows that $G_i \subseteq \mathscr{G}_i$ for all nonnegative integers i, which implies $\bar{G}_i \subseteq \bar{\mathscr{G}}_i$.

Next, we show that $\mathscr{G}_i \subseteq \bar{G}_i$ for all $i \in \mathbb{N}$ which implies $\bar{\mathscr{G}}_i \subseteq \bar{G}_i$. To this end, it suffices to prove that $\mathscr{G}_i \subseteq \operatorname{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$ since $\operatorname{Lie}_{C^\infty(\mathbb{R}^n)}(G_i) \subseteq \bar{G}_i$. It is obvious that $\mathscr{G}_0 \subseteq \operatorname{Lie}_{C^\infty(\mathbb{R}^n)}(G_0)$ and

$$\begin{aligned}
\mathscr{G}_1 &= G_0 + [\mathscr{G}_f, G_0] \\
&= \operatorname{span}\{g_1, \dots, g_m, ad_f g_1, \dots, ad_f g_m, [g, g_1], \dots, [g, g_m] : g \in G_0\} \\
&\subseteq \operatorname{Lie}_{C^\infty(\mathbb{R}^n)}(G_1).
\end{aligned}$$

For the induction, assume that, for some positive integer $I \geq 2$,

$$\mathscr{G}_{i-1} \subseteq \operatorname{Lie}_{C^\infty(\mathbb{R}^n)}(G_{i-1}), \quad i \in \{2, \dots, I\}.$$

We must show that $\mathscr{G}_i = \mathscr{G}_{i-1} + [\mathscr{G}_f, \mathscr{G}_{i-1}] \subseteq \operatorname{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$ for $i \in \{2, \dots, I\}$. It is enough to prove that $[\mathscr{G}_f, \mathscr{G}_{i-1}] = [f, \mathscr{G}_{i-1}] + [G_0, \mathscr{G}_{i-1}] \subseteq \operatorname{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$. However, since $[G_0, \mathscr{G}_{i-1}] \subseteq \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_{i-1}) \subseteq \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$, all we are left to show is that $[f, \mathscr{G}_{i-1}] \subseteq \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$.

Let $\tau_1 = g_1,\ \tau_2 = g_2,\ \ldots,\ \tau_{im-1} = ad_f^{i-1} g_{m-1},\ \tau_{im} = ad_f^{i-1} g_m$. Then a general vector field in $\mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_{i-1})$ is a $C^\infty(\mathbb{R}^n)$-linear combination of vector fields of the form

$$\vartheta = \left[\tau_{j_k}, \left[\tau_{j_{k-1}}, \cdots, [\tau_{j_2}, \tau_{j_1}]\right]\right], \tag{4.3}$$

$1 \leq j_k \leq im$, $1 \leq k < \infty$. By assumption, any vector field in $\mathscr{G}_{i-1}$ can also be expressed in this way. Take any vector field $h \in \mathscr{G}_{i-1}$ and consider

$$[f, h] = [f, \sum_{i \in \mathcal{I}} c_i \vartheta_i] = \sum_{i \in \mathcal{I}} [f, c_i \vartheta_i]$$

where $\mathcal{I}$ is some finite index set, $c_i \in C^\infty(\mathbb{R}^n)$, and ϑ_i are of the form (4.3). Each term in the above summation can be expressed as $[f, c_i \vartheta_i] = c_i [f, \vartheta_i] + (L_f c_i)\, \vartheta_i$, so it is enough to show that $[f, \vartheta] \in \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$ where ϑ is of the form (4.3).

When $k = 1$, i.e. $\vartheta = \tau_{j_1}$, then $[f, \tau_{j_1}] \in G_i \subseteq \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$. Next assume that

$$\vartheta = \left[\tau_{j_{k-1}}, \left[\tau_{j_{k-2}}, \cdots, [\tau_2, \tau_1]\right]\right]$$

is such that $[f, \vartheta] \in \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$. We will show that $[f, [\tau_{j_k}, \vartheta]] \in \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$. Clearly, $[\tau_{j_k}, \vartheta] \in \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_{i-1})$ for any $1 \leq j_k \leq im$. By the Jacobi identity

$$[f, [\tau_{j_k}, \vartheta]] = [[\vartheta, f], \tau_{j_k}] + [[f, \tau_{j_k}], \vartheta],$$

and since $[\vartheta, f] \in \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$ and $\tau_{j_k} \in G_{i-1}$, it follows that $[[\vartheta, f], \tau_{j_k}] \in \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$. Also, $[f, \tau_{j_k}] \in G_i$ so that $[[f, \tau_{j_k}], \vartheta] \in \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i)$. This induction argument shows that $[f, \mathscr{G}_{i-1}] \subseteq \mathrm{Lie}_{C^\infty(\mathbb{R}^n)}(G_i) \subseteq \bar{G}_i$ as required. ∎

COROLLARY 4.2. *The transverse controllability indices of system* (2.1) *with respect to a set* $\Gamma^\star$ *are invariant under coordinate and feedback transformations.*

Proof. The push forward map $F_\star$ associated with any $F \in \mathrm{Diff}(U)$ is an isomorphism at each $p \in \Gamma^\star \cap U$. It follows from the definition of the integers $\rho_0, \ldots, \rho_i, \ldots$ that they do not change under coordinate transformations. By Lemma 4.1,

$$\rho_i(p) = \dim(T_p\Gamma^\star + \mathscr{S}_i(p)) - \dim(T_p\Gamma^\star + \bar{\mathscr{G}}_{i-1}(p)).$$

In [15, Proposition 2] it is shown that $\mathscr{S}_i$ and $\bar{\mathscr{G}}_{i-1}$ are feedback-invariant, and so the integers $\rho_0, \ldots, \rho_i, \ldots$ are also invariant under feedback transformations. ∎

LEMMA 4.3. *Suppose that LTFLP is solvable at* $p_0 \in \Gamma^\star$. *Then the transverse controllability indices of* (2.1) *with respect to* $\Gamma^\star$ *coincide with the controllability indices of* (A, B) *in* (3.1).

Proof. This lemma will be proved by direct calculation of the integers ρ_i in (z, ξ) coordinates. Let $V = \Xi(\Gamma^\star \cap U)$. By the properties of the normal form (3.1), for any $\tilde{p} \in \Gamma^\star \cap U$, $\Xi(\tilde{p}) = \mathrm{col}(p, 0)$. Hence, in (z, ξ) coordinates we have that for any $p \in V$ and any $i \in \mathbf{n} - \mathbf{n}^\star$

$$T_pV + G_i(\mathrm{col}(p, 0)) = \mathrm{Im}\left(\begin{bmatrix} I_{n^\star} & \star & \star & \ldots & \star \\ 0_{n-n^\star \times n^\star} & B & AB & \ldots & A^iB \end{bmatrix}\right). \tag{4.4}$$

In (z, ξ) coordinates, consider the collection of constant distributions Δ_i, $i \in \mathbf{n} - \mathbf{n}^\star$, given by

$$\Delta_i = \mathrm{Im}\left(I_{n^\star} \oplus \begin{bmatrix} B & AB & \cdots A^iB \end{bmatrix}\right).$$

At each $p \in V$, $\Delta_i(p) = T_pV + G_i(\mathrm{col}(p,0))$. Furthermore, since each Δ_i is (trivially) involutive and $G_i|_V \subset \Delta_i$, it follows that $\bar{G}_i\big|_V \subseteq \Delta_i$. This shows that for all $i \in \mathbf{n} - \mathbf{n}^\star$

$$TV + \bar{G}_i \subseteq \Delta_i = TV + G_i.$$

On the other hand, $TV + G_i \subseteq TV + \bar{G}_i$ always holds, and so we have shown that $\Delta_i = TV + G_i = TV + \bar{G}_i$. Calculating the integers ρ_i we have

$$\begin{aligned}
\rho_i(p) &= \dim(T_p\Gamma^\star + \bar{G}_{i-1}(p) + ad_f^i G_0(p)) - \dim(T_p\Gamma^\star + \bar{G}_{i-1}(p)) \\
&= \dim(T_p\Gamma^\star + G_{i-1}(p) + ad_f^i G_0(p)) - \dim(T_p\Gamma^\star + G_{i-1}(p)) \\
&= \dim(T_p\Gamma^\star + G_i(p)) - \dim(T_p\Gamma^\star + G_{i-1}(p)) \\
&= \mathrm{rank}(\Delta_i) - \mathrm{rank}(\Delta_{i-1}) \\
&= \mathrm{rank}([B \ \cdots \ A^iB]) - \mathrm{rank}([B \ \cdots A^{i-1}B]).
\end{aligned}$$

The claim follows from the definition of the integers $\{k_1, \dots, k_{\rho_0}\}$. $\square$

When the transverse controllability indices are constant on an open subset of $\Gamma^\star$ and the distributions $\bar{G}_i$ are regular, the next two lemmas establish the existence of a feedback transformation yielding a particularly useful set of local generators for each $\bar{G}_i$.

LEMMA 4.4. *Let $\tilde{U}$ be an open subset of $\mathbb{R}^n$ such that $\tilde{V} := \tilde{U} \cap \Gamma^\star \neq \emptyset$. Assume that, for all $i \in \mathbf{n} - \mathbf{n}^\star$,*

$$\begin{aligned}
&(\forall p \in \tilde{V}) \ \dim(T_p\Gamma^\star + G_i(p)) = \dim(T_p\Gamma^\star + \bar{G}_i(p)) = \ \textit{constant} \\
&(\forall p \in \tilde{U}) \ \dim(\bar{G}_i(p)) = \nu_i = \ \textit{constant.}
\end{aligned}$$

Then, $\rho_0 \geq \rho_1 \geq \cdots \geq \rho_{n-n^\star-1}$ and there exist an open set $U \subseteq \tilde{U}$ and a regular static feedback (α, β) on U such that, letting $V := U \cap \Gamma^\star$, for all $p \in V$ and for all $i \in \mathbf{n} - \mathbf{n}^\star$ the following holds

$$T_p\Gamma^\star + \bar{G}_i(p) = T_p\Gamma^\star \oplus \left(\bigoplus_{j=0}^{i} \mathrm{span}\{ad_{\tilde{f}}^j \tilde{g}_k : 1 \leq k \leq \rho_j\}(p) \right). \tag{4.5}$$

Proof. Choose an open set $U \subseteq \tilde{U}$ such that $V := U \cap \Gamma^\star \neq \emptyset$ and such that V is covered by a coordinate chart in the atlas of $\Gamma^\star$. Apply the preliminary feedback transformation $(u^\star, I_m)$ defined on V. Let $\tilde{f} = f + gu^\star$. On V, define the distribution (i.e. a subbundle of $T\mathbb{R}^n|_V$ defined using the natural orthogonal structure on $\mathbb{R}^n$)

$$\mathcal{G}_0 = [\bar{G}_0 \cap TV]^\perp \cap \bar{G}_0.$$

On V, $\bar{G}_0 \cap TV$ is constant dimensional since

$$\dim(\bar{G}_0 \cap TV) = \dim(TV) + \dim(\bar{G}_0) - \dim(TV + \bar{G}_0).$$

Since $\bar{G}_0$ and TV are regular distributions and their intersection is constant dimensional, it follows from Lemma 2.4 that, by possibly shrinking U (and hence V), $\bar{G}_0 \cap TV$ is smooth and so too is $[\bar{G}_0 \cap TV]^\perp$. Thus, $\mathcal{G}_0$ is the intersection of smooth, regular distributions. Furthermore, $\mathcal{G}_0$ has constant dimension on V since, for each $p \in V$,

$$\begin{aligned}
&\dim(\mathcal{G}_0(p)) \\
&= n - \dim(\bar{G}_0(p) \cap T_pV) + \dim(\bar{G}_0(p)) - \dim\left([\bar{G}_0(p) \cap T_pV]^\perp + \bar{G}_0(p)\right) \\
&= \dim(\bar{G}_0(p)) - \dim(\bar{G}_0(p) \cap T_pV) \\
&= \rho_0.
\end{aligned}$$

Since $\mathcal{G}_0 \subseteq \bar{G}_0$ and $\mathcal{G}_0 \cap TV = (\bar{G}_0^\perp + TV^\perp) \cap (\bar{G}_0^\perp + TV^\perp)^\perp = 0$ we have

$$(\forall p \in V)\ T_pV \oplus \mathcal{G}_0(p) = T_pV + \bar{G}_0(p) = T_pV + G_0(p).$$

By construction, V is covered by a coordinate chart, so there exist $n^\star$ vector fields on V such that at each $p \in V$, $T_pV = \operatorname{span}\{v_1, \ldots, v_{n^\star}\}(p)$. Moreover, by possibly shrinking U (and hence V), there exist ρ_0 vector fields $w_1, \ldots, w_{\rho_0} : V \to T\mathbb{R}^n|_V$ such that $\mathcal{G}_0 = \operatorname{span}\{w_1, \ldots, w_{\rho_0}\}$ so that, on V,

$$TV \oplus \mathcal{G}_0 = \operatorname{span}\{v_1, \ldots, v_{n^\star}\} \oplus \operatorname{span}\{w_1, \ldots, w_{\rho_0}\}.$$

Using the fact that $\mathcal{G}_0 \subset TV + G_0$, we write

$$w_j = \sum_{k=1}^{n^\star} \alpha_k^j v_k + \sum_{k=1}^{m} \beta_k^j g_k, \quad j = 1, \ldots, \rho_0, \tag{4.6}$$

where $\alpha_k^j : V \to \mathbb{R}$, $\beta_k^j : V \to \mathbb{R}$ are $C^\infty(V)$ functions. Let β_0 be the $m \times \rho_0$ matrix of real-valued functions whose (k, j)-th element is β_k^j and let

$$\begin{bmatrix} \tilde{g}_1 & \cdots & \tilde{g}_{\rho_0} \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix} \beta_0.$$

We now show that $\tilde{g}_1, \ldots, \tilde{g}_{\rho_0}$ are linearly independent which implies that β_0 is full rank. Suppose there exist ρ_0 functions $c_i \in C^\infty(V)$ such that $\sum_{i=1}^{\rho_0} c_i \tilde{g}_i = 0$. Then, by (4.6), $\sum_{i=1}^{\rho_0} c_i w_i \in TV$ which implies $c_i = 0$, $i = 1, \ldots, \rho_0$, since $\mathcal{G}_0 \cap TV = 0$. Note that this argument also shows that $\operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_0}\} \cap TV = 0$. For, if this were false, then there would exist a linear combination of the w_i's in (4.6) which, point-wise, belongs to T_pV.

Next, we seek $m - \rho_0$ vector fields $\tilde{g}_{\rho_0+1}, \ldots, \tilde{g}_m$ which belong to TV and such that, on V, $\operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_m\} = G_0$. By possibly shrinking U (and hence V), there exists a set of smooth local generators, $\tilde{g}_{\rho_0+1}, \ldots, \tilde{g}_m$, for $G_0 \cap TV$. We now have the desired decomposition on V

$$TV + G_0 = TV + \bar{G}_0 = TV \oplus \operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_0}\},$$

where, in the new basis for G_0, at each $p \in V$

$$\begin{aligned} &\tilde{g}_1(p), \ldots, \tilde{g}_{\rho_0}(p) \in G_0(p), \\ &\tilde{g}_{\rho_0+1}(p), \ldots, \tilde{g}_m(p) \in T_pV. \end{aligned} \tag{4.7}$$

On V, $\tilde{f}(p) \in T_pV$, so we have that $ad_{\tilde{f}}^j \tilde{g}_k \in TV + \bar{G}_{j-1}$, $\rho_0 + 1 \leq k \leq m$, $j = 0, 1, \ldots$ and hence $\rho_0 \geq \rho_1, \ldots, \rho_{n-n^\star-1}$.

Now we perform the induction step. Assume that, for some positive integer I, and any $i \in \{0, \ldots, I\}$, there exists a basis $\{\hat{g}_1, \ldots, \hat{g}_m\}$ for G_0 such that

$$\text{(a) } TV + \bar{G}_{i-1} = TV \oplus \left(\bigoplus_{j=0}^{i-1} \operatorname{span}\{ad_{\tilde{f}}^j \hat{g}_k : 1 \leq k \leq \rho_j\} \right)$$

$$\text{(b) } (\forall k \in \{\rho_{i-1} + 1, \ldots, m\})\ ad_{\tilde{f}}^{i-1} \hat{g}_k \in TV + \bar{G}_{i-2}.$$

Property (b) implies that $\rho_{i-1} \geq \rho_i, \ldots, \rho_{n-n^\star-1}$. We now seek a basis $\{\tilde{g}_1, \ldots, \tilde{g}_m\}$ for G_0 such that for any $i \in \{0, \ldots, I\}$,

$$\text{(a)}' \;\; TV + \bar{G}_i = TV \oplus \left(\bigoplus_{j=0}^{i} \operatorname{span}\{ad_{\tilde{f}}^j \tilde{g}_k : 1 \leq k \leq \rho_j\} \right)$$

$$\text{(b)}' \;\; (\forall k \in \{\rho_i + 1, \ldots, m\}) \; ad_{\tilde{f}}^i \tilde{g}_k \in TV + \bar{G}_{i-1}.$$

On V, define the distribution

$$\mathcal{G}_i = \left[\bar{G}_i \cap (TV + \bar{G}_{i-1})\right]^\perp \cap \bar{G}_i.$$

Note that $\bar{G}_i \cap (TV + \bar{G}_{i-1})$ is constant dimensional since

$$\dim(\bar{G}_i \cap (TV + \bar{G}_{i-1})) = \dim(\bar{G}_i) + \dim(TV + \bar{G}_{i-1}) - \dim(TV + \bar{G}_i).$$

The distribution $\bar{G}_i$ is regular on U and $TV + \bar{G}_{i-1}$ is constant dimensional on V. Since their intersection is constant dimensional, it follows from Lemma 2.4 that the orthogonal complement of their intersection is smooth, and thus $\mathcal{G}_i$, being the intersection of two smooth and regular distributions, is smooth. Furthermore

$$\begin{aligned} \dim \mathcal{G}_i &= n - \dim(\bar{G}_i) - \dim(TV + \bar{G}_{i-1}) + \dim(TV + \bar{G}_i) \\ &\qquad + \dim(\bar{G}_i) - \dim\left(\left[\bar{G}_i \cap (TV + \bar{G}_{i-1})\right]^\perp + \bar{G}_i \right) \\ &= \dim(TV + \bar{G}_i) - \dim(TV + \bar{G}_{i-1}) \\ &= \rho_i. \end{aligned}$$

By construction, $\mathcal{G}_i \subseteq \bar{G}_i$ and $\mathcal{G}_i \cap (TV + \bar{G}_{i-1}) = 0$ so by dimensionality we have that

$$\left(TV + \bar{G}_{i-1}\right) \oplus \mathcal{G}_i = TV + \bar{G}_i = TV + G_i. \tag{4.8}$$

By possibly shrinking U (and hence V), there exist ρ_i smooth vector fields $w_1, \ldots, w_{\rho_i}$ such that on V, $\mathcal{G}_i = \operatorname{span}\{w_1, \ldots, w_{\rho_i}\}$. Hence, by (4.8) we can write

$$w_j = \bar{w} + \sum_{k=1}^{\rho_{i-1}} \beta_k^j ad_{\tilde{f}}^i \hat{g}_k + \sum_{k=\rho_{i-1}+1}^{m} \beta_k^j ad_{\tilde{f}}^i \hat{g}_k, \quad j \in \{1, \ldots, \rho_i\}, \tag{4.9}$$

where $\bar{w} \in TV + \bar{G}_{i-1}$ and each $\beta_k^j : V \to \mathbb{R}$ is a $C^\infty(V)$ function. By property (b), for all $k \in \{\rho_{i-1} + 1, \ldots, m\}$, $ad_{\tilde{f}}^i \hat{g}_k \in TV + \bar{G}_{i-1}$. Let

$$\hat{w}_j := \sum_{k=1}^{\rho_{i-1}} \beta_k^j ad_{\tilde{f}}^i \hat{g}_k, \; j = 1, \ldots, \rho_i. \tag{4.10}$$

Notice that $\operatorname{span}\{\hat{w}_1, \ldots, \hat{w}_{\rho_i}\} \cap \left(TV + \bar{G}_{i-1}\right) = 0$. For, if this were false, then there would exist a $C^\infty(V)$-linear combination of the w_j which belongs to $TV + \bar{G}_{i-1}$ which, by (4.8), is not possible. Furthermore, $\hat{w}_1, \ldots, \hat{w}_{\rho_i}$ are linearly independent because if there exist ρ_i functions $c_i \in C^\infty(V)$ such that on V $c_1 \hat{w}_1 + \cdots + c_{\rho_i} \hat{w}_{\rho_i} = 0$, then, for some $w^* \in TV + \bar{G}_{i-1}$, $c_1 w_1 + \cdots + c_{\rho_i} w_{\rho_i} - w^* = 0$, thus, $c_1 w_1 + \cdots + c_{\rho_i} w_{\rho_i} = 0$

implying $c_1 = \cdots = c_{\rho_i} = 0$. Now let β_i be the $\rho_{i-1} \times \rho_i$ matrix of smooth functions whose (k,j)-th element is β_k^j obtained from (4.10) so that

$$\left[\begin{array}{ccc} \hat{w}_1 & \cdots & \hat{w}_{\rho_i} \end{array}\right] = \left[\begin{array}{ccc} ad_{\tilde{f}}^i \hat{g}_1 & \cdots & ad_{\tilde{f}}^i \hat{g}_{\rho_{i-1}} \end{array}\right] \beta_i.$$

The vector fields $\hat{w}_1, \ldots, \hat{w}_{\rho_i}$ are linearly independent and are generated as the image of ρ_{i-1} linearly independent vector fields under β_i. Therefore β_i is full rank. We can now write

$$TV + \bar{G}_i = \left(TV + \bar{G}_{i-1}\right) \oplus \operatorname{span}\{\hat{w}_1, \ldots, \hat{w}_{\rho_i}\}.$$

Let

$$\left[\begin{array}{ccc} \tilde{g}_1 & \cdots & \tilde{g}_{\rho_i} \end{array}\right] = \left[\begin{array}{ccc} \hat{g}_1 & \cdots & \hat{g}_{\rho_{i-1}} \end{array}\right] \beta_i.$$

The vector fields $\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_i}\}$ are linearly independent because the vector fields $\{\hat{g}_1, \ldots, \hat{g}_{\rho_{i-1}}\}$ are linearly independent and β_i is full rank. Moreover, since $\operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_i}\} \subseteq \operatorname{span}\{\hat{g}_1, \ldots, \hat{g}_{\rho_{i-1}}\}$ and both are constant dimensional, we can find (making U, and hence V, smaller if necessary) $\rho_{i-1} - \rho_i$ vector fields $\tilde{g}_{\rho_i+1}, \ldots, \tilde{g}_{\rho_{i-1}}$ such that $\operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_{i-1}}\} = \operatorname{span}\{\hat{g}_1, \ldots, \hat{g}_{\rho_{i-1}}\}$ (hence preserving property (a) from the induction assumption) and $ad_f^i \tilde{g}_{\rho_i+1}, \ldots, ad_f^i \tilde{g}_{\rho_{i-1}} \in TV + \bar{G}_{i-1}$. This is done by finding local generators for the smooth distribution

$$\operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_i}\}^{\perp} \cap \operatorname{span}\{\hat{g}_1, \ldots, \hat{g}_{\rho_{i-1}}\}$$

which has constant dimension $\rho_{i-1} - \rho_i$. Finally, let $\tilde{g}_{\rho_{i-1}+1} = \hat{g}_{\rho_{i-1}+1}, \ldots, \tilde{g}_m = \hat{g}_m$. We have therefore obtained a basis $\{\tilde{g}_1, \ldots, \tilde{g}_m\}$ for G_0 in which properties (a)$'$ and (b)$'$ hold. In summary, the induction process gives a basis for G_0 in which the input vector fields are arranged in such a way that for $i \in \mathbf{n} - \mathbf{n}^\star$,

$$\left(TV + \bar{G}_{i-1} + ad_f^i G_0\right) / \left(TV + \bar{G}_{i-1}\right) \simeq \operatorname{span}\{ad_{\tilde{f}}^i \tilde{g}_1, \ldots, ad_{\tilde{f}}^i \tilde{g}_{\rho_i}\}.$$

We are left to show that this arrangement can be achieved using a regular static feedback and that the arrangement is valid on U, and not just V as is presently the case. To this end, let $\tilde{g} = \left[\begin{array}{ccc} \tilde{g}_1 & \cdots & \tilde{g}_m \end{array}\right]$ and define a regular static feedback defined on V by $(\hat{\alpha}, \hat{\beta})$ where $\hat{\alpha} = u^\star$ and $\hat{\beta} = \left(g^\top g\right)^{-1} g^\top \tilde{g}$. To obtain a feedback transformation defined off of $\Gamma^\star$, we can, by possibly shrinking U (and hence V), and applying Lemma 2.3 introduce a retraction $r : U \to V$ of U onto V. Then, let $\alpha = \hat{\alpha} \circ r$ and $\beta = \hat{\beta} \circ r$. The regular static feedback (α, β) has the desired properties. ∎

In order to identify directions in the intersection $T_pV \cap \bar{G}_i(p)$ which are not contained in the intersection $T_pV \cap \bar{G}_{i-1}(p)$, it is useful to define the integers

$$\begin{aligned} \mu_0(p) &:= \dim(T_pV \cap \bar{G}_0(p)) \\ \mu_i(p) &:= \dim(T_pV \cap \bar{G}_i(p)) - \dim(T_pV \cap \bar{G}_{i-1}(p)), \end{aligned}$$

for $i \in \mathbb{N}$, and let

$$n_i(p) := \sum_{j=0}^{i} \mu_j(p),$$

so that $\dim(T_pV \cap \bar{G}_i(p)) = n_i(p)$. Under the assumption of Lemma 4.4, we have that $\dim(T_pV \cap \bar{G}_i(p)) = n^\star + \nu_i - \dim(T_pV + \bar{G}_i(p))$ and hence the μ_i are constant for all $i \in \mathbf{n} - \mathbf{n}^\star$ and we have the following result.

LEMMA 4.5. *Let $\tilde{U}$ be an open subset of $\mathbb{R}^n$ such that $\tilde{V} := \tilde{U} \cap \Gamma^\star \neq \emptyset$. Assume that, for all $i \in \mathbf{n} - \mathbf{n}^\star$, the conditions of Lemma 4.4 hold. Then, there exists an open set $U \subseteq \tilde{U}$ and $n_{n-n^\star-1}$ vector fields $v^j_\ell \in \mathsf{V}(U)$, $0 \le j \le n - n^\star - 1$, $1 \le \ell \le \mu_j$, such that, after the feedback transformation of Lemma 4.4, letting $V := U \cap \Gamma^\star$, and*

$$G_i^{\|} := \operatorname{span}\{v_1^0, \ldots, v_{\mu_0}^0, \ldots\ldots, v_1^i, \ldots, v_{\mu_i}^i\}$$

one has that, for all $i \in \mathbf{n} - \mathbf{n}^\star$, on U,

$$\bar{G}_i = G_i^{\|} \oplus \left(\bigoplus_{j=0}^{i} \operatorname{span}\{ad_{\tilde{f}}^j \tilde{g}_k : 1 \le k \le \rho_j\} \right)$$

$$\text{and} \quad G_i^{\|}\Big|_V = TV \cap \bar{G}_i.$$

Proof. Suppose the feedback transformation (α, β) of Lemma 4.4 has been applied which is valid on $U \subseteq \tilde{U}$ as defined therein. Since every point p of U is a regular point for the distributions $\bar{G}_i$, we can, by possibly shrinking U (and hence V) find a set of local generators $X_1^i, \ldots, X_{\nu_i}^i$ valid on U for $\bar{G}_i$, $i \in \mathbf{n} - \mathbf{n}^\star$.

On V, define the distribution $Q_0 = TV \cap \bar{G}_0$. By assumption, Q_0 has constant dimension μ_0. Moreover, since Q_0 is the intersection of two smooth, regular distributions and constant dimensional, it is, by Lemma 2.4, smooth. By shrinking U (and hence V) we can find a basis such that, on V, $Q_0 = \operatorname{span}\{\hat{v}_1, \ldots, \hat{v}_{\mu_0}\}$. By construction, $Q_0 \subset \bar{G}_0$ so that each $\hat{v}_k \in Q_0$ can be expressed as

$$\hat{v}_k = \sum_{j=1}^{\nu_0} \hat{c}_{j0}^k X_j^0, \quad k \in \{1, \ldots, \mu_0\},$$

where each $\hat{c}_{j0}^k : V \to \mathbb{R}$ is a $C^\infty(V)$ function. Next, we apply Lemma 2.3 and, by possibly shrinking U (and hence V) introduce a retraction $r : U \to V$ of U onto V. Let $c_{j0}^k = \hat{c}_{j0}^k \circ r$ so that

$$v_k^0 := \sum_{i=1}^{\nu_0} c_{i0}^k X_i^0, \quad k \in \{1, \ldots \mu_0\},$$

are now vector fields defined on U and let $G_0^{\|} := \operatorname{span}\{v_1^0, \ldots, v_{\mu_0}^0\}$. It follows that $G_0^{\|} \subset \bar{G}_0$ in U and $G_0^{\|}\Big|_V = Q_0$. By Lemma 4.4, $TV + \bar{G}_0 = TV \oplus \operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_0}\}$ so that $Q_0 \cap \operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_0}\} = 0$. The distribution $\bar{G}_0$ has dimension $\nu_0 = n_0 + \rho_0$ throughout U so that

$$\begin{aligned}
&(\forall p \in U)\ \bar{G}_0(p) \supseteq \operatorname{span}\{v_1^0, \ldots, v_{\mu_0}^0\}(p) + \operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_0}\}(p) \\
&(\forall p \in V)\ \bar{G}_0(p) = \operatorname{span}\{v_1^0, \ldots, v_{\mu_0}^0\}(p) \oplus \operatorname{span}\{\tilde{g}_1, \ldots, \tilde{g}_{\rho_0}\}(p) \\
&\text{where} \quad \operatorname{span}\{v_1^0, \ldots, v_{\mu_0}^0\} \subseteq TV.
\end{aligned}$$

The vector fields $v_1^0, \ldots, v_{\mu_0}^0, \tilde{g}_1, \ldots, \tilde{g}_{\rho_0}$ are linearly independent on V, therefore they remain linearly independent in some open neighborhood of V in $\mathbb{R}^n$, without loss of generality U. Therefore, they are local generators for $\bar{G}_0$ on U. Next, we perform the induction step. Assume that, for some positive integer I, and any $i \in \{0, \ldots, I\}$, on U,

$$\begin{aligned} \bar{G}_{i-1} &= G^{\|}_{i-1} \oplus \left(\bigoplus_{j=0}^{i-1} \mathrm{span}\{ad^j_{\tilde{f}} \tilde{g}_k : 1 \le k \le \rho_j\} \right) \\ \text{and} \quad G^{\|}_{i-1}\Big|_V &= TV \cap \bar{G}_{i-1}. \end{aligned} \tag{4.11}$$

We want to show the existence of μ_i vector fields $v^i_1, \ldots, v^i_{\mu_i}$ such that, for any $i \in \{0, \ldots, I\}$, letting $G^{\|}_i = G^{\|}_{i-1} \oplus \mathrm{span}\{v^i_1, \ldots, v^i_{\mu_i}\}$, one has that, on U,

$$\begin{aligned} \bar{G}_{i} &= G^{\|}_{i} \oplus \left(\bigoplus_{j=0}^{i} \mathrm{span}\{ad^j_{\tilde{f}} \tilde{g}_k : 1 \le k \le \rho_j\} \right) \\ \text{and} \quad G^{\|}_{i}\Big|_V &= TV \cap \bar{G}_{i}. \end{aligned} \tag{4.12}$$

On V, define the distribution Q_i by $Q_i = (TV \cap \bar{G}_i) \cap (TV \cap \bar{G}_{i-1})^{\perp}$. The distribution Q_i is the intersection of two smooth, regular distributions. Furthermore, for all $p \in V$,

$$\begin{aligned} \dim(Q_i(p)) &= \dim(T_pV \cap \bar{G}_i(p)) - \dim(T_pV \cap \bar{G}_{i-1}(p)) \\ &= \mu_i \end{aligned}$$

is constant by assumption, thus, by Lemma 2.4, Q_i is a smooth regular distribution. Locally, by making U (and hence V) smaller if necessary, there exist local generators $\hat{v}_k$, $k \in \{1, \ldots, \mu_i\}$, for Q_i. By construction, $Q_i \subset \bar{G}_i$ so that each $\hat{v}_k \in Q_i$ can be expressed as

$$\hat{v}_k = \sum_{j=1}^{\nu_i} \hat{c}^k_{ji} X^i_j, \quad k \in \{1, \ldots, \mu_i\},$$

where each $\hat{c}^k_{ji} : V \to \mathbb{R}$ is a $C^\infty(V)$ function. Let $c^k_{ji} = \hat{c}^k_{ji} \circ r$ so that

$$v^i_k := \sum_{j=1}^{\nu_i} c^k_{ji} X^i_j, \quad k \in \{1, \ldots, \mu_i\}$$

are vector fields defined on U and let $G^{\|}_{i/i-1} := \mathrm{span}\{v^i_1, \ldots, v^i_{\mu_i}\}$. It follows that $G^{\|}_{i/i-1} \subset \bar{G}_i$ and $G^{\|}_{i/i-1}\Big|_V = Q_i$. By the definition of Q_i and by (4.11) it follows that, $G^{\|}_{i-1}\Big|_V \cap G^{\|}_{i/i-1}\Big|_V = \big(TV \cap \bar{G}_{i-1}\big) \cap Q_i = 0$. Furthermore, since $(TV \cap \bar{G}_{i-1}) \subset (TV \cap \bar{G}_i)$, we have that $TV \cap \bar{G}_i = G^{\|}_{i-1} \oplus G^{\|}_{i/i-1} =: G^{\|}_i$. In addition, by Lemma 4.4, on V,

$$G^{\|}_i \cap \left(\bigoplus_{j=0}^{i} \mathrm{span}\{ad^j_{\tilde{f}} \tilde{g}_k : 1 \le k \le \rho_j\} \right) = 0.$$

Finally, since $\dim(\bar{G}_i) = n_i + \sum_{j=0}^{i} \rho_j$ we have that (4.12) holds on V. Thus, since $G^{\|}_i \subset \bar{G}_i$ on U, (4.12) also holds in a neighborhood of V, without loss of generality, U. ▯

Lemmas 4.4 and 4.5 elucidate the fact that when the $\bar{G}_i$ are regular and the integers ρ_i are constant, it is possible to find a local basis for each $\bar{G}_i$ distinguishing between the tangential component and transversal component. Specifically, we have that for $i \in \mathbf{n} - \mathbf{n}^\star$, on U,

$$
\begin{aligned}
\bar{G}_i &= G_i^{\|} \oplus G_i^{\pitchfork} \\
&= \left(G_0^{\|} + G_{1/0}^{\|} + \cdots + G_{i/i-1}^{\|}\right) \oplus \left(G_0^{\pitchfork} + G_{1/0}^{\pitchfork} + \cdots + G_{i/i-1}^{\pitchfork}\right)
\end{aligned}
\tag{4.13}
$$

where $G_i^{\|}\Big|_V = TV \cap \bar{G}_i$ and

$$
\begin{aligned}
G_{i/i-1}^{\|} &= \operatorname{span}\{v_j^i : 1 \le j \le \mu_i\} \\
G_{i/i-1}^{\pitchfork} &= \operatorname{span}\{ad_{\tilde{f}}^i \tilde{g}_j : 1 \le j \le \rho_i\}.
\end{aligned}
\tag{4.14}
$$

The distributions $G_{i/i-1}^{\|}$ and $G_{i/i-1}^{\pitchfork}$ span, respectively, the tangential and transversal directions contained in $\bar{G}_i$ but *not* contained in $\bar{G}_{i-1}$. An immediate consequence of Lemma 4.4 is that when $\sum k_i = n - n^\star$, i.e.,

$$TV + \bar{G}_{k_1 - 1} = T\mathbb{R}^n,$$

then, after feedback transformation,

$$TV \oplus \operatorname{span}\{ad_{\tilde{f}}^j \tilde{g}_k : 0 \le j \le n - n^\star - 1, 1 \le k \le \rho_j\} = T\mathbb{R}^n. \tag{4.15}$$

As a result, Lemma 4.5 yields the following array[2] of n independent vector fields on U.

$$
\begin{array}{l|l}
1 & G_0^{\|}, G_0^{\pitchfork}; \ldots; G_{k_{\rho_0}-1/k_{\rho_0}-2}^{\|}, G_{k_{\rho_0}-1/k_{\rho_0}-2}^{\pitchfork}; \\
2 & G_{k_{\rho_0}/k_{\rho_0}-1}^{\|}, G_{k_{\rho_0}/k_{\rho_0}-1}^{\pitchfork}; \ldots; G_{k_{\rho_0-1}-1/k_{\rho-1}-2}^{\|}, G_{k_{\rho_0-1}-1/k_{\rho_0-1}-2}^{\pitchfork}; \\
 & . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
\rho_0 - 1 & G_{k_3/k_3-1}^{\|}, G_{k_3/k_3-1}^{\pitchfork}; \ldots; G_{k_2-1/k_2-2}^{\|}, G_{k_2-1/k_2-2}^{\pitchfork}; \\
\rho_0 & G_{k_2/k_2-1}^{\|}, G_{k_2/k_2-1}^{\pitchfork}; \ldots; G_{k_1-1/k_1-2}^{\|}, G_{k_1-1/k_1-2}^{\pitchfork}; \\
\rho_0 + 1 & v_1, , \ldots, v_{n^\star - n_{k_1-1}}.
\end{array}
\tag{4.16}
$$

In (4.16), each block delimited by semicolons in rows 1 to ρ_0 contains independent vector fields in some $\bar{G}_k$ which are not contained in $\bar{G}_{k-1}$. The vector fields in rows 1 through j, $1 \le j \le \rho_0$, span $\bar{G}_{k_{\rho_0 - j+1}-1}$. The vector fields in row $\rho_0 + 1$ are solely defined on $V \subset \Gamma^\star$ and are not contained in any of the $\bar{G}_i$'s so that at each $p \in V$, $\operatorname{span}\{v_1, \ldots, v_{n^\star - n_{k_1-1}}\}(p) \simeq (T_pV + \bar{G}_{k_1-1}(p))/\bar{G}_{k_1-1}(p)$. They are chosen to complete the basis for T_pV, so that

$$(\forall p \in V)\ T_pV = \operatorname{span}\{v_1, \ldots, v_{n^\star - n_{k_1-1}}\}(p) \oplus G_{k_1-1}^{\|}(p).$$

[2]In the array we use the symbols $G_{i/i-1}^{\|}$ and $G_{i/i-1}^{\pitchfork}$ to mean a family of vector fields and not the span of the vector fields.

5. Proof of the Main Result (Theorem 3.2). Suppose that LTFLP is solvable at $p_0 \in \Gamma^\star$. Let $V = \Xi(\Gamma^\star \cap U)$ and consider the expression (4.4) for $TV + G_i$ in local coordinates. It is clear from (4.4) that the subspace $T_pV + G_i(\mathrm{col}(p, 0))$ has constant dimension $n^\star + \mathrm{rank}([B \ \cdots \ A^i B])$. Since the pair (A, B) is controllable, we have that $\mathrm{rank}([B \ \cdots \ A^{n-n^\star-1}B]) = n - n^\star$ and condition (a) holds. As far as condition (b) is concerned, we have already shown in the proof of Lemma 4.3 that $TV + G_i = TV + \bar{G}_i$.

Conversely, suppose conditions (a) and (b) hold. These two conditions along with the regularity of $\bar{G}_i$, $i \in \mathbf{n} - \mathbf{n}^\star - \mathbf{1}$, allow one to invoke Lemmas 4.4 and 4.5. Specifically, there exist a neighborhood $\tilde{U}$ of p_0 in $\mathbb{R}^n$, a regular static feedback (α, β) on $\tilde{U}$, and n_{k_1-1} vector fields defined on $\tilde{U}$ such that, letting $\tilde{V} := \tilde{U} \cap \Gamma^\star$, the distributions $\bar{G}_i$ have the representation given in (4.13), (4.14) on $\tilde{U}$ and at each $p \in \tilde{V}$ the n vector fields of the array (4.16) are linearly independent.

Having applied the static feedback (α, β), we will denote $\tilde{f}$ and $\tilde{g}$ by, respectively, f and g to simplify notation. We construct ρ_0 functions $\alpha_i : U \to \mathbb{R}$ satisfying Theorem 3.1. Pick the point p_0 as the origin for the s-coordinate system to be generated from the vector fields in (4.16) (see, [25], [27]). Compose the flows generated by the vector fields in (4.16) starting from the bottom row. Consider the mapping

$$\begin{aligned} F_\emptyset &: W_\emptyset \subset \mathbb{R}^{n^\star - n_{k_1-1}} \to V \\ &: S_\emptyset = (s_1^{\parallel}, \dots, s_{n^\star - n_{k_1-1}}^{\parallel}) \quad \mapsto \phi^{v_{n^\star - n_{k_1-1}}}_{s^{\parallel}_{n^\star - n_{k_1-1}}} \circ \cdots \circ \phi^{v_1}_{s_1^{\parallel}}(p_0). \end{aligned}$$

We continue by moving upwards in the array (4.16) to generate a sequence of mappings similar to $F_\emptyset$. To each pair $(G^{\pitchfork}_{i/i-1}, G^{\parallel}_{i/i-1})$, $0 \le i \le k_1 - 1$, we associate a set of times[3] $S_{i/i-1} = (S^{\pitchfork}_{i/i-1}; S^{\parallel}_{i/i-1}) := (s^{\pitchfork}_{(i/i-1,1)}, \dots, s^{\pitchfork}_{(i/i-1,\rho_i)};\ s^{\parallel}_{(i/i-1,1)}, \dots, s^{\parallel}_{(i/i-1,\mu_i)})$ and a mapping

$$\begin{aligned} F_{i/i-1} &: W_{i/i-1} \subset \mathbb{R}^{\mu_i + \rho_i} \to \mathbb{R}^n \\ &: (S^{\parallel}_{i/i-1}, S^{\pitchfork}_{i/i-1}) \mapsto \Phi^{\parallel}_{i/i-1} \circ \Phi^{\pitchfork}_{i/i-1}(p). \end{aligned}$$

Here $\Phi^{\parallel}_{i/i-1}$ is the composition of flows generated by vector fields spanning $G^{\parallel}_{i/i-1}$, and $\Phi^{\pitchfork}_{i/i-1}$ is the composition of flows generated by the vector fields spanning $G^{\pitchfork}_{i/i-1}$. Specifically,

$$\begin{aligned} \Phi^{\parallel}_{i/i-1} &= \phi^{v^i_{\mu_i}}_{s^{\parallel}_{(i/i-1,\mu_i)}} \circ \cdots \circ \phi^{v^i_1}_{s^{\parallel}_{(i/i-1,1)}} \\ \Phi^{\pitchfork}_{i/i-1} &= \phi^{ad^i_f g_{\rho_i}}_{s^{\pitchfork}_{(i/i-1,\rho_i)}} \circ \cdots \circ \phi^{ad^i_f g_1}_{s^{\pitchfork}_{(i/i-1,1)}}. \end{aligned}$$

Let

$$s = \mathrm{col}(S_\emptyset; S_{k_1-1/k_1-2}; \dots; S_{1/0}; S_0). \tag{5.1}$$

and let $W \subset \mathbb{R}^n$ be a neighborhood of $s = 0$, sufficiently small, to ensure that the map

$$\begin{aligned} F &: W \to F(W) \\ &s \mapsto F_0 \circ F_{1/0} \circ \cdots \circ F_{k_1-2/k_1-3} \circ F_{k_1-1/k_1-2} \circ F_\emptyset(p_0). \end{aligned} \tag{5.2}$$

[3] We define $i/(i-1) := 0$ when $i = 0$ to be consistent with the array (4.16)

is a diffeomorphism onto its image and that $F(W) \subset \tilde{U}$. The existence of W is guaranteed by the inverse function theorem and the fact that the differential of F at $s = 0$,

$$dF_0 = \begin{bmatrix} v_1 & \cdots & v_{n^\star - n_{k_1-1}} & ad_f^{k_1-1} g_1 & \cdots\cdots & g_1 & \cdots & g_{\rho_0} & v_1^0 & \cdots & v_{\mu_0}^0 \end{bmatrix} (p_0). \tag{5.3}$$

is an $n \times n$ square matrix whose columns span the subspace $T_{p_0}\Gamma^\star + G_{k_1-1}(p_0)$ which, by condition (a), has dimension n. As candidate (virtual) output functions, let α_i, $i \in \{1, \ldots, \rho_0\}$ be the time spent flowing along $ad_f^{k_i-1} g_i$, i.e,

$$\alpha_i(x) = s^\pitchfork_{(k_i-1/k_i-2,\ i)}(x), \quad i \in \{1, \ldots, \rho_0\}. \tag{5.4}$$

The image of $\tilde{V}$ under F^{-1} is the hyper-plane

$$F^{-1}(\tilde{V}) = \{s \in W : S_0^\pitchfork = 0,\ S_{1/0}^\pitchfork = 0, \ldots,\ S_{k_1-1/k_1-2}^\pitchfork = 0\}.$$

Since the chosen functions $\alpha_1, \ldots, \alpha_{\rho_0}$ are a subset of the functions whose zero level set define $F(\tilde{V})$, the α_i are identically zero on $F(\tilde{V})$ and hence condition (1) of Theorem 3.1 is satisfied. Next, we must show that $\alpha = \mathrm{col}(\alpha_1, \ldots, \alpha_{\rho_0})$ yields a well-defined vector relative degree of $(k_1, \ldots, k_{\rho_0})$ at $p_0 = F^{-1}(0)$. As per [9], this entails showing that

(VRD1) $L_{ad_f^k g_j} \alpha_i(x) = 0$ for all $1 \le j \le \rho_0$, for all $0 \le k \le k_i - 2$, for all $1 \le i \le \rho_0$ and for all x in a neighborhood of p_0.

(VRD2) The $\rho_0 \times \rho_0$ matrix

$$\begin{pmatrix} L_{ad_f^{k_1-1} g_1} \alpha_1(p_0) & \cdots & L_{ad_f^{k_1-1} g_{\rho_0}} \alpha_1(p_0) \\ L_{ad_f^{k_2-1} g_1} \alpha_2(p_0) & \cdots & L_{ad_f^{k_2-1} g_{\rho_0}} \alpha_2(p_0) \\ \cdots & \cdots & \cdots \\ L_{ad_f^{k_{\rho_0}-1} g_1} \alpha_{\rho_0}(p_0) & \cdots & L_{ad_f^{k_{\rho_0}-1} g_{\rho_0}} \alpha_{\rho_0}(p_0) \end{pmatrix} \tag{5.5}$$

is non-singular at $p = p_0$ (if this matrix is non-singular, then the decoupling matrix has full rank).

First we show that VRD1 holds. Fix a set of times $S_\emptyset = c_\emptyset$, $S_{k_1-1/k_1-2} = c_{k_1-1/k_1-2}$, $\ldots$, $S_{k_i-1/k_i-2} = c_{k_i-1/k_i-2}$, where each c_j is a constant vector, to uniquely determine the hyper-plane

$$H_i = \{s \in W : S_\emptyset = c_\emptyset, S_{k_1-1/k_1-2} - c_{k_1-1/k_1-2}, \ldots, S_{k_i-1/k_i-2} = c_{k_i-1/k_i-2}\}.$$

Consider the point $s = \mathrm{col}(c_\emptyset, \ldots, c_{k_i-1/k_i-2}, 0, \ldots, 0) \in H_i$ and let $x = F(s) \in \tilde{U}$. Through x there passes an integral submanifold of each $\bar{G}_i$, $i \in \mathbf{k_1 - 1}$, which we denote by $L_i(x)$. Consider the map $F_0 \circ F_{1/0} \circ \cdots \circ F_{k_i-2/k_i-3}(x)$. It is the composition of the flows defined by vector fields which are local generators for $\bar{G}_{k_i-2}$. Therefore the image of this map is the ν_{k_i-2} dimensional manifold $L_{k_i-2}(x) \cap \tilde{U}$. On the other hand, the image of this map in s-coordinates is the hyper-plane H_i, i.e., $H_i = F^{-1}(L_{k_i-2}(x) \cap \tilde{U})$. Therefore for each $s \in H_i$, $T_s H_i = \left(F^{-1}\right)_\star \bar{G}_{k_i-2}(s) = \mathrm{Im}\left(\mathrm{col}(0, I_{\nu_{k_i-2}})\right)$. The function α_i is among those fixed times which define the hyper-plane H_i. Therefore, $d\alpha_i \in \mathrm{ann}(\bar{G}_{k_i-2}) \subset \cdots \subset \mathrm{ann}(G_0)$ and hence VRD1 holds in a sufficiently small neighborhood of p_0.

Next we show that VRD2 holds. Treating $T\mathbb{R}^n$ as an orthogonal bundle with the usual inner product, the value of the (i,j)-th entry of (5.5) is equal to

$$\left\langle d\alpha_i, ad_f^{k_i-1} g_j \right\rangle (p_0).$$

From the expression (5.3) for dF_0 it follows that

$$ad_f^{k_i-1} g_j(p_0) = \left[F_\star \left(\frac{\partial}{\partial s^{\pitchfork}_{(k_i-1/k_i-2,\ j)}} \right) \right] \Bigg|_{s=0}, \quad 1 \le i \le j \le \rho_0$$

so that

$$\frac{\partial}{\partial s^{\pitchfork}_{(k_i-1/k_i-2,\ j)}} = \left[F_\star^{-1} \left(ad_f^{k_i-1} g_j(x) \right) \right] \Big|_{x=p_0}, \quad 1 \le i \le j \le \rho_0.$$

In light of this and the definition of α_i, $i \in \{1, \ldots, \rho_0\}$, given by (5.4), in s-coordinates the values of the entries of (5.5), along and below the diagonal, at p_0 are

$$\left\langle ds^{\pitchfork}_{k_i-1/k_i-2,\ i}, \frac{\partial}{\partial s^{\pitchfork}_{(k_i-1/k_i-2,\ j)}} \right\rangle = \delta_{ij}, \quad 1 \le i \le j \le \rho_0,$$

where δ_{ij} is the Kronecker delta function. Thus, at $0 = F^{-1}(p_0)$ the matrix (5.5), in s-coordinates, has ones along its diagonal and zeros below. Therefore it is non-singular at $s = 0$ which is equivalent to being non-singular at $p_0 = F(0)$. $\square$.

6. Conclusions. We have determined necessary and sufficient conditions under which a multi-input nonlinear control-affine system is locally transversally feedback linearizable with respect to a given invariant submanifold. Our main conditions are checkable, though we do not present a constructive procedure for finding the coordinate and feedback transformations.

One can similarly pose the global transverse feedback linearization problem (GTFLP) in which one, roughly speaking, seeks a single coordinate and feedback transformation such that (2.1) is feedback equivalent to the normal form (3.1) in a tubular neighborhood of $\Gamma^\star$. Clearly the geometry of $\Gamma^\star$ will play an increased role in characterizing the solution. In [19], we provided sufficient conditions for the solvability of GTFLP in the single-input case. GTFLP for multi-input systems remains an open problem.

Acknowledgments. Both authors are supported by the National Science and Engineering Research Council (NSERC) of Canada. Christopher Nielsen was partially supported by the Ontario Graduate Scholarship (OGS).

REFERENCES

[1] J.P. Aubin. *Viability Theory.* Birkhauser, Boston, 1991.
[2] A. Banaszuk and J. Hauser. Feedback linearization of transverse dynamics for periodic orbits. *Systems and Control Letters*, 26(2):95–105, Sept. 1995.
[3] R.W. Brockett. Feedback invariants for nonlinear systems. In *Proc. IFAC World Congress*, pages 1115–1120, Helsinki, 1978.
[4] P. Brunovský. A classification of linear controllable systems. *Kybernetika*, 3(6):173–187, 1970.
[5] V. Guillemin and A. Pollack. *Differential Topology.* Prentice Hall, New Jersey, 1974.
[6] M.W. Hirsch. *Differential Topology.* Graduate Texts in Math., 33. Springer - Verlag, New York, 1976.

[7] R. M. Hirschorn. $(A, \mathscr{B})$-invariant distributions and disturbance decoupling of nonlinear systems. *SIAM J. Control and Optimization*, 19(1):1–19, January 1981.

[8] L.R. Hunt, R. Su, and G. Meyer. Design for multi-input nonlinear systems. In R.W. Brockett, R.S. Millman, and H. Sussmann, editors, *Differential Geometric Control Theory*, pages 268–298. Birkhauser, Boston, 1983.

[9] A. Isidori. *Nonlinear Control Systems.* Springer, New York, 3^{rd} edition, 1995.

[10] A. Isidori, A. J. Krener, C. Gori-Giorgi, and S. Monaco. Nonlinear decoupling via feedback: A differential geometric approach. *IEEE Transactions on Automatic Control*, 26(2):331–345, April 1981.

[11] A. Isidori and A.J. Krener. On feedback equivalence of nonlinear systems. *Systems and Control Letters*, 2(2):118–121, 1982.

[12] B. Jakubczyk and W. Respondek. On linearization of control systems. *Bull. Acad. Polon. Sci. Sér. Sci. Math*, 28(9–10):517–522, 1980.

[13] A. J. Krener. On the equivalence of control systems and the linearization of nonlinear systems. *SIAM J. Control and Optimization*, 11(4):670–676, November 1973.

[14] A.J. Krener, A. Isidori, and W. Respondek. Partial and robust linearization by feedback. In *22^{nd} IEEE Conference on Decision and Control*, pages 126–130, San Antonio, December 1983.

[15] R. Marino. On the largest feedback linearizable subsystem. *Systems and Control Letters*, 6(5):345–351, 1986.

[16] R. Marino, W.M. Boothby, and D.L. Elliott. Geometric properties of linearizable control systems. *Mathematical Systems Theory*, 18(1):97–123, December 1985.

[17] R. Marino and P. Tomei. *Nonlinear Control Design*. Prentice Hall, Toronto, 1995.

[18] C. Nielsen and M. Maggiore. Maneuver regulation via transverse feedback linearization: Theory and examples. In *Proceedings of the IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, Stuttgart, Germany, September 2004.

[19] C. Nielsen and M. Maggiore. Output stabilization and maneuver regulation: A geometric approach. *Systems Control Letters*, 55:418–427, 2006.

[20] H. Nijmeijer. Controlled invariance for affine control systems. *International Journal of Control*, 34(4):825–833, October 1981.

[21] H. Nijmeijer and A. van der Schaft. *Nonlinear Dynamical Control Systems.* Springer - Verlag, New York, 1990.

[22] H. Poincaré. Sur les propérties des fonctions définies par les équations aux différences partielles. In *Œuvres*, volume 1, pages xcix–cx. Gauthier-Villars, Paris, 1929.

[23] W. Respondek. Partial linearization, decompositions and fibre linear systems. In C.I. Byrnes and A. Lindquist, editors, *Theory and Applications of Nonlinear Control Systems*, pages 137–154. North-Holland, 1986.

[24] A. Shiriaev, J.W. Perram, and C. Canudas de Wit. Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. *IEEE Transactions on Automatic Control*, 50(8):1164–1176, August 2005.

[25] M. Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 1. Publish or Perish, Houston, 3^{rd} edition, 2005.

[26] R. Su. On the linear equivalents of nonlinear systems. *Systems and Control Letters*, 2(1):48–52, July 1982.

[27] R. Su and L.R. Hunt. A canonical expansion for nonlinear systems. *IEEE Transactions on Automatic Control.*, 31(7):670–673, 1986.

[28] W.M. Wonham. *Linear Multivariable Control: A Geometric Approach.* Springer-Verlag, New York, 3^{rd} edition, 1985.

[29] W.M. Wonham and A.S. Morse. Feedback invariants of linear multivariable systems. *Automatica*, 8(1):93–100, 1972.

[30] Z. Xu and L.R. Hunt. On the largest input-output linearizable subsystem. *IEEE Transactions on Automatic Control*, 41(1):128–132, January 1996.

[31] Z. Xu and L.R. Hunt. Finding maximal linear subsystems of nonlinear systems with outputs. *IEEE Transactions on Automatic Control*, 45(9):1701–1704, September 2000.