University of Toronto
Department of Electrical and Computer Engineering

ECE411S Real-Time Computer Control
Spring 2004-Course Notes

## Chapter 1

## Analysis of Discrete Time Linear Systems

### 1.1 Introduction

There are 3 common ways to describe discrete time linear systems: difference equation models, transfer function models, and state space models. We shall study how to use each of these models for analysis, and show how you can move readily from one description to another. In most of our work, we shall study only single-input single-output systems, although many of the results generalize to multivariable systems as well.

### 1.2 Difference Equations

Consider the following difference equation with constant coefficients:

$$
\begin{equation*}
y(k)+a_{1} y(k-1)+\cdots+a_{n} y(k-n)=b_{0} u(k)+b_{1} u(k-1)+\cdots+b_{m} u(k-m) \tag{1.1}
\end{equation*}
$$

Here $u$ is the given input, and $y$ is the output to be determined. We can, in principle, solve this equation by recursion, starting with known initial conditions $y(-1), y(-2), \ldots, y(-n)$, and $u(-1), u(-2)$, $\ldots, u(-m)$. To do this, simply re-write the equation as

$$
y(k)=-a_{1} y(k-1)-\cdots-a_{n} y(k-n)+b_{0} u(k)+b_{1} u(k-1)+\cdots+b_{m} u(k-m)
$$

It is clear that the output $y(k)$ can be determined from the past inputs and outputs and the current input $u(k)$. However, we often would like to determine the analytical solution for $y$ for a given $u$. Similar to the case of differential equations with constant coefficients, the general solution of (1.1) can be written as

$$
y(k)=y_{h}(k)+y_{p}(k)
$$

where $y_{h}$ is the solution to the homogeneous equation

$$
\begin{equation*}
y(k)+a_{1} y(k-1)+\cdots+a_{n} y(k-n)=0 \tag{1.2}
\end{equation*}
$$

and $y_{p}$ is a particular solution to (1.1). If we take $p^{k}$ to be a trial solution to (1.2), we see that $p$ must satisfy the auxiliary equation

$$
\begin{equation*}
p^{n}+a_{1} p^{n-1}+\cdots+a_{n}=0 \tag{1.3}
\end{equation*}
$$

Each distinct root of the auxiliary equation gives rise to a distinct solution of the homogeneous equation. Suppose there are $n$ distinct roots $p_{1}, p_{2}, \cdots, p_{n}$ to (1.3). The general solution to (1.2) is then given by

$$
\begin{equation*}
y_{h}(k)=\alpha_{1} p_{1}^{k}+\alpha_{2} p_{2}^{k}+\cdots+\alpha_{n} p_{n}^{k} \tag{1.4}
\end{equation*}
$$

The particular solution $y_{p}(k)$ can often be determined by guessing the form of the solution and matching coefficients. The procedure is so close to that of solving higher-order inhomogeneous differential equations that we shall simply illustrate with an example.
Example 1.
Consider the following simple difference equation:

$$
\begin{equation*}
y(k)-2 y(k-1)=k \quad y(-1)=1 \tag{1.5}
\end{equation*}
$$

Rewriting it in the form

$$
y(k)=2 y(k-1)+k
$$

we see easily that the general solution is given by

$$
y(k)=2^{k} \alpha+y_{p}(k)
$$

For the particular solution $y_{p}$, try

$$
y_{p}(k)=A k+B
$$

Substituting in (1.5), we obtain

$$
A k+B-2[A(k-1)+B]=k
$$

This gives, on matching coefficients,

$$
2 A-B=0
$$

and

$$
\begin{gathered}
-A k=k \\
\Rightarrow A=-1 \quad B=-2
\end{gathered}
$$

The particular solution is therefore given by

$$
y_{p}(k)=-(k+2)
$$

The general solution is then

$$
y(k)=2^{k} \alpha-(k+2)
$$

On putting $k=0$, we get

$$
y(0)=2 y(-1)=2
$$

Substituting into the general solution, we find

$$
\alpha=y(0)+2=2\left(y_{-1}+1\right)=4
$$

The complete solution is given by

$$
\begin{aligned}
y(k) & =4 \times 2^{k}-(k+2) \\
& =2^{k+1}\left(y_{-1}+1\right)-(k+2)
\end{aligned}
$$

While it is possible to give a more general treatment of solutions of linear higher-order difference equations, including the variation of parameters formula for inhomogeneous equations, the above approach often gives an effective method of solution. We refer you to F.B. Hildebrand, Finite Difference Equations and Simulations for further details.

### 1.3 Z-transforms

The z-transform is the analogue of the Laplace transform for analyzing discrete time signals. Assume that the discrete time sequence $x_{k}$ satisfies

$$
\left|x_{k}\right| \leq c r_{0}^{k}
$$

i.e. $x_{k}$ is exponentially (geometrically) bounded. Then for all $r>r_{0}$

$$
\begin{gathered}
\Sigma\left|x_{k}\right| r^{-k} \\
\leq c \Sigma\left(\frac{r_{0}}{r}\right)^{k}<\infty
\end{gathered}
$$

Define $z$-transform of $x_{k}$ as

$$
X(z)=\sum_{k=0}^{\infty} x_{k} z^{-k}
$$

We see that $X(z)$ converges in $|z|>r_{0}$

$X(z)$ is then an analytic function in the region of convergence. For convenience, we often use the symbol $\mathcal{Z}$ to denote the z -transform operator.

## Example:

$$
\begin{gathered}
x_{k}=a^{k} \quad k \geq 0 \\
X(z)=\sum_{k=0}^{\infty}\left(a z^{-1}\right)^{k}=\frac{1}{1-a z^{-1}} \quad|z|>|a|
\end{gathered}
$$

For notational convenience, we indicate $a^{k}$ and $\frac{1}{1-a z^{-1}}$ are $z$-transform pairs by writing $\mathcal{Z}\left(a^{k}\right)=\frac{1}{1-a z^{-1}}$, or $\mathcal{Z}^{-1}\left[\frac{1}{1-a z^{-1}}\right]=a^{k}$.

Next we examine some basic properties and results in connection with z-transforms.

## Inversion integral:

$$
\begin{equation*}
x_{k}=\frac{1}{2 \pi j} \oint X(z) z^{k-1} d z \tag{1.6}
\end{equation*}
$$

with the circular path of the contour integral inside region of convergence. The validity of this formula can be seen from

$$
\begin{aligned}
& \frac{1}{2 \pi j} \oint \Sigma x_{n} z^{-n} z^{k-1} d z \\
& \quad=\frac{1}{2 \pi j} \Sigma \oint x_{n} z^{-(n-k)} \frac{d z}{z}
\end{aligned}
$$

(convergence uniform to permit integration term by term)

$$
\begin{aligned}
& =x_{k} \\
& =\Sigma \text { residues of } X(z) z^{k-1} \text { inside } C
\end{aligned}
$$

## Example:

$$
\begin{aligned}
& X(z)=\frac{1}{1-a z^{-1}} \quad|z|>|a| \\
& x_{k}=\frac{1}{2 \pi j} \oint \frac{z^{k-1}}{1-a z^{-1}} d z \\
& \quad=\frac{1}{2 \pi j} \oint \frac{z^{k}}{z-a} d z=a^{k} \quad k \geq 0
\end{aligned}
$$

Note the importance of knowing the region of convergence. If the contour had been chosen in $|z|<|a|$, the integral would be 0 .

We can also do an infinite series expansion to get

$$
\frac{1}{1-a z^{-1}}=\Sigma a^{k} z^{-k}
$$

from which we can recognize that $a^{k}$ is the time sequence.
Since a discrete-time signal in computer control is usually defined for $k \geq 0$, it invariably gives rise to a z-transform with a region of convergence being the exterior of a circle with a sufficiently large radius. For
this reason, the region of convergence for a transform $X(z)$ is often omitted with the understanding that it will enclose all the poles of $X(z)$.

Using the inversion integral, one can show that

$$
\begin{equation*}
\mathcal{Z}^{-1}\left[\frac{z}{(z-p)^{i+1}}\right]=\frac{k!}{i!(k-i)!} p^{k-i} \quad \text { for all } i \geq 0 \tag{1.7}
\end{equation*}
$$

This is a very useful formula which, as we shall see, will help us to invert many z-transforms quickly. Two cases of particular interest are:

$$
\begin{gathered}
i=0: \quad \mathcal{Z}^{-1}\left[\frac{z}{z-p}\right]=\mathcal{Z}^{-1}\left[\frac{1}{1-p z^{-1}}\right]=p^{k} \\
i=1: \quad \mathcal{Z}^{-1}\left[\frac{z}{(z-p)^{2}}\right]=k p^{k-1}
\end{gathered}
$$

Beyond the basic definition of z-transforms and the inversion integral, there are a number of useful properties of $z$-transforms which we quickly survey.
Convolution of (causal) time sequences:

$$
\begin{gathered}
w_{k}=\sum_{l=0}^{k} x_{l} y_{k-l} \\
W(z)=\sum_{k=0}^{\infty} \sum_{l=0}^{k} x_{l} y_{k-l} z^{-k} \\
=\sum_{l=0}^{\infty} \sum_{k=l}^{\infty} x_{l} y_{k-l} z^{-(k-l)} z^{-l} \\
= \\
\sum_{l=0}^{\infty} \sum_{j=0}^{\infty} x_{l} y_{j} z^{-j} z^{-l}=X(z) Y(z)
\end{gathered}
$$

We often describe this result as convolution in the time domain corresponds to multiplication in the ztransform domain.

## Multiplication of time sequences:

$$
w_{k}=x_{k} y_{k}
$$

where $X(z)$ has region of convergence $|z|>R_{0}$ and $Y(z)$ has region of convergence $|z|>R_{1}$.

$$
\begin{aligned}
W(z) & =\sum_{k} x_{k} y_{k} z^{-k}=\sum x_{k} z^{-k} \frac{1}{2 \pi j} \oint Y(\zeta) \zeta^{k-1} d \zeta \\
& =\frac{1}{2 \pi j} \oint \sum x_{k}\left(\frac{z}{\zeta}\right)^{k} Y(\zeta) \frac{d \zeta}{\zeta}=\frac{1}{2 \pi j} \oint X\left(\frac{z}{\zeta}\right) Y(\zeta) \frac{d \zeta}{\zeta}
\end{aligned}
$$

where the contour integral is over a circle $|\zeta|>R_{1}$. Since we require

$$
\left|\frac{z}{\zeta}\right|>R_{0}, \quad|\zeta|>R_{1} \Rightarrow|z|>R_{0} R_{1} \text { is the region of convergence }
$$

It can also be expressed as

$$
W(z)=\frac{1}{2 \pi j} \oint X(\zeta) Y\left(\frac{z}{\zeta}\right) \frac{d \zeta}{\zeta}
$$

where the contour integral is over a circle $|\zeta|>R_{0}$. This result is the dual of the previous one. We often refer to it as multiplcation in the time domain corresponds to convolution in the z -transform domain. Multiplication by $a^{k}$ :

$$
\mathcal{Z}\left\{a^{k} x_{k}\right\}=X\left(\frac{z}{a}\right)
$$

since

$$
\sum a^{k} x_{k} z^{-k}=\sum x_{k}\left(\frac{z}{a}\right)^{-k}
$$

The region of convergence can be readily determined as follows: If

$$
\begin{aligned}
\left|x_{k}\right| & \leq c r_{0}^{k} \\
& \Rightarrow\left|a^{k} x_{k}\right| \leq c\left[|a| r_{0}\right]^{k}
\end{aligned}
$$

Hence $|z|>|a| r_{0}$ is the region of convergence for $X\left(\frac{z}{a}\right)$.
From the point of view of solving difference equations, the most important property of z-transforms is the following.

## Translation:

Backward shift:

$$
\begin{aligned}
\sum_{k=0}^{\infty} x_{k-m} z^{-k} & =\sum_{k=0}^{m-1} x_{k-m} z^{-k}+\sum_{k=m}^{\infty} x_{k-m} z^{-(k-m)} z^{-m} \\
& =\sum_{k=0}^{m-1} x_{k-m} z^{-k}+z^{-m} X(z) \\
& =z^{-m} X(z)+x_{-m}+z^{-1} x_{-m+1}+\ldots+x_{-1} z^{-m+1}
\end{aligned}
$$

Forward shift:

$$
\begin{aligned}
\sum_{k=0}^{\infty} x_{k+m} z^{-k} & =\sum_{k=0}^{\infty} x_{k+m} z^{-(k+m)} z^{m} \\
& =\sum_{l=m}^{\infty} x_{l} z^{-l} z^{m}=z^{m} X(z)-\sum_{l=0}^{m-1} x_{l} z^{-l} z^{m} \\
& =z^{m} X(z)-\left\{z^{m} x_{0}+z^{m-1} x_{1}+\ldots+z x_{m-1}\right\}
\end{aligned}
$$

Two additional properties which we do not use very often are included for completeness.
Initial Value Theorem: The initial value of a sequence $x_{k}$ with z-transform $X(z)$ is given by

$$
x_{0}=\lim _{z \rightarrow \infty} X(z)
$$

Final Value Theorem: Assume $f_{k} \xrightarrow[k \rightarrow \infty]{ } A<\infty$. Then

$$
\begin{aligned}
& \lim _{z \rightarrow 1}^{z \rightarrow 1} \gg 1 \text { real } \\
& \gg 1 \\
& >
\end{aligned}(z-1) F(z)=A
$$

Solving difference equations: Consider the difference equation

$$
y_{k}+a_{1} y_{k-1}+\cdots+a_{n} y_{k-n}=b_{0} u_{k}+\cdots+b_{m} u_{k-m} \text { with } u_{k}=0, k<0 .
$$

Putting $a_{0}=1$, we can write the above equation as

$$
\sum_{j=0}^{n} a_{j} y_{k-j}=\sum_{j=0}^{m} b_{j} u_{k-j}
$$

Suppose $\left|u_{k}\right| \leq \beta r_{u}^{k}$ for some $\beta \geq 0, r_{u}>0$. Almost all inputs in practice will satisfy some such geometric bound. Then the solution $y_{k}$ will satisfy also a geometric bound and hence z -transformable. Taking $z$-transform of the left hand side gives

$$
\begin{aligned}
\sum_{k=0}^{\infty} \sum_{j=0}^{n} a_{j} y_{k-j} z^{-k} & =Y(z)+a_{1} z^{-1} Y(z)+\ldots \\
& =A\left(z^{-1}\right) Y(z)+\sum_{j=1}^{n} \sum_{k=0}^{j-1} a_{j} y_{k-j} z^{-k}
\end{aligned}
$$

where

$$
\begin{align*}
& A\left(z^{-1}\right)=\sum_{j=0}^{n} a_{j} z^{-j}, \text { with } a_{0}=1 \\
& \therefore Y(z)=\frac{I(z)}{A\left(z^{-1}\right)}+\frac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)} U(z) \tag{1.8}
\end{align*}
$$

where $I(z)$ is a polynomial depending on the initial condition. Let

$$
Y_{i}(z)=\frac{I(z)}{A\left(z^{-1}\right)}
$$

and

$$
Y_{e}(z)=\frac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)} U(z)
$$

In terms of the terminology of Section $1.2, Y_{i}(z)$ is the transform of a homogeneous solution, and $Y_{e}(z)$ is the transform of a particular solution. The solution $y_{k}$ can then be obtained by taking the inverse z-transform.
As an example, we solve the difference equation (1.5) using $z$-transforms. First note that in terms of our polynomial notation,

$$
A\left(z^{-1}\right)=1-2 z^{-1}
$$

$$
B\left(z^{-1}\right)=1
$$

Since

$$
\begin{aligned}
& U(z)=\sum_{k=0}^{\infty} k z^{-k}=-z \frac{d}{d z} \sum_{k=0}^{\infty} z^{-k} \\
& =-z \frac{d}{d z} \frac{1}{1-z^{-1}}=-z \frac{d}{d z}\left(\frac{z}{z-1}\right) \\
& =-z \frac{(z-1)-z}{(z-1)^{2}}=\frac{z}{(z-1)^{2}}
\end{aligned}
$$

we can write (1.5) in the form

$$
\begin{gathered}
Y(z)-2 z^{-1} Y(z)-2 y_{-1}=\frac{z}{(z-1)^{2}} \\
Y(z)=\frac{2 y_{-1}}{1-2 z^{-1}}+\frac{z}{(z-1)^{2}\left(1-2 z^{-1}\right)}
\end{gathered}
$$

In the terminology of (1.8),

$$
Y_{i}(z)=\frac{I(z)}{A\left(z^{-1}\right)}=\frac{2 y_{-1}}{1-2 z^{-1}}=\frac{2}{1-2 z^{-1}}
$$

and

$$
Y_{e}(z)=\frac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)} U(z)=\frac{z}{(z-1)^{2}\left(1-2 z^{-1}\right)}
$$

Inverting $Y_{i}(z)$ readily gives

$$
y_{i}(k)=2 \times 2^{k}
$$

To invert $Y_{e}(z)$, we shall make use of (1.7). We first perform a partial-fraction expansion of $\frac{Y_{e}(z)}{z}$ :

$$
\begin{aligned}
\frac{Y_{e}(z)}{z} & =\frac{z}{(z-1)^{2}(z-2)} \\
& =\frac{2}{z-2}+\frac{\alpha z+\beta}{(z-1)^{2}} \\
& =\frac{2(z-1)^{2}+\alpha z^{2}+(\beta-2 \alpha) z-2 \beta}{(z-1)^{2}(z-2)} \\
& =\frac{2\left(z^{2}-2 z+1\right)+\alpha z^{2}+(\beta-2 \alpha) z-2 \beta}{(z-1)^{2}(z-2)}
\end{aligned}
$$

On matching coefficients, we have

$$
\alpha=-2 \quad \beta=1
$$

Putting everything together, we obtain

$$
\begin{aligned}
\frac{Y_{e}(z)}{z} & =\frac{2}{z-2}+\frac{-2 z}{(z-1)^{2}}+\frac{1}{(z-1)^{2}} \\
& =\frac{2}{z-2}-\frac{2(z-1)+2}{(z-1)^{2}}+\frac{1}{(z-1)^{2}} \\
& =\frac{2}{z-2}-\frac{2}{z-1}-\frac{1}{(z-1)^{2}}
\end{aligned}
$$

Hence

$$
Y_{e}(z)=\frac{2 z}{z-2}-\frac{2 z}{z-1}-\frac{z}{(z-1)^{2}}
$$

Now each term of $Y_{e}(z)$ can be inverted using (1.7) to give

$$
\begin{gathered}
y_{e}(k)=2 \times 2^{k}-(k+2) \\
\therefore y(k)=y_{i}(k)+y_{e}(k)=4 \times 2^{k}-(k+2)
\end{gathered}
$$

which is the same result as before.
The solution via z-transform often involves expansion the z-transform into partial fractions. A convenient way to compute partial fractions, when there are repeated poles, say of order $m$ at the point $p$, is to expand

$$
\frac{Y(z)}{z}=\frac{c_{1}}{z-p}+\frac{c_{2}}{(z-p)^{2}}+\ldots+\frac{c_{m}}{(z-p)^{m}}+g(z)
$$

where $g(z)$ is analytic at $p$ and

$$
\begin{aligned}
c_{m} & =\left.\left[(z-p)^{m} \frac{Y(z)}{z}\right]\right|_{z=p} \\
c_{m-1} & =\left.\frac{d}{d z}\left[(z-p)^{m} \frac{Y(z)}{z}\right]\right|_{z=p} \\
& \vdots \\
c_{1} & =\left.\frac{1}{(m-1)!} \frac{d^{m-1}}{d z^{m-1}}\left[(z-p)^{m} \frac{Y(z)}{z}\right]\right|_{z=p}
\end{aligned}
$$

From this expansion, we can write

$$
Y(z)=c_{1} \frac{z}{z-p}+c_{2} \frac{z}{(z-p)^{2}}+\ldots+c_{m} \frac{z}{(z-p)^{m}}+z g(z)
$$

so that the first $m$ terms in the expansion of $Y(z)$ (i.e. not including $z g(z)$ ), corresponding to the contribution of the poles at $p$ to $y(k)$, can be writtne down with the help of (1.7).

### 1.4 State Space Analysis of Linear Systems

The third method for analysing linear time-invariant discrete-time systems that we shall study is state space analysis. Here the analysis of the system response is via the state equation, which we shall examine first.

The state equation for a linear time-invariant discrete-time system is given by

$$
\begin{gather*}
x(k+1)=A x(k)+B u(k)  \tag{1.9}\\
y(k)=C x(k)+D u(k) \tag{1.10}
\end{gather*}
$$

By recursive substitution, we find that the solution is given by

$$
\begin{gather*}
x(k)=A^{k-k_{0}} x\left(k_{0}\right)+\sum_{j=k_{0}}^{k-1} A^{k-j-1} B u_{( }(j)  \tag{1.11}\\
y(k)=C A^{k-k_{0}} x\left(k_{0}\right)+\sum_{j=k_{o}}^{k-1} C A^{k-j-1} B u(j)+D u(k) \tag{1.12}
\end{gather*}
$$

It is clear that the solution for $x(k)$ depends on $A^{k}$, which we consider next.

### 1.5 Computing $A^{k}$

We examine 2 methods: diagonalization and $z$-transform.

## I. Diagonalization

Assume that the matrix $A$ can be diagonalized (for example, when $A$ has n distinct eigenvalues or is symmetric). Then there exists a nonsingular matrix $T$ such that

$$
T^{-1} A T=\Lambda
$$

where $\Lambda$ is the diagonal matrix consisting of the eigenvalues of $A$. Raising $\Lambda$ to the $k$ th power gives

$$
\Lambda^{k}=T^{-1} A^{k} T=\left[\begin{array}{cccc}
\lambda_{1}^{k} & 0 & \cdots & 0 \\
0 & \lambda_{2}^{k} & 0 & \vdots \\
\vdots & & \ddots & 0 \\
0 & \cdots & 0 & \lambda_{n}^{k}
\end{array}\right]
$$

so that

$$
A^{k}=T\left[\begin{array}{cccc}
\lambda_{1}^{k} & 0 & \cdots & 0  \tag{1.13}\\
0 & \lambda_{2}^{k} & 0 & \vdots \\
\vdots & & \ddots & 0 \\
0 & \cdots & 0 & \lambda_{n}^{k}
\end{array}\right] T^{-1}
$$

## Example 1.

$$
A=\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right]
$$

$$
\begin{aligned}
& \operatorname{det}(z I-A)=\operatorname{det}\left[\begin{array}{cc}
z & -1 \\
2 & z+3
\end{array}\right] \\
& =z^{2}+3 z+2=(z+2)(z+1)
\end{aligned}
$$

Since $A$ has distinct eigenvalues, the matrix $T$ consisting of the linearly independent eigenvectors of $A$ as its columns will diagonalize $A$. We next determine the eigenvectors.

$$
\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right]\left[\begin{array}{l}
v_{1} \\
v_{2}
\end{array}\right]=-2\left[\begin{array}{l}
v_{1} \\
v_{2}
\end{array}\right]
$$

Solving for $v_{1}$ and $v_{2}$ yields

$$
\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right]\left[\begin{array}{r}
-2 \\
4
\end{array}\right]=-2\left[\begin{array}{r}
-2 \\
4
\end{array}\right]
$$

Similarly,

$$
\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right]\left[\begin{array}{r}
-1 \\
1
\end{array}\right]=-1\left[\begin{array}{r}
-1 \\
1
\end{array}\right]
$$

We can verify that

$$
T=\left[\begin{array}{rr}
-1 & -2 \\
1 & 4
\end{array}\right]
$$

does diagonalize $A$ :

$$
\begin{gathered}
{\left[\begin{array}{rr}
-1 & -2 \\
1 & 4
\end{array}\right]^{-1}\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right]\left[\begin{array}{rr}
-1 & -2 \\
1 & 4
\end{array}\right]} \\
=-\frac{1}{2}\left[\begin{array}{rr}
4 & 2 \\
-1 & -1
\end{array}\right]\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right]\left[\begin{array}{rr}
-1 & -2 \\
1 & 4
\end{array}\right] \\
=-\frac{1}{2}\left[\begin{array}{rr}
4 & 2 \\
-1 & -1
\end{array}\right]\left[\begin{array}{rr}
1 & 4 \\
-1 & -8
\end{array}\right]=-\frac{1}{2}\left[\begin{array}{ll}
2 & 0 \\
0 & 4
\end{array}\right] \\
=\left[\begin{array}{rr}
-1 & 0 \\
0 & -2
\end{array}\right]
\end{gathered}
$$

Using (1.13), we obtain

$$
\begin{aligned}
A^{k}= & -\frac{1}{2}\left[\begin{array}{rr}
-1 & -2 \\
1 & 4
\end{array}\right]\left[\begin{array}{cc}
(-1)^{k} & 0 \\
0 & (-2)^{k}
\end{array}\right]\left[\begin{array}{rr}
4 & 2 \\
-1 & -1
\end{array}\right] \\
& =-\frac{1}{2}\left[\begin{array}{rr}
-1 & -2 \\
1 & 4
\end{array}\right]\left[\begin{array}{cc}
4(-1)^{k} & 2(-1)^{k} \\
-(-2)^{k} & -(-2)^{k}
\end{array}\right]
\end{aligned}
$$

$$
\begin{gathered}
=-\frac{1}{2}\left[\begin{array}{cc}
-4(-1)^{k}+2(-2)^{k} & -2(1)^{k}+2(-2)^{k} \\
4(-1)^{k}-4(-2)^{k} & 2(1)^{k}-4(-2)^{k}
\end{array}\right] \\
\\
{\left[\begin{array}{cc}
2(-1)^{k}-(-2)^{k} & (-1)^{k}-(-2)^{k} \\
-2(-1)^{k}+2(-2)^{k} & -(-1)^{k}+2(-2)^{k}
\end{array}\right]}
\end{gathered}
$$

## II. Solution by z-transform

The second method for solving state equations is by use of z-transforms. A state equation is a first-order vector-valued difference equation. Solving it using $z$-transforms is a natural procedure. Taking z -transforms of both sides of (1.9), we obtain

$$
\begin{align*}
z X(z)-z x_{0} & =A X(z)+B U(z)  \tag{1.14}\\
X(z) & =(z I-A)^{-1} z x_{0}+(z I-A)^{-1} B U(z)  \tag{1.15}\\
& =\left(I-z^{-1} A\right)^{-1} x_{0}+z^{-1}\left(I-z^{-1} A\right)^{-1} B U(z) \tag{1.16}
\end{align*}
$$

Comparing this with (1.11), and using $\mathcal{Z}^{-1}$ to denote the inverse z-transform operation, we see that

$$
A^{k}=\mathcal{Z}^{-1}\left(I-z^{-1} A\right)^{-1}
$$

It is of interest to note that $X(z)$ has a power series expansion in $z^{-1}$ of the form

$$
X(z)=\sum A^{k} z^{-k} x_{0}+\sum_{l=0}^{\infty} A^{l} z^{-(l+1)} B \sum_{j=0}^{\infty} u_{j} z^{-j}
$$

Re-arranging, we get

$$
\begin{aligned}
& X(z)=\sum_{k=0}^{\infty} A^{k} z^{-k} x_{0}+\sum_{k=1}^{\infty} \sum_{j=0}^{k-1} A^{k-j-1} B u(j) z^{-k} \\
& \therefore \quad x(k)=A^{k} x_{0}+\sum_{j=0}^{k-1} A^{k-j-1} B u(j) \quad k \geq 1
\end{aligned}
$$

which is the same as (1.11) (for $k_{0}=0$ ).
Example 2.
Let us determine $A^{k}$ for the matrix $A$ in Example 1 using the z-transform method.

$$
\begin{aligned}
& \mathcal{Z}\left(A^{k}\right)=\left(I-z^{-1} A\right)^{-1}=\left[\begin{array}{cc}
1 & -z^{-1} \\
2 z^{-1} & 1+3 z^{-1}
\end{array}\right]^{-1} \\
& \quad=\frac{\left[\begin{array}{cc}
1+3 z^{-1} & z^{-1} \\
2 z^{-1} & 1
\end{array}\right]}{1+3 z^{-1}+2 z^{-2}}=\frac{\left[\begin{array}{cc}
1-3 z^{-1} & z^{-1} \\
-2 z^{-1} & 1
\end{array}\right]}{\left(1+z^{-1}\right)\left(1+2 z^{-1}\right)}
\end{aligned}
$$

$$
=\left[\begin{array}{cc}
\frac{2}{1+z^{-1}}+\frac{-1}{1+2 z^{-1}} & \frac{1}{1+z^{-1}}+\frac{-1}{1+2 z^{-1}} \\
\frac{-2}{1+z^{-1}}+\frac{2}{1+2 z^{-1}} & \frac{-1}{1+z^{-1}}+\frac{2}{1+2 z^{-1}}
\end{array}\right]
$$

Inversion gives

$$
=\left[\begin{array}{ll}
2(-1)^{k}+(-1)(-2)^{k} & (-1)^{k}-(-2)^{k} \\
-2(-1)^{k}+2(-2)^{k} & -(-1)^{k}+2(-2)^{k}
\end{array}\right]
$$

which is the same result as before.

## Example 3.

We can also use the analytical formula for the solution of the state equation, (1.11), to solve the difference equation of Example 1. We first re-write it in state equation form:

$$
\begin{aligned}
& y(k+1)=2 y(k)+(k+1) \\
& y(k)=2^{k} y_{0}+\sum_{j=0}^{k-1} 2^{k-j-1}(j+1) \\
& =2^{k} y_{0}+\sum_{l=0}^{k-1} 2^{l}+2^{k-1} \sum_{j=0}^{k-1} j 2^{-j} \\
& =2^{k} y_{0}+\left(2^{k-1}\right)+2^{k-1} \sum_{j=0}^{k-1} j 2^{-j}
\end{aligned}
$$

We first determine

$$
\begin{gathered}
\sum_{j=0}^{k-1} j \beta^{j}=\beta \frac{d}{d \beta} \sum_{j=0}^{k-1} \beta^{j} \\
=\beta \frac{d}{d \beta}\left(\frac{1-\beta^{k}}{1-\beta}\right)=\beta \frac{d}{d \beta}\left(\frac{\beta^{k}-1}{\beta-1}\right) \\
=\beta \frac{k(\beta-1) \beta^{k-1}-\left(\beta^{k}-1\right)}{(\beta-1)^{2}} \\
=\beta \frac{k \beta^{k}-k \beta^{k-1}-\beta^{k}+1}{(\beta-1)^{2}} \\
=\beta \frac{k \beta^{k-1}(\beta-1)-\left(\beta^{k}-1\right)}{(\beta-1)^{2}}
\end{gathered}
$$

On setting $\beta=\frac{1}{2}$,

$$
\begin{gathered}
\sum_{j=0}^{k-1} j 2^{-j}=\frac{1}{2} \frac{k\left(\frac{1}{2}\right)^{k-1}\left(-\frac{1}{2}\right)-\left[\left(\frac{1}{2}\right)^{k}-1\right]}{\frac{1}{4}} \\
=-k\left(\frac{1}{2}\right)^{k-1}-2\left[\left(\frac{1}{2}\right)^{k}-1\right] \\
\therefore \quad y(k)=2^{k} y_{0}+2^{k}-1+2^{k-1}\left[-k\left(\frac{1}{2}\right)^{k-1}-\left(\frac{1}{2}\right)^{k-1}+2\right] \\
=2^{k} y_{0}+2 \times 2^{k}-2-k \\
=4 \times 2^{k}-(k+2)
\end{gathered}
$$

which is the same result as before.
Alternatively, we can also apply z-transform to solve the equation. Details are similar to the previous z-transform calculation and are omitted.

### 1.6 State Space to Input-Output and Transfer Function Descriptions

Let $k_{0}=0$. Then

$$
\begin{align*}
& y(k)=C A^{k} x(0)+\sum_{j=0}^{k-1} C A^{k-j-1} B u(j)+D u(k)  \tag{1.17}\\
& y(k)=C A^{k} x(0)+\sum_{j=0}^{k} h(k-j) u(j) \\
& \text { where } \\
& h(k)=C A^{k-1} B \\
& k>0 \\
& =D \quad k=0 \\
& =0 \quad k<0
\end{align*}
$$

$h(k)$ is called the impulse response or the weighting function.
The z-transform of the output, $Y(z)$, can similarly be expressed in terms of $x_{0}$ and $U(z)$ by using (1.15).

$$
\begin{equation*}
Y(z)=C X(z)+D U(z)=C(z I-A)^{-1} z x_{0}+\left[C(z I-A)^{-1} B+D\right] U(z) \tag{1.18}
\end{equation*}
$$

The transfer function from $u$ to $y$ is therefore given by

$$
\begin{equation*}
G(z)=C(z I-A)^{-1} B+D \tag{1.19}
\end{equation*}
$$

Recall that a proper (scalar) rational function is a ratio of 2 polynomials with the degree of the numerator polynomial $\leq$ the degree of the denominator polynomial. A proper rational function is strictly proper if
the degree of the numerator polynomial < the degree of the denominator polynomial. In the single-input single-output case, i.e., both $u$ and $y$ are scalar-valued, we can express

$$
C(z I-A)^{-1} B=\frac{\operatorname{Cadj}(z I-A) B}{\operatorname{det}(z I-A)}
$$

which is a strictly proper rational function, where $\operatorname{adj}(A)$ denotes the adjoint of the matrix $A$. Hence the transfer function $G(z)$ is proper but not strictly proper if and only if $D \neq 0$.

There is a very convenient interpretation of the $z$ variable as a shift operator, namely,

$$
\begin{array}{cl}
\left(z^{-1} x\right)(k)=x(k-1) & \text { backward shift } \\
(z x)(k)=x(k+1) & \text { forward shift }
\end{array}
$$

Using the shift operator interpretaion, we can rewrite the higher order difference equation (1.1) in the form

$$
\begin{equation*}
A\left(z^{-1}\right) y(k)=B\left(z^{-1}\right) u(k) \tag{1.20}
\end{equation*}
$$

where

$$
A\left(z^{-1}\right)=\sum_{j=0}^{n} a_{j} z^{-j}
$$

with the leading coefficient $a_{0}=1$ and

$$
B\left(z^{-1}\right)=\sum_{j=0}^{m} b_{j} z^{-j}
$$

We can then write

$$
\begin{equation*}
y(k)=\frac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)} u(k) \tag{1.21}
\end{equation*}
$$

Note also that in terms of $z$-transforms

$$
\begin{equation*}
Y(z)=\frac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)} U(z) \tag{1.22}
\end{equation*}
$$

where now $z$ is a complex variable. These 2 interpretations of $\frac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)}$ allow us to go immediately from z-transform to difference equation, and vice versa.

## Example 4:

Consider a state space system with

$$
\begin{aligned}
A=\left[\begin{array}{cc}
0 & 1 \\
-2 & -3
\end{array}\right] \quad B & =\left[\begin{array}{l}
0 \\
1
\end{array}\right] \quad C=\left[\begin{array}{ll}
1 & 2
\end{array}\right] \quad D=0 \\
(z I-A)^{-1} & =\left[\begin{array}{cc}
z & -1 \\
2 & z+3
\end{array}\right]^{-1} \\
& =\frac{\left[\begin{array}{cc}
z+3 & 1 \\
-2 & z
\end{array}\right]}{z^{2}+3 z+2}
\end{aligned}
$$

so that

$$
G(z)=\frac{2 z+1}{z^{2}+3 z+2}
$$

Interpreted as a difference equation, with $z$ as the forward shift, we can also write the input-output relation as

$$
y_{k+2}+3 y_{k+1}+2 y_{k}=2 u_{k+1}+u_{k}
$$

or equivalently

$$
y_{k}+3 y_{k-1}+2 y_{k-2}=2 u_{k-1}+u_{k-2}
$$

### 1.7 Connecting the Different Models

It is of interest to connect the 3 different methods of analysis so that one can move easily from one description to another. We have already shown the connection between difference equations and z-transforms. Since it is straightforward to obtain the transfer function from the state equation (see (1.19)), we know how to go from state equations to an input-output description. To complete the connections, we show here how one can write down a state equation corresponding to a higher-order difference equation.

Difference equation to state models
Suppose the inputs and outputs are related by the difference equatioSuppose the inputs and outputs are related by the difference equation

$$
y(k)+a_{1} y(k-1)+\cdots+a_{n} y(k-n)=b_{0} u(k)+\cdots+b_{n} u(k-n)
$$

We write down the various components of the state vector $x(k)$ :

$$
\begin{equation*}
x_{n-j}(k)=-\sum_{i=j+1}^{n} a_{i} z^{-(i-j)} y(k)+\sum_{i=j+1}^{n} b_{i} z^{-(i-j)} u(k) \tag{1.23}
\end{equation*}
$$

Using the difference equation, it is readily seen that the output $y(k)$ is given by

$$
\begin{align*}
y(k) & =x_{n}(k)+b_{0} u(k) \\
& =[0 \cdots 01] x(k)+b_{0} u(k) \tag{1.24}
\end{align*}
$$

To see the state equation which this definition gives rise to, we note that

$$
\begin{aligned}
x_{n-j}(k+1)= & -\sum_{i=j+1}^{n} a_{i} z^{-(i-j-1)} y(k)+\sum_{i=j+1}^{n} b_{i} z^{-(i-j-1)} u(k) \\
= & -a_{j+1} y(k)+b_{j+1} u(k) \\
& -\sum_{i=j+2}^{n} a_{i} z^{-(i-(j+1))} y(k)+\sum_{i=j+2}^{n} b_{i} z^{-(i-(j+1))} u(k) \\
= & x_{n-j-1}(k)-a_{j+1} y(k)+b_{j+1} u(k) \\
= & x_{n-j-1}(k)-a_{j+1}\left(x_{n}(k)+b_{0} u(k)\right)+b_{j+1} u(k)
\end{aligned}
$$

Putting everything together, we finally get

$$
x(k+1)=\left[\begin{array}{c}
x_{1}(k+1)  \tag{1.25}\\
\vdots \\
x_{n}(k+1)
\end{array}\right]=\left[\begin{array}{cccc}
0 & \cdots & 0 & -a_{n} \\
1 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
\cdots & \cdots & 1 & -a_{1}
\end{array}\right]\left[\begin{array}{c}
x_{1}(k) \\
\vdots \\
x_{n}(k)
\end{array}\right]+\left[\begin{array}{c}
b_{n}-b_{0} a_{n} \\
\vdots \\
b_{1}-b_{0} a_{1}
\end{array}\right] u(k)
$$

If $b_{0}=0$, the equation simplifies to

$$
\begin{gather*}
x(k+1)=\left[\begin{array}{cccc}
0 & \cdots & 0 & -a_{n} \\
1 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
\cdots & \cdots & 1 & -a_{1}
\end{array}\right] x(k)+\left[\begin{array}{c}
b_{n} \\
\vdots \\
b_{1}
\end{array}\right] u(k)  \tag{1.26}\\
y(k)=[0 \cdots 01] x(k) \tag{1.27}
\end{gather*}
$$

Since this is a single-input single-output system, the transfer function is a scalar rational function. Thus if we take the transpose of the transfer function, which does not change the transfer function, we see immediately that the following state equation

$$
\begin{gather*}
x(k+1)=\left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & & & 1 \\
-a_{n} & -a_{n-1} & & \cdots & -a_{2} & -a_{1}
\end{array}\right] x(k)+\left[\begin{array}{c}
0 \\
\vdots \\
0 \\
1
\end{array}\right] u(k)  \tag{1.28}\\
y(k)=\left[b_{n} \cdots b_{1}\right] x(k) \tag{1.29}
\end{gather*}
$$

is also a realization of the difference equation. The state space realization, (1.26), (1.27) is referred to as being in observable canonical form, while the state space realization, (1.28), (1.29) is referred to as being in controllable canonical form. The reasons for these names will become clear when we study design of control systems based on state space methods.

