Bang Bang Hybrid Stabilization of Perturbed Double Integrators

Edoardo Serpelloni, Manfredi Maggiore and Christopher Damaren

***Abstract*— A bang bang hybrid controller is presented that globally practically stabilizes the origin of a double-integrator with bounded disturbances. The proposed controller has finite switching frequency and, when the disturbances are absent, it reduces to the time-optimal bang-bang controller for the double-integrator.**

I. INTRODUCTION

In this paper we consider the perturbed double-integrator system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x,t) + u, \end{aligned} \tag{1}$$

where $(x_1, x_2) \in \mathbb{R}\times\mathbb{R}$, $u \in U := \{-\bar{u}, 0, +\bar{u}\}$, with $\bar{u} > 0$, and $f(x,t)$ is a map in the class $\mathcal{F}$ of functions $\mathbb{R}^2\times\mathbb{R} \to \mathbb{R}$ that are locally Lipschitz with respect to x, measurable with respect to t, and uniformly bounded by a constant $\bar{f} > 0$, i.e., $\sup|f| \leq \bar{f}$. We investigate the following stabilization problem.

Stabilization by Constant Controls Problem (SCCP). Consider system (1), where $f \in \mathcal{F}$. Design a feedback controller with values in $U = \{-\bar{u},\ 0,\ +\bar{u}\}$ such that the following properties hold:

- (i) If $f \equiv 0$, there exist controller parameters such that $x = 0$ is reached in finite time.
- (ii) For all $f \in \mathcal{F}$, the point $x = 0$ is *globally practically stable* for the closed-loop system, i.e., for all $r > 0$ there exist controller parameters such that, for all $x_0 \in \mathbb{R}^2$ and all $t_0 \in \mathbb{R}$, the solution $x(t)$ with initial condition $x(t_0) = x_0$ is such that there exists $T > 0$ such that $x(t) \in B_r(0)$ for all $t \geq T$.
- (iii) The controller has a *finite switching frequency,* i.e., given any compact time interval $[t_0, t_1]$, u switches value a finite number of times.

The control of double integrators plays an important role in control theory. Many aerospace and robotic systems, for example, can be modeled as cascades of linear or rotational double integrators, [1]. For some of these applications bang-bang solutions are of particular interest in that many actuators can be modeled as on-off functions, [2], [3]. It is well-known that the unperturbed double integrator admits a bang bang time optimal solution, [4], [5]. As shown in [5], this controller is robust to a certain class of bounded disturbances, but it induces a sliding mode along the switching boundary, therefore violating specification (iii) of SCCP. Sliding mode controllers with hysteresis bands alleviate the issues related to the switching frequency. It is shown in [6] that when the perturbation $f(x,t)$ is known, the controller's switching frequency can be kept constant by adjusting the size of the hysteresis band. This constrains the size of the ball that the controller can stabilize. Moreover, as shown in [7], the use of hysteresis loops with linear sliding surfaces results only in local stabilization properties. In [8], [9], a discontinuous controller has been proposed that guarantees the convergence of the state trajectory to the origin in finite time. Although the authors only prove convergence to the origin and not stability, we believe that a simple modification to the controller in [8], [9] can be adopted to solve SCCP. It has been shown in [10], [11], [12] that hybrid feedback is advantageous over discontinuous feedback since it has the potential of being robust with respect to measurement noise. Motivated by this observation, we seek a hybrid solution to SCCP. When the perturbation $f \equiv 0$, our hybrid controller recovers the time-optimal bang-bang stabilizer for the double-integrator. The idea of our solution is roughly this. We consider the switching boundary of the time-optimal bang-bang stabilizer, and add another switching boundary, the set $\{x_2 = 0\}$. We then define a finite state machine that selectively enables and disables switching surfaces in such a way that the resulting sequence of switching points contracts to the origin, and sliding modes are avoided.

The paper is organized as follows. In Section II we present the solution of SCCP and state the main result, Theorem 2.1. In Section III we review a basic result from [13] characterizing the boundary of attainable sets of planar nonlinear systems, and apply it to the perturbed double-integrator (1). In Section IV we find conditions for the existence of a well-defined switching sequence. The proof of the main theorem is presented in Section V. Section VI presents simulation results.

Notation: We denote $B_\epsilon(0) = \{x \in \mathbb{R}^2 : (x^T x)^{1/2} < \epsilon\}$ and $\bar{B}_\epsilon(0) = \{x \in \mathbb{R}^2 : (x^T x)^{1/2} \leq \epsilon\}$. These definitions imply that the set $B_0(0)$ is empty, while $\bar{B}_0(0) = \{0\}$. The boundary of a set A is defined as $\partial A = \bar{A} \setminus \operatorname{int} A$ where $\bar{A}$ is the closure of A and $\operatorname{int} A$ is its interior. We denote by A^c the set $A^c = \mathbb{R}^2 \setminus A$.

M. Maggiore and C. Damaren were supported by the Natural Sciences and Engineering Research Council of Canada (NSERC). E. Serpelloni was supported by the Dean's Fund, University of Toronto

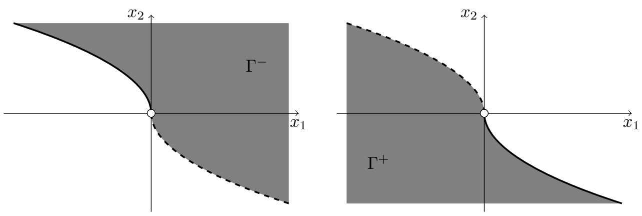
E. Serpelloni and M. Maggiore are with the Department of Electrical and Computer Engineering, University of Toronto, 10 Kings College Road, M5S 3G4, Toronto, Ontario, Canada edoardo.serpelloni@mail.utoronto.ca maggiore@control.utoronto.ca

Christopher Damaren is with the University of Toronto Institute for Aerospace Studies, University of Toronto, 4925 Dufferin St., M3H 5T6, Toronto, Ontario, Canada. damaren@utias.utoronto.ca

II. Control Law

Referring to Figures 1(a) and 1(b), define **initialization sets** Γ^+, Γ^- as

$$\begin{aligned}\Gamma^+ =&\{(x_1,x_2) : x_1 < 0,\ x_2 < \sqrt{-2\bar{u}x_1}\} \cup \\ &\{(x_1,x_2) : x_1 > 0,\ x_2 \le -\sqrt{2\bar{u}x_1}\}, \\ \Gamma^- =&\{(x_1,x_2) : x_1 < 0,\ x_2 \ge \sqrt{-2\bar{u}x_1}\} \cup \\ &\{(x_1,x_2) : x_1 > 0,\ x_2 > -\sqrt{2\bar{u}x_1}\}.\end{aligned}$$



(a) Initialization set Γ^-. (b) Initialization set Γ^+.

Fig. 1.

Referring to Figure 2, define **switching sets** Λ^+, Λ^- as

$$\begin{aligned}\Lambda^+ =&\{(x_1,x_2) : x_1 < 0,\ x_2 \le 0\} \cup \\ &\{(x_1,x_2) : x_1 > 0,\ x_2 \le -\sqrt{2\bar{u}x_1}\}, \\ \Lambda^- =&\{(x_1,x_2) : x_1 > 0,\ x_2 \ge 0\} \cup \\ &\{(x_1,x_2) : x_1 < 0,\ x_2 \ge \sqrt{-2\bar{u}x_1}\}.\end{aligned} \tag{2}$$

The boundaries of sets Λ^+ and Λ^- are given by

$$\begin{aligned}\partial\Lambda^+ = S^+ \cup \{(x_1,0) : x_1 \le 0\} \\ \partial\Lambda^- = S^- \cup \{(x_1,0) : x_1 \ge 0\},\end{aligned} \tag{3}$$

where S^+, S^- are half-parabolas

$$\begin{aligned}S^+ = \{(x_1, -\sqrt{2\bar{u}x_1}) : x_1 > 0\}, \\ S^- = \{(x_1, \sqrt{-2\bar{u}x_1}) : x_1 < 0\}.\end{aligned}$$

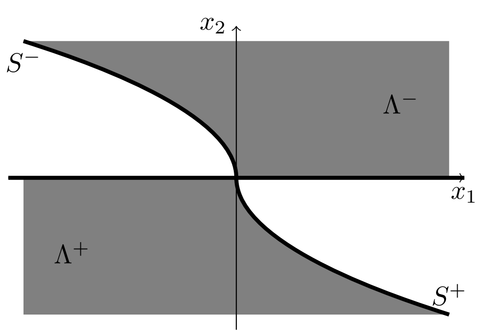


Fig. 2. The switching sets Λ^-, Λ^+.

The proposed control law solving SCCP is described by the automaton $\mathcal{A}$ in Figure 3, and is characterized by discrete states $Q = \{q_1, q_2, q_3\}$ and continuous states $x \in \mathbb{R}^2$. The automaton depends on two design parameters δ_1, δ_2, with $0 \le \delta_1 < \delta_2$. A state transition from state q_i to state q_j, with $j \neq i$, will be denoted as $q_i \to q_j$. The control value is given by the hybrid feedback $u^\star : Q \to \mathbb{R}$ defined as

$$\begin{aligned}u^\star(q_1) &= -\bar{u} \\ u^\star(q_2) &= +\bar{u} \\ u^\star(q_3) &= 0.\end{aligned} \tag{4}$$

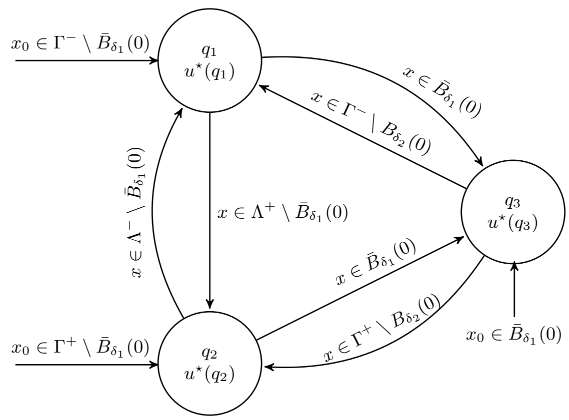


Fig. 3. The automaton $\mathcal{A}$ representing the proposed controller.

The discrete states q_1 and q_2 in the automaton $\mathcal{A}$ activate and deactivate the switching sets Λ^+ and Λ^-, so that a switch in the control value is allowed only when the trajectory enters the switching set which is currently active. This mutually exclusive activation of the switching sets eliminates sliding modes. Moreover, referring to Figure 2, the gap between Λ^+ and Λ^- (white region) guarantees that when trajectories are away from the origin, the switching frequency is bounded. Near the origin, an hysteresis mechanism, regulated by two nested balls $B_{\delta_1}(0) \subset B_{\delta_2}(0)$ and by the discrete state q_3, guarantees the boundedness of the switching frequency. To illustrate, referring to Figure 3, if $x_0 \in \Gamma^- \setminus \bar{B}_{\delta_1}(0)$ the automaton is initialized with discrete state at q_1. The associated control value is therfore $u^\star(q_1) = -\bar{u}$. A state transition from q_1 is allowed either when $x(t)$ enters $\bar{B}_{\delta_1}(0)$ ($q_1 \to q_3$), switching the control value to $u^\star(q_3) = 0$, or when $x(t)$ enters $\Lambda^+ \setminus \bar{B}_{\delta_1}(0)$, ($q_1 \to q_2$), switching $u^\star$ to $u^\star(q_2) = +\bar{u}$. Therefore, if the discrte state is at q_1, the switching set Λ^- is disabled. Similarly in q_2, the switching set Λ^+ is disabled. A state transition occurs only when the state enters $\bar{B}_{\delta_1}(0)$ or when it enters set $\Lambda^- \setminus \bar{B}_{\delta_1}(0)$. If the discrete state is at q_3, the controller is turned off, i.e. $u^\star = 0$. The controller will be turned on again only if the state exits $B_{\delta_2}(0)$. The following state transition will depend on the location of such exiting point: the control value will switch to $+\bar{u}$ if $x(t)$ enters $\Gamma^+ \setminus B_{\delta_2}(0)$ ($q_3 \to q_2$), and to $-\bar{u}$ if $x(t)$ enters $\Gamma^- \setminus B_{\delta_2}(0)$, ($q_3 \to q_1$).

For appropriate choices of $\bar{u}$, δ_1,δ_2 controller (4) solves SCCP.

Theorem 2.1: Controller (4) solves SCCP if, and only if, $\bar{u}$ satisfies $\bar{u} > \bar{f}(1+\sqrt{5})/2$. In particular, if this inequality holds, then for any $r > 0$ there exist scalars δ_1, δ_2 with $0 < \delta_1 < \delta_2 < r$, such that for all solutions through x_0 there is $T_{x_0} > 0$ such that $x(t) \in B_r(0)$ for all $t \ge T_{x_0}$.

Remark 2.2: When the perturbation $f \equiv 0$, by setting $\delta_1 = 0$ and $\delta_2 > 0$ the proposed hybrid feedback reduces to the time-optimal bang-bang controller for the double-integrator. Moreover, for arbitrary $f \in \mathcal{F}$, it can be shown that setting $\delta_1 = 0$ and $\delta_2 > 0$ makes the origin globally finite-time stable, but the switching frequency becomes infinite when solutions reach the origin, violating requirement (ii) of SCCP. Finally, the proposed feedback is robust against measurement noise. The proof of this fact will be shown elsewhere.

III. Boundaries of attainable sets

In this section we characterize the boundaries of attainable sets of system (1) under the hybrid feedback (4), assuming that the control value $\bar{u}$ is larger than the bound $\bar{f}$ on the perturbation $f \in \mathcal{F}$. We begin with some preliminary notions taken from [13].

Consider the planar system

$$\dot{x} = \lambda(x,t) F_1(x) + (1 - \lambda(x,t))\, F_2(x) \tag{5}$$

where $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are planar C^1 vector fields and $\lambda : \mathbb{R}^2 \times \mathbb{R} \to [0,1]$ is a function that is locally Lipschitz with respect to x and measurable with respect to t. Define sets $\mathcal{R}^-$ and $\mathcal{R}^+$ as

$$\begin{aligned} \mathcal{R}^- &= \{x \in \mathbb{R}^2 : \det [F_1(x)\ F_2(x)] < 0\}, \\ \mathcal{R}^+ &= \{x \in \mathbb{R}^2 : \det [F_1(x)\ F_2(x)] > 0\}. \end{aligned} \tag{6}$$

Definition 3.1: The **extremal vector fields** $F_L(x)$ and $F_R(x)$ are defined as

$$\begin{aligned} F_L(x) &= \begin{cases} F_1(x), & x \in \mathcal{R}^+ \\ F_2(x), & x \in \mathcal{R}^- \end{cases} \\ F_R(x) &= \begin{cases} F_2(x), & x \in \mathcal{R}^+ \\ F_1(x), & x \in \mathcal{R}^-. \end{cases} \end{aligned} \tag{7}$$

The solutions at time t of the extremal vector fields $F_L(x)$ and $F_R(x)$ are called **extremal solutions** and are denoted by $\phi_L(t, x_0)$ and $\phi_R(t, x_0)$, respectively. The images of extremal solutions on the plane are called **extremal arcs**. In particular, the **L-arc (resp. R-arc) through** x_0, denoted by $\gamma_L(x_0)$ (resp. $\gamma_R(x_0)$), is the image of the map $t \mapsto \phi_L(t, x_0)$ (resp. $t \mapsto \phi_R(t, x_0)$) for t ranging over some interval over which the map is defined. △

Vector field $F_L(x)$ (resp. $F_R(x)$) is said to be **extremal**, in that it represents the vector field of (5) of maximum (resp. minimum) slope. Similarly, extremal arcs of (5) are the phase curves of (5) with minimum and maximum slope.

Definition 3.2: The **attainable set** $\mathcal{A}(x_0, t)$ **from** x_0 **at time** t of system (5) is the set

$$\mathcal{A}(x_0, t) = \{x(t) : x(t) \text{ is a solution of (5) through } x_0 \text{ for some admissible } \lambda(x,t)\}$$

The **attainable set** $\mathcal{A}(x_0)$ **from** x_0 of system (5) is the set $\mathcal{A}(x_0) = \bigcup_{t \geq 0} \mathcal{A}(x_0, t)$. △

The next lemma states that extremal arcs form the boundary of attainable sets. Before stating the lemma we recall that system (5) is said to be **small time locally controllable (STLC) from** x_0 if, for all $T > 0$, x_0 lies in the interior of $\mathcal{A}(x_0, [0, T])$.

Lemma 3.3 ([13]): Let $x_0 \in \mathbb{R}^2$ be such that system (5) is not STLC from x_0. Suppose that for some $T > 0$ a solution $x(t)$ of (5) with initial conditions x_0 has the property that $x(t) \in \partial\mathcal{A}(x_0, t)$ for all $t \in [0, T]$ and that system (5) is not STLC from $x(t)$, for all $t \in [0, T]$. Then $x(t)$ is a concatenation of extremal solutions.

Thus, extremal arcs form the boundaries of attainable sets of system (5) through x_0. Now we return to the perturbed double integrator (1) with the hybrid feedback (4). For each value[1] q_i, $i \in \{1, 2\}$, of the discrete state, the closed-loop double integrator can be rewritten as

$$\dot{x} = \lambda(x,t) F_1^{q_i}(x) + (1 - \lambda(x,t)) F_2^{q_i}(x), \tag{8}$$

with $\lambda(x,t) = \left(\bar{f} - f(x,t)\right)/(2\bar{f}) \in [0,1]$ and

$$F_1^{q_i}(x) = \begin{bmatrix} x_2 \\ -\bar{f} + (-1)^i \bar{u} \end{bmatrix}, \quad F_2^{q_i}(x) = \begin{bmatrix} x_2 \\ \bar{f} + (-1)^i \bar{u} \end{bmatrix}.$$

If $\bar{u} > \bar{f}$ the system is not STLC from x_0, since $\dot{x}_2$ is bounded away from zero. Therefore, Lemma 3.3 is applicable to system (8). The sets $\mathcal{R}^+$, $\mathcal{R}^-$ are given by

$$\mathcal{R}^+ = \{(x_1, x_2) : x_2 > 0\},\ \mathcal{R}^- = \{(x_1, x_2) : x_2 < 0\}.$$

For each fixed q_i, $i \in \{1, 2\}$, the extremal vector fields of (8) are given by

$$\begin{aligned} F_L^{q_i}(x) &= \begin{bmatrix} x_2 \\ -\operatorname{sign}(x_2)\bar{f} + (-1)^i \bar{u} \end{bmatrix} \\ F_R^{q_i}(x) &= \begin{bmatrix} x_2 \\ \operatorname{sign}(x_2)\bar{f} + (-1)^i \bar{u} \end{bmatrix}. \end{aligned} \tag{9}$$

The associated extremal solutions $\phi_L^{q_i}(s, x_0)$ and $\phi_R^{q_i}(s, x_0)$ through x_0 for $s \geq 0$, can be computed analytically. They are concatenations of arcs of parabolas $X_s^{q_i}(x_0)$ and $Y_s^{q_i}(x_0)$ defined as

$$\begin{aligned} X_s^{q_i}(x_0) &= \begin{bmatrix} \left(-\bar{f} + (-1)^i \bar{u}\right) \frac{s^2}{2} + x_{20} s + x_{10} \\ \left(-\bar{f} + (-1)^i \bar{u}\right) s + x_{20} \end{bmatrix} \\ Y_s^{q_i}(x_0) &= \begin{bmatrix} \left(\bar{f} + (-1)^i \bar{u}\right) \frac{s^2}{2} + x_{20} s + x_{10} \\ \left(\bar{f} + (-1)^i \bar{u}\right) s + x_{20} \end{bmatrix}, \end{aligned}$$

where the concatenation occurs when the solution hits $\{x_2 = 0\}$. More precisely, for all $x_0 \in \mathcal{R}^-$, we have

$$\begin{aligned} \phi_L^{q_i}(s, x_0) &= \begin{cases} Y_s^{q_i}(x_0), & \text{if } Y_s^{q_i}(x_0) \in \mathcal{R}^- \\ X_{s - s_Y^i(x_0)}^{q_i} \circ Y_{s_Y^i(x_0)}^{q_i}(x_0), & \text{if } Y_s^{q_i}(x_0) \in \mathcal{R}^+ \end{cases} \\ \phi_R^{q_i}(s, x_0) &= \begin{cases} X_s^{q_i}(x_0), & \text{if } X_s^{q_i}(x_0) \in \mathcal{R}^- \\ Y_{s - s_X^i(x_0)}^{q_i} \circ X_{s_X^i(x_0)}^{q_i}(x_0), & \text{if } X_s^{q_i}(x_0) \in \mathcal{R}^+, \end{cases} \end{aligned} \tag{10}$$

[1]We do not characterize the attainable set for the discrete state q_3, as it is not needed in our analysis.

while for all $x_0 \in \mathcal{R}^+$, we have

$$\phi_L^{q_i}(s,x_0) = \begin{cases} X_s^{q_i}(x_0), & \text{if } X_s^{q_i}(x_0) \in \mathcal{R}^+ \\ Y^{q_i}_{s-s^i_X(x_0)} \circ X^{q_i}_{s^i_X(x_0)}(x_0), & \text{if } X_s^{q_i}(x_0) \in \mathcal{R}^- \end{cases}$$

$$\phi_R^{q_i}(s,x_0) = \begin{cases} Y_s^{q_i}(x_0), & \text{if } Y_s^{q_i}(x_0) \in \mathcal{R}^+ \\ X^{q_i}_{s-s^i_Y(x_0)} \circ Y^{q_i}_{s^i_Y(x_0)}(x_0), & \text{if } Y_s^{q_i}(x_0) \in \mathcal{R}^- \end{cases} \tag{11}$$

where

$$s^i_X(x_0) = -\frac{x_{02}}{(-1)^i\bar{u} - \bar{f}},\ s^i_Y(x_0) = -\frac{x_{02}}{(-1)^i\bar{u} + \bar{f}}.$$

The existence of extremal solutions for each $x_0 \in \mathbb{R}^2$ and each fixed q_i, $i \in \{1,2\}$, is guaranteed by the theory of Filippov in [14], and the fact that the vector fields $F_1^{q_i}, F_2^{q_i}$ are parallel to each other on the line $\{x_2 = 0\}$, and they are both tranversal to this line. We denote by $\gamma_L^{q_i}(x_0)$ and $\gamma_R^{q_i}(x_0)$ the extremal arcs generated by $\phi_L^{q_i}(s,x_0)$ and $\phi_R^{q_i}(s,x_0)$, respectively, and we denote by $\mathcal{A}^{q_i}(x_0)$ the attainable set from x_0 of system (8) for fixed q_i, $i \in \{1,2\}$.

IV. Existence of a switching sequence

In this section we present necessary and sufficient conditions on the control value $\bar{u}$ in order that any solution of the double-integrator (1) with hybrid feedback (4) gives rise to a well-defined sequence of switching points $\{x^i\}$, with $i \in I \subset \mathbb{N}$, defined below. This result, stated in Proposition 4.3 below, is useful because it allows us to reduce the problem of proving convergence to the origin of state trajectories to the much simpler study of convergence of a sequence of switching points.

Definition 4.1: Let $x(t)$ be a solution of system (1) with hybrid feedback (4). A time instant t_i is called a **switching time** if $x(t_i) \in \big(S^+ \cup S^- \cup \bar{B}_{\delta_1}(0)\big)$ and at time $t = t_i$ a state transition $q_i \to q_k$, with $i,k \in \{1,2,3\}$, $i \neq k$ occurs. The value of the state at a switching time, $x^i = x(t_i)$ is called a **switching point**. △

Lemma 4.2: Let $0 \leq \delta_1 < \delta_2$. If, and only if, $\bar{u} > \bar{f}$, then for all $x_0 \in (\bar{B}_{\delta_1}(0))^c$ and all $f \in \mathcal{F}$, there exists a finite $\tau > 0$ such that at time τ there is a state transition $q_i \to q_j$, for some $i,j \in \{1,2,3\}$, $i \neq j$.

Lemma 4.2 will now be used to prove the following.

Proposition 4.3: Let $0 \leq \delta_1 < \delta_2$. If, and only if, $\bar{u} > \bar{f}$, then for any $f \in \mathcal{F}$ and any initial condition in $(\bar{B}_{\delta_1}(0))^c$, the solution $x(t)$ of (1) with hybrid feedback (4) induces a switching sequence $\{x^i\}$, $i \in I \subset \mathbb{N}$ nonempty, with the following property:

$$(x^1, \ldots, x^i \in (\bar{B}_{\delta_1}(0))^c) \implies i+1 \in I. \tag{12}$$

In other words, as long as the solution $x(t)$ does not enter $\bar{B}_{\delta_1}(0)$, there will be new switching points. Therefore, $x(t) \to \infty$ if and only if $I = \mathbb{N}$ and $x^i \to \infty$, and $x(t)$ enters $\bar{B}_{\delta_1}(0)$ if and only if $\{x_i\}$ enters $\bar{B}_{\delta_1}(0)$.

Proof: ($\Leftarrow$) Omitted for brevity.

($\Rightarrow$) If $\bar{u} > \bar{f}$, then by Lemma 4.2 for any initial condition in $(\bar{B}_{\delta_1}(0))^c$ there exists a finite time $\tau > 0$ at which there is a state transition in the automaton $\mathcal{A}$. The solution at time τ, $x(\tau)$, must lie in $\partial\Lambda^+$, or in $\partial\Lambda^-$, or in $\bar{B}_{\delta_1}(0)$.

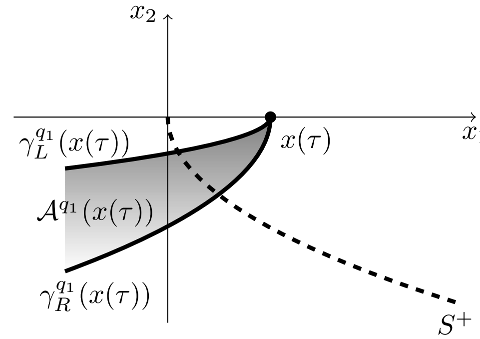


Fig. 4. Attainable set from $x(\tau)$, $\mathcal{A}^{q_1}(x(\tau))$.

If $x(\tau) \in \bar{B}_{\delta_1}(0)$, then $x(\tau)$ is a switching point according to Definition 4.1, and the singleton $\{x(\tau)\}$ trivially meets property (12). Therefore, the only case of interest is when $x(\tau) \in (\partial\Lambda^+ \cup \partial\Lambda^-) \setminus \bar{B}_{\delta_1}(0)$. We will consider the case, $x(\tau) \in \partial\Lambda^- \setminus \bar{B}_{\delta_1}(0)$, the other case being completely analogous. Either $x(\tau) \in S^-$ or $x(\tau) \in \{(x_1, 0) : x_1 \geq 0\}$. In the former case, $x(\tau)$ is a switching point according to Definition 4.1. In the latter case, $x(\tau)$ is not a switching point, but it induces a state transition $q_2 \to q_1$, and a new control value $u^\star(q_1) = -\bar{u}$. Lemma 4.2 guarantees the existence of time $t_1 > \tau$ at which a new state transition occurs. We claim that $x(t_1) \in S^+$, and therefore $x(t_1)$ is a switching point. Indeed, the extremal arcs from $x(\tau)$ are arcs of parabolas with negative concavity, shown in Figure 4, that intersect S^+. This fact and Lemma 3.3 imply that $x(t_1) \in S^+$. The proof of sufficiency follows by induction. ■

A byproduct of Proposition 4.3 is that, when $\bar{u} > \bar{f}$, only three types of switching points are possible. They are classified in the next definition.

Definition 4.4: Let $x^i \in (S^+ \cup S^-) \setminus \bar{B}_{\delta_1}(0)$ be a switching point of a solution $x(t)$ of (1) with hybrid feedback (4) and $\bar{u} > \bar{f}$, and consider the next switching point x^{i+1}, whose existence is guaranteed by Proposition 4.3.

- x^{i+1} is a **1-switch from** x_i if one of the points x^i, x^{i+1} belongs to S^+, and the other one belongs to S^-.
- x^{i+1} is a **2-switch from** x_i if x^i, x^{i+1} belong to the same arc of parabola, S^+ or S^-.
- x^{i+1} is a **0-switch from** x_i if $x^{i+1} \in \bar{B}_{\delta_1}(0)$.

Figure 5 illustrates a 1-switch and a 2-switch from a point $x^i \in S^+$.

V. Solution of SCCP

In this section we present the proof of Theorem 2.1. In Proposition 4.3 we have shown that any solution of (1) with hybrid feedback (4) induces a sequence of switching points $\{x^i\}_{i \in I}$. We begin by showing that this sequence is contracting (i.e., there exists $\alpha \in (0,1)$ such that $\|x^{i+1}\| \leq \alpha\|x^i\|$ for all $i \in I$) for sufficiently large control value $\bar{u}$.

Lemma 5.1: Consider system (1) with hybrid feedback (4), and pick δ_1, δ_2 such that $0 \leq \delta_1 < \delta_2$. The following are equivalent:

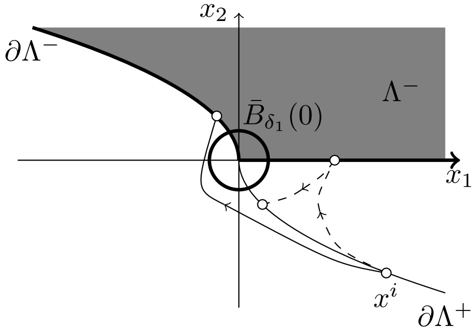


Fig. 5. Types of switching points induced by hybrid feedback (4). The 1-switch is depicted with a solid line, while the 2-switch by a dashed line.

(i) There exists $\alpha \in (0,1)$ such that for any $f \in \mathcal{F}$ and any initial condition, the sequence $\{x^i\}_{i\in I}$ of switching points induced by the solution $x(t)$ of (1) with hybrid feedback (4) is contracting as long as $x^i \notin \bar{B}_{\delta_1}(0)$: $x^i, x^{i+1} \in (\bar{B}_{\delta_1}(0))^c \implies \|x^{i+1}\| \le \alpha\|x^i\|$;

(ii) $\bar{u} > \bar{f}\left(1+\sqrt{5}\right)/2$.

Proof:

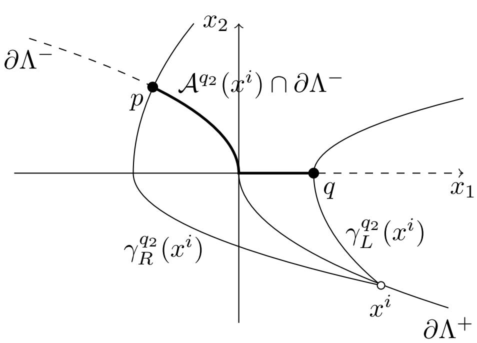


Fig. 6. Attainable switching set from x^i.

(ii) ⇒ (i). Assume that $x^i \in S^+$, so that the automaton $\mathcal{A}$ is at q_2 (the argument for the case $x^i \in S^-$ is analogous). If $x^{i+1} \in \bar{B}_{\delta_1}(0)$, then part (i) trivially holds. Suppose that $x^{i+1} \notin \bar{B}_{\delta_1}(0)$. Either $x^{i+1} \in S^-$ (i.e., x^{i+1} is a 1-switch from x^i) or $x^{i+1} \in S^+$ (i.e., x^{i+1} is a 2-switch from x^i).

Suppose first that $x^{i+1} \in S^-$, from which it follows that $x^{i+1} \in \mathcal{A}^{q_2}(x^i) \cap S^-$. Let $p = \gamma_R^{q_2}(x^i) \cap S^-$ (see Figure 6). Then x^{i+1} lies on the arc of parabola S^- delimited by 0 and p, implying that $\|x^{i+1}\| \le \|p\|$. Using the expression for $\phi_R^{q_2}(s, x^i)$ in (10) one can show that p exists and its first component p_1 is related to the first component x_1^i of x^i as $p_1 = -\alpha_1^2 x_1^i$, where

$$\alpha_1 = \left(\frac{\bar{f}(\bar{f}+\bar{u})}{(\bar{u}-\bar{f})(2\bar{u}+\bar{f})}\right)^{1/2}. \tag{13}$$

Since $\bar{u} > \bar{f}\left(1+\sqrt{5}\right)/2$, it holds that $\alpha_1 \in (0,1)$, and therefore

$$\|p\|^2 = (\alpha_1^2 x_1^i)^2 + 2\bar{u}c^2 x_1^i \le \alpha_1^2\left((x_1^i)^2 + 2\bar{u}x_1^i\right) \le \alpha_1^2\|x^i\|^2.$$

Hence $\|x^{i+1}\| \le \|p\| \le \alpha_1\|x^i\|$, with $\alpha_1 \in (0,1)$.

Now suppose that $x^{i+1} \in S^+$ is a two-switch from x^i. The switching point x^{i+1} is reached from x^i through the following sequence of events. (A) The solution from x^i with $u^\star(q_2) = \bar{u}$ hits the positive x_1 axis at a point z, a state transition $q_2 \to q_1$ occurs, and the control value becomes $u^\star(q_1) = -\bar{u}$. (B) The solution from z intersects S^+ in x^{i+1}. Consider the point $q = \gamma_L^{q_2}(x^i) \cap \{(x_1, 0) : x_1 \ge 0\}$ depicted in Figure 6. The point z defined above must lie on the segment of the x_1 axis delimited by 0 and q. Therefore, $\|z\| \le \|q\|$. Using the expression for $\phi_L^{q_2}(x^i)$ from (10) it can be shown that the first component q_1 of q satisfies $q_1 = \alpha_2^2 x_1^i$, where $\alpha_2 \in (0,1)$ is given by $\alpha_2 = \left(1 - \bar{u}/(\bar{u}+\bar{f})\right)^{1/2}$. Therefore, $\|z\| \le \|q\| \le \alpha_2^2 x_1^i$. Now we turn our attention to event (B) above. The point x^{i+1} lies in the segment $S^+ \cap \mathcal{A}^{q_1}(z)$. The extremal solutions from z are arcs of parabolas given by $\phi_L^{q_1}(s,z) = Y_s^{q_1}(z)$ and $\phi_R^{q_1}(s,z) = X_s^{q_1}(z)$, defined in Section III. In particular, the first component of both functions is decreasing with s. This implies that the first component x_1^{i+1} of x^{i+1} satisfies $x_1^{i+1} < z_1 \le \alpha_2^2 x_1^i$. We thus have

$$\begin{aligned}\|x^{i+1}\| &= [(x_1^{i+1})^2 + 2\bar{u}x_1^{i+1}]^{1/2} \le [\alpha_2^4 (x_1^i)^2 + 2\bar{u}\alpha_2^2 x_1^i]^{1/2} \\ &\le \alpha_2[(x_1^i)^2 + 2\bar{u}x_1^i]^{1/2} = \alpha_2\|x^i\|.\end{aligned}$$

By setting $\alpha = \max\{\alpha_1, \alpha_2\}$, and noting that $\alpha \in (0,1)$, the proof of sufficiency is complete.

(i) ⇒ (ii). Let $\{x^i\}$ be a contracting switching sequence and suppose, by way of contradiction, that $\bar{u} \le \bar{f}\left(1+\sqrt{5}\right)/2$. Let $x^i, x^{i+1} \in (\bar{B}_{\delta_1}(0))^c$. Assume $x^i \in S^+$ and let $f \in \mathcal{F}$ be defined as $f(x,t) = \bar{f}\,\mathrm{sign}(x_2(t))$. Then $x(t) = \phi_R^{q_2}(t - t_i, x^i)$ for all $t \in [t_i, t_{i+1}]$. Therefore $x^{i+1} = p \in S^-$, as defined in the proof of sufficiency. Recall that $p_1 = -\alpha_1^2 x_1^i$, with α_1 defined in (13). Since $\bar{u} \le \bar{f}(1+\sqrt{5})/2$ we have $\alpha_1 \ge 1$ which contradicts the hypothesis that the switching sequence is contracting. ∎

Next we show that for any $r > 0$, there exists a compact positively invariant subset of $B_r(0)$. This will be used to prove practical stability.

Lemma 5.2: Consider system (1) with the hybrid feedback (4). For any $r > 0$ there exist δ_1, δ_2, $0 < \delta_1 < \delta_2 < r$, and a compact positively invariant set $\mathcal{Q} \subset B_r(0)$ such that $\bar{B}_{\delta_2}(0) \subset \mathcal{Q}$.

Remark 5.3: δ_2 can be chosen as follows. If $(\bar{u} - \bar{f})^2 + \bar{u}\bar{f} - \bar{f}\sqrt{\bar{u}^2 + c^2r^2} \le 0$ then choose $\delta_2 < \sqrt{2\bar{f}(-\bar{u} + \sqrt{\bar{u}^2 + c^2r^2}) - (\bar{u} - \bar{f})^2}$, otherwise choose $\delta_2 < \frac{\bar{f}}{\bar{u}-\bar{f}}(-\bar{u} + \sqrt{\bar{u}^2 + c^2r^2})$, where $c = \min\{1, (\bar{u} - \bar{f})^2/\bar{f}^2 + 2\bar{u}(\bar{u} - \bar{f})/(r\bar{f})\}$. Let $\delta_1 \in (0, \delta_2)$.

Remark 5.4: Note that δ_2 has to be chosen so that $\delta_2 < r$: state trajectories with initial conditions in $\bar{B}_{\delta_2}(0)$ will eventually leave $\bar{B}_{\delta_2}(0)$, without, however, leaving $\mathcal{Q} \subset B_r(0)$.

We are ready to prove the main result of this paper.

Proof of Theorem 2.1.

(⇒) Similar to the proof of necessity of Lemma 5.1.

(⇐) The proof that the proposed controller meets specifications (i) and (iii) of SCCP is straightforward and is omitted. We now prove global practical asymptotic stability.

For any $r > 0$, by Lemma 5.2 there exists a positively invariant set $\mathcal{Q} \subset B_r(0)$ and $0 < \delta_1 < \delta_2$ such that $\bar{B}_{\delta_2} \subset \mathcal{Q}$. By Proposition 4.3, for any initial condition in $(\bar{B}_{\delta_1}(0))^c$ and any $f \in \mathcal{F}$, the solution $x(t)$ gives rise to a well-defined switching sequence $\{x^i\}_{i \in I}$. By Lemma 5.1, this sequence is contracting as long as $x^i \notin \bar{B}_{\delta_1}(0)$. Since $\bar{B}_{\delta_1}(0) \subset \bar{B}_{\delta_2}(0) \subset \mathcal{Q}$, $x^i \in \mathcal{Q}$ for sufficiently large i. By Lemma 5.2, $x(t) \in \mathcal{Q} \subset B_r(0)$ for all $t \geq t_i$. ■

VI. Simulations

Let $\bar{f} = 1.2 \cdot 10^{-3}$ and let the disturbance function be given by $f(x,t) = (\bar{f}/2)\left(\sin(0.7t - \pi/4) + \cos(0.1x_1)\right)$. $f(x,t)$ is bounded by $\bar{f}$. Suppose the actuator provides an input of $\bar{u} = 2 \cdot 10^{-3} > \bar{f}\left(1 + \sqrt{5}\right)/2$. We choose $r = 5 \cdot 10^{-3}$, $\delta_1 = 5 \cdot 10^{-4}$ and $\delta_2 = 2.5 \cdot 10^{-3}$ with initial conditions $x_0 = (0.5,\ 0.02)$. Figure 7 shows the approach of the state trajectory to the desired neighborhood $B_r(0)$, while Figure 8 confirms that $B_r(0)$ is indeed stabilized by controller (4). It can be seen in Figure 8 that whenever the trajectory hits $\partial B_{\delta_2}(0)$, the controller turns on and forces the trajectory toward $\bar{B}_{\delta_1}(0)$ again, without ever leaving $B_r(0)$. Figure 9 presents the time history of the control value $u/\bar{u}$.

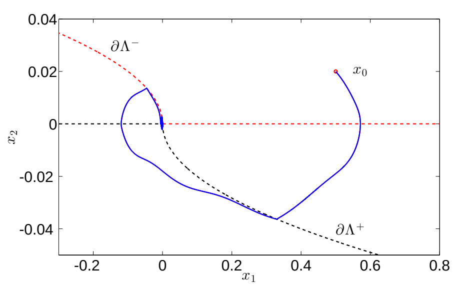


Fig. 7. Convergence of the state trajectory to $B_r(0)$.

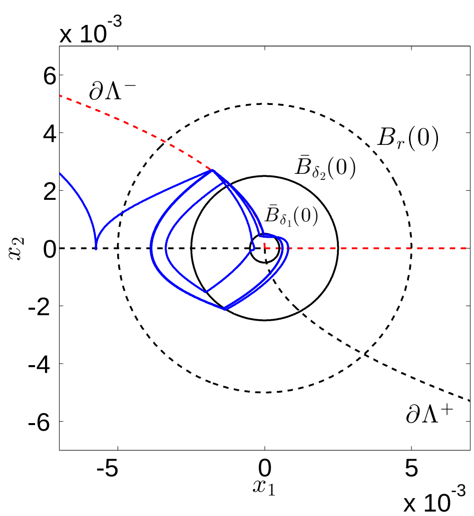


Fig. 8. Stabilization of $B_r(0)$.

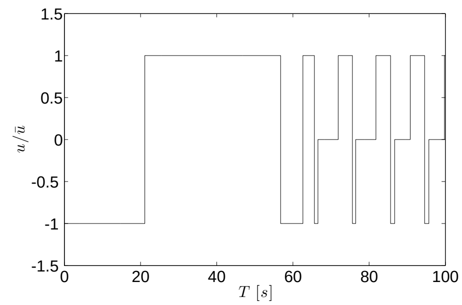


Fig. 9. Switching function $u/\bar{u}$.

VII. Conclusions

In the paper a hybrid control law has been proposed that solves the problem of global practical asymptotic stabilization of a double integrator affected by unknown bounded disturbances by means of constant controls. The proposed controller does not have sliding modes and it undergoes a finite number of switches in the control value over any compact time interval. Necessary and sufficient stability conditions were provided in terms of the control magnitude.

References

[1] P. Hughes, *Spacecraft attitude dynamics*. Wiley, 1986.

[2] M. Harris and B. Akmee, "Minimum time rendezvous of multiple spacecraft using differential drag," *Journal of Guidance, Control, and Dynamics*, vol. 37, no. 2, pp. 365–373, 2014.

[3] L. Mazal, G. Mingotti, and P. Gurfil, "Optimal on-off cooperative maneuvers for long-term satellite cluster flight," *Journal of Guidance, Control, and Dynamics*, vol. 37, no. 2, pp. 391–402, 2014.

[4] A. Bryson and Y. Ho, *Applied Optimal Control: Optimization, Estimation and Control*. Hemisphere, 1975.

[5] V. Rao and D. Bernstein, "Naive control of the double integrator," *Control Systems, IEEE*, vol. 21, no. 5, pp. 86–97, 2001.

[6] H. Lee and V. Utkin, "Chattering suppression methods in sliding mode control systems," *Annual Reviews in Control*, vol. 31, no. 2, pp. 179 – 188, 2007.

[7] P. Marti, M. Velasco, A. Camacho, E. Martin, and J. Fuertes, "Networked sliding mode control of the double integrator system using the event-driven self-triggered approach," in *Industrial Electronics (ISIE), 2011 IEEE International Symposium on*, June 2011, pp. 2031–2036.

[8] G. Bartolini, A. Ferrara, and E. Usai, "Output tracking control of uncertain nonlinear second-order systems," *Automatica*, vol. 33, no. 12, pp. 2203 – 2212, 1997.

[9] ——, "Chattering avoidance by second-order sliding mode control," *Automatic Control, IEEE Transactions on*, vol. 43, no. 2, pp. 241–246, Feb 1998.

[10] C. Mayhew and A. Teel, "Hybrid control of planar rotations," in *American Control Conference (ACC), 2010*, 2010, pp. 154–159.

[11] R. Sanfelice, A. Teel, and R. Goebel, "Supervising a family of hybrid controllers for robust global asymptotic stabilization," in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, 2008, pp. 4700–4705.

[12] C. Mayhew, R. Sanfelice, and A. Teel, "Robust global asymptotic attitude stabilization of a rigid body by quaternion-based hybrid feedback," in *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, 2009, pp. 2522–2527.

[13] M. Maggiore, B. Rawn, and P. Lehn, "Invariance kernels of single-input planar nonlinear systems," *SIAM J. Control and Optimization*, vol. 50, no. 2, pp. 1012–1037, 2012.

[14] A. F. Filippov, "Differential equations with discontinuous righthand sides," *Mathematics and Its Applications*, 1988.