# Chapter 2

# ANALYSIS OF LINEAR STOCHASTIC SYSTEMS

## 2.1 Discrete Time Stochastic Processes

We shall deal only with processes which evolve at discrete instances of time. Typically, the time index can be $k_0, k_0+1, \ldots, N$, with $k_0$ and $N$ both finite, or it can be the nonnegative integers $\mathcal{Z}_+ = 0, 1, \ldots$, or it can be all the integers $\mathcal{Z}$. Let $\mathcal{T}$ be such an index set, which can be finite or countably infinite. Assume also that there is an underlying probability space $(\Omega, \mathcal{S}, P)$ with respect to which all random variables are defined. A discrete time stochastic process is just a family of random variables $w_k,\ k \in \mathcal{T}$. This means that for each fixed $k$, $w_k$ is a random variable on the sample space $\Omega$, and that the family $w_k,\ k \in \mathcal{T}$ is jointly defined on $\Omega$. We also use the notation $w(k),\ k \in \mathcal{T}$ to denote the stochastic process.

The stochastic process $w$ is thus a function of 2 variables, $k$ denoting the evolution of time, and $\omega$ denoting the point in the sample space that the random variables are to be evaluated. If we fix $\omega$, $w_k(\omega)$ is a function of $k$ only. These are called the sample paths or realizations of the stochastic process. If we take arbitrary, but finite collections of points in $\mathcal{T}$, $i_1, i_2, \ldots, i_n$, the family of joint distributions $F_{w_{i_1}, w_{i_2}, \cdots, w_{i_n}}(w_1, w_2, \cdots, w_n)$ is called the finite dimensional distributions of $w$. It can be shown that given a family of distributions which satisfy certain consistency properties, there exists a stochastic process $w$ which has the given family of distributions as its finite dimensional distributions.

**Example 1:** A Gaussian process $w$ is one whose finite dimensional distributions are multidimensional Gaussian distributions, i.e., $w_{i_1}, w_{i_2}, \cdots, w_{i_n}$ are jointly Gaussian random variables.

**Example 2:** Let $w_k$ be an independent identically distributed sequence of random variables satisfying

$$P(w_k = 1) = p$$

$$P(w_k = -1) = 1 - p = q$$

where $0 < p < 1$. Now consider the process $x_n$ satisfying the equation

$$x_{k+1} = x_k + w_k, \quad k \geq 0 \tag{2.1}$$

where $x_0 = \alpha$ for some given integer $\alpha \geq 0$. The process $x_n$ is called the simple random walk. It can be interpreted as the fortune of a gambler who gambles by flipping a coin with $P(Head) = p$. He wins \$1 if the outcome of the coin flip is $Head$, and loses \$1 if the outcome is $Tail$. The value of $x_k$ corresponds to the gambler's fortune at time $k$ if he starts with an initial fortune of $\alpha$ at time 0.

Although for any fixed $N$, a stochastic process defined on $[0\ N]$ can be interpreted as a random vector, we shall often be interested in the behaviour of the process over an unbounded interval, e.g., the nonnegative integers. This requires us to consider a possibly infinite collection of random variables, jointly distributed on the same probability space. This situation is fundamentally different from that of a finite collection of random vectors.

## 2.2 Stochastic Difference Equations

We shall now concentrate on the case when $\mathcal{T} = \mathcal{Z}_+$. Suppose we are given a random variable $x_0$, a stochastic process $w_k$, $k \in \mathcal{Z}_+$, and a fixed deterministic time sequence $u_k$, $k \in \mathcal{Z}_+$. In general, $x_0$ is $n$-dimensional, $u_k$ is $m$-dimensional, and $w_k$ is $l$-dimensional. Consider the difference equation

$$x_{k+1} = f_k(x_k, u_k, w_k) \quad k = 0, 1, \dots \tag{2.2}$$

Here $f_k$ are given functions, and $x_0$ serves as the initial condition to the difference equation. We can recursively calculate the solution sequence as

$$\begin{aligned} x_1 &= f_0(x_0, u_0, w_0) \\ x_2 &= f_1(x_1, u_1, w_1) = f_1(f_0(x_0, u_0, w_0), u_1, w_1) \end{aligned}$$

and so on. Notice that for fixed $u$, $x_1$ is a function of the random variables $x_0$ and $w_0$, and hence itself random. Similiarly, $x_2$ is a function of the random variables $x_0$, $w_0$, and $w_1$. If we use the notation $w_{k_0}^{k_1}$, or $w_{k_0:k_1}$, to denote the sequence $w_{k_0}, w_{k_0+1}, \dots, w_{k_1}$, we see that for $k \geq 1$, $x_k$ depends on $x_0$ and $w_0^{k-1}$. Thus the underlying basic random variable $x_0$ and stochastic process $w$ generate a solution process $x$ which is itself a stochastic process.

**Example:** The simple random walk $x_k$ satisfies a stochastic difference equation.

**Example:** A popular inventory model is the following:

Let $x_k$ be the store inventory at the beginning of day $k$. $u_k$ is inventory ordered and delivered at the beginning of day $k$, assumed to be deterministic. $w_k$ is the (random) amount of inventory sold during day $k$. Then the equation describing the amount of inventory at the beginning of day $k+1$ is given by

$$x_{k+1} = x_k + u_k - w_k$$

If inventory is assumed to be nonnegative, the equation then becomes

$$x_{k+1} = max(x_k + u_k - w_k, 0)$$

So far, we have not made any assumptions about the properties of $x_0$ and $w$. One reasonable assumption that is often made is the following:

**Assumption M:** The process $w$ is an independent sequence (i.e. $w_k$ and $w_j$ are independent for all $k \neq j$), and is independent of $x_0$.

Many physical systems perturbed by totally unpredictable disturbances satisfy this assumption. Let us explore the implications of this assumption. For simplicity, we shall consider equations with no input $u_k$.

$$x_{k+1} = f_k(x_k, w_k)$$

We then have

$$\begin{aligned} P(x_{k+1} \leq x | x_0^k) &= P[f_k(x_k, w_k) \leq x | x_0^k] \\ &= P[f_k(x_k, w_k) \leq x | x_k] \end{aligned}$$

since conditioned on $x_k$, $f_k(x_k, w_k)$ is a function of $w_k$ only, which is independent of $x_0^{k-1}$, by Assumption M. Hence

$$P(x_{k+1} \leq x | x_0^k) = P(x_{k+1} \leq x | x_k) \tag{2.3}$$

We call a stochastic process $x_k$ satisfying (2.3) a Markov process and the property expressed by (2.3) the Markov property. Note that if $u_k$ is a known deterministic input, it can be incorporated into the time dependence of the function $f_k$. We therefore conclude that a process generated by the first order stochastic difference equation (2.2) satisfying Assumption M with a known input is a Markov process.

**Example:** The simple random walk is a Markov process, since $w_k$ is an independent sequence. However, it is instructive to show this explicitly. First note that for any $k, m \geq 0$, the solution to (2.1) for $x_k$ is given by

$$x_{k+m} = x_m + \sum_{l=m}^{k+m-1} w_l$$

Hence

$$P(x_{k+m} = j | x_m, x_{m-1}, \cdots, x_0) = P(\sum_{l=m}^{k+m-1} w_l = j - x_m | x_m, x_{m-1}, \cdots, x_0)$$

Since for $m \leq l \leq k+m-1$, $w_l$ is independent of $x_0, \cdots, x_m$, we obtain

$$P(x_{k+m} = j | x_m, x_{m-1}, \cdots, x_0) = P(\sum_{l=m}^{k+m-1} w_l = j - x_m | x_m) = P(x_{k+m} = j | x_m)$$

which shows that $x_k$ is Markov.

For the simple random walk, owing to its relatively simple structure, it is possible to derive various results concerning its sample path behaviour. For example, if we take the gambling interpretation of the simple random walk, we can ask what is the probability that the gambler will lose all his fortune. This is called the gambler's ruin problem. We now describe its solution.

Let $p_k$ be the probability of ruin if the gambler starts with fortune $k$. Then $p_k$ satisfies the equation

$$p_k = p p_{k+1} + q p_{k-1} \tag{2.4}$$

with the boundary conditions $p_0 = 1$, $p_N = 0$. Putting $\theta^k$ as the trial solution to (2.4) gives the equation

$$\theta = p\theta^2 + q$$

There are 2 distinct roots $1, \frac{q}{p}$ if $p \neq \frac{1}{2}$. In this case, the general solution to (2.4) is given by

$$p_k = A_1 + A_2 (\frac{q}{p})^k$$

Applying the boundary conditions yield the equations

$$\begin{aligned} A_1 + A_2 &= 1 \\ A_1 + A_2 (\frac{q}{p})^N &= 0 \end{aligned}$$

Solving, we obtain

$$A_2 = \frac{1}{1-(\frac{q}{p})^N}$$

and

$$A_1 = \frac{-(\frac{q}{p})^N}{1-(\frac{q}{p})^N}$$

so that

$$p_k = \frac{(\frac{q}{p})^k - (\frac{q}{p})^N}{1-(\frac{q}{p})^N}$$

If $p = q = \frac{1}{2}$, the roots coincide, and the general solution for $p_k$ is then given by

$$p_k = A_1 + A_2 k$$

Applying the boundary conditions gives $A_1 = 1$ and $A_2 = -\frac{1}{N}$, resulting in

$$p_k = 1 - \frac{k}{N}$$

For more information about the gambler's ruin problem, see the classic probability text, *W. Feller, An Introduction to Probability Theory and Its Applications, Vol. I, 3rd Ed.*

Properties about the sample path behaviour of stochastic processes are usually quite difficult to establish. If we are only interested in the average behaviour, we need only to determine the moments. This problem becomes relatively straightforward in the case of linear systems. The rest of Chapter 2 focuses on the moment properties of linear stochastic systems. We in fact do not need and **will not make the Assumption M for the rest of Chapter 2**.

## 2.3 Linear Stochastic Systems

At the degree of generality of (2.2), there is not much more one can say about the properties of the process $x_k$ based on those of $x_0$ and $w$. For the rest of this chapter, we shall concentrate on second order analysis of linear stochastic systems. We shall see that quite a lot of concrete results can be obtained in this case.

A linear stochastic system is described by the equation

$$x_{k+1} = A_k x_k + B_k u_k + G_k w_k \quad k = 0, 1, \ldots \tag{2.5}$$

where for each $k$, $A_k$, $B_k$, and $G_k$ are given matrices of dimensions $n \times n$, $n \times m$, and $n \times l$, respectively. It is readily verified that the solution of (2.5) is given by the following formula:

$$x_k = \Phi(k;0)x_0 + \sum_{j=0}^{k-1} \Phi(k; j+1)(B_j u_j + G_j w_j) \tag{2.6}$$

where

$$\begin{aligned} \Phi(k;j) &= A_{k-1}A_{k-2}\cdots A_j \quad \text{for } k > j \\ \Phi(j;j) &= I \end{aligned} \tag{2.7}$$

This explicit solution allows us to obtain various additional properties of the solution process $x$.

**Assumption I:** The process $w$ has zero mean, i.e. $Ew_k = 0$ for all $k$.

Let $m_k$ denote $Ex_k$ with $m_0 = Ex_0$ given. Then by taking expectation on both sides of (2.6), we obtain the following formula for $m_k$:

$$m_k = \Phi(k;0)m_0 + \sum_{j=0}^{k-1} \Phi(k;j+1)B_j u_j \tag{2.8}$$

Equivalently, we can consider $m_k$ as satisfying the difference equation

$$m_{k+1} = A_k m_k + B_k u_k$$

for which the solution is given by (2.8). Note that if we assume we know $Ew_k$, the assumption that it is zero for all $k$ is without loss of generality. A nonzero $Ew_k$ will just result in an additional term on the right hand side of the $m_k$ equation.

To analyze the second order properties of $x_k$, we make the following additional assumptions:

**Assumption II:** $w_k$ satisfies the property $Ew_k w_j^T = Q_k \delta_{kj}$, where $\delta_{kj}$ is the Kronecker delta function: $\delta_{kj} = 1$ when $k = j$, $\delta_{kj} = 0$ when $k \neq j$.

Assumption II can be relaxed by adding dynamics to the system.

The process $w$ satisfying the above assumption is often referred to as (wide-sense) white noise. We make one further assumption:

**Assumption III:** $Ex_0 w_k^T = 0$ for all $k$.

Assumption III is a reasonable assumption as there is usually no reason to expect that the system noise is correlated with the initial condition.

Assumptions II and III will allow us to develop equations for the covariance matrix of $x_k$. To that end, let $\tilde{x}_k = x_k - m_k$. It is easily verified that $\tilde{x}_k$ satisfies the equation

$$\tilde{x}_{k+1} = A_k \tilde{x}_k + G_k w_k \tag{2.9}$$

Solving, we obtain

$$\tilde{x}_k = \Phi(k;0)\tilde{x}_0 + \sum_{j=0}^{k-1} \Phi(k;j+1)G_j w_j \tag{2.10}$$

Let $\Sigma_k$ denote the covariance matrix of $x_k$, i.e. $\Sigma_k = E\tilde{x}_k \tilde{x}_k^T$. Direct substitution into (2.10) gives

$$\Sigma_k = \Phi(k;0)\Sigma_0 \Phi(k;0)^T + \sum_{j=0}^{k-1}\sum_{l=0}^{k-1} \Phi(k;j+1)G_j E(w_j w_l^T) G_l^T \Phi(k;l+1)^T$$

where we have used Assumption III. On using Assumption II and simplifying, we get

$$\Sigma_k = \Phi(k;0)\Sigma_0 \Phi(k;0)^T + \sum_{j=0}^{k-1} \Phi(k;j+1)G_j Q_j G_j^T \Phi(k;j+1)^T \tag{2.11}$$

(2.11) gives the explicit solution for the covariance matrix $\Sigma_k$. $\Sigma_k$ can also be shown to satisfy a difference equation. First we make the observation that

$$E\tilde{x}_k w_k^T = 0 \quad \text{for all } k \tag{2.12}$$

This comes from the fact that $\tilde{x}_k$ depends linearly on $\tilde{x}_0$ and $w_0^{k-1}$, as can be seen from (2.10). By Assumption II, $E\tilde{x}_k w_k^T = 0$. Now compute $E\tilde{x}_{k+1}\tilde{x}_{k+1}^T$ using (2.9), and use the observation (2.12). We find that

$$\Sigma_{k+1} = A_k \Sigma_k A_k^T + G_k Q_k G_k^T \tag{2.13}$$

Naturally, the solution to (2.13) is given by (2.11).

## 2.4 Sampling a Continuous Time Linear Stochastic System

Discrete time processes often arise from sampling of continuous time processes. Although we focus on discrete time processes in this course, we briefly discuss how sampling of a linear stochastic differential equation driven by white noise give rise to a discrete time linear stochastic system.

A continuous time zero mean white noise process is formally defined as a zero mean stochastic process $v(t)$ having a covariance function $Ev(t)v^T(s) = V\delta(t-s)$. Here, $\delta(t)$ is the Dirac delta function having the properties

(i) $\delta(t) = 0, \quad t \neq 0$

(ii) $\int_{-\infty}^{\infty} \delta(t)dt = 1$

(iii) $\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$

While a such process $v(t)$ cannot exist in a physical sense as $Ev(t)v^T(t)$ has an infinite value due to the delta function, it turns out to be very useful when used as an input to linear systems. For simplicity, we discuss only linear time-invariant continuous time systems.

Consider the stochastic differential equation

$$\dot{x}(t) = Ax(t) + Bv(t) \tag{2.14}$$

where $v$ is a zero mean continuous time white noise process having a covariance function $Ev(t)v^T(s) = V\delta(t-s)$. While (2.14) is a formal description, it can be made rigorous using the "differential" version

$$dx(t) = Ax(t)dt + Bdw(t)$$

where $w(t)$ is a Wiener process (See, e.g., M.H.A. Davis, *Linear Estimation and Stochastic Control*).

From standard results on linear differential equations, the solution of (2.14) is given by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bv(s)ds \tag{2.15}$$

Suppose we sample the $x(t)$ process at the sampling times $kT$, for integers $k \geq 0$. Using (2.15), we can write

$$x(kT+T) = e^{AT}x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-s)}Bv(s)ds \tag{2.16}$$

We can write (2.16) as

$$z_{k+1} = Fz_k + w_k \tag{2.17}$$

where $z(k) = x(kT)$, $F = e^{AT}$, and $w_k$ denotes the 2nd term on the R.H.S. of (2.16). We will now show that $w_k$ is a zero mean discrete time white noise process with $Ew_kw_j^T = Q\delta_{kj}$, for some $Q$.

It is clear that since $\mathrm{E}v(t) = 0$ that $\mathrm{E}w_k = 0$ also. For $j \neq k$,

$$\begin{aligned} \mathrm{E}w_kw_j^T &= \mathrm{E}\int_{kT}^{kT+T} e^{A(kT+T-s)}Bv(s)ds[\int_{jT}^{jT+T} e^{A(jT+T-\tau)}Bv(\tau)d\tau]^T \\ &= \int_{kT}^{kT+T}\int_{jT}^{jT+T} e^{A(kT+T-s)}BVB^Te^{A^T(jT+T-\tau)}\delta(s-\tau)dsd\tau \end{aligned} \tag{2.18}$$

Note that since the intervals of integration in (2.18) do no overlap, there are no values of $s$ and $\tau$ for which $s-\tau = 0$. Hence for $j \neq k$,

$$\mathrm{E}w_kw_j^T = 0$$

For $j = k$, we obtain from (2.18), that

$$\begin{aligned}
\mathrm{E} w_k w_k^T &= \int_{kT}^{kT+T} \int_{kT}^{kT+T} e^{A(kT+T-s)} BVB^T e^{A^T(kT+T-\tau)} \delta(s-\tau) ds d\tau \\
&= \int_{kT}^{kT+T} e^{A(kT+T-s)} BVB^T e^{A^T(kT+T-s)} ds \\
&= \int_0^T e^{A\tau} BVB^T e^{A^T\tau} d\tau \qquad (2.19)
\end{aligned}$$

Hence $w_k$ is a discrete time zero mean white noise process with a covariance $Q = \int_0^T e^{A\tau} BVB^T e^{A^T\tau} d\tau$

These results show that sampling a stochastic differential equation driven by white noise results in a standard discrete time state model for the sampled process.

## 2.5 Analysis of Linear Time-Invariant Stochastic Systems

More complete results can be obtained if we assume that the matrices $A$, $B$, $G$, and $Q$ are constant. In this case, the solution to (2.5) takes the form

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1}(Bu_j + Gw_j) \qquad (2.20)$$

The mean value sequence $m_k$ now satisfy

$$m_{k+1} = Am_k + Bu_k$$

with explicit solution given by

$$m_k = A^k m_0 + \sum_{j=0}^{k-1} A^{k-j-1} Bu_j \qquad (2.21)$$

The covariance equation is now given by

$$\Sigma_{k+1} = A\Sigma_k A^T + GQG^T \qquad (2.22)$$

with explicit solution

$$\Sigma_k = A^k \Sigma_0 (A^T)^k + \sum_{j=0}^{k-1} A^{k-j-1} GQG^T (A^T)^{k-j-1} \qquad (2.23)$$

The explicit solutions involve evaluation of $A^k$, which we shall briefly discuss. There are generally 2 methods: diagonalization and inverse z-transform.

**Diagonalization:**

Suppose $A$ can be diagonalized by a nonsingular, possibly complex, matrix $V$. That is,

$$V^{-1}AV = \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix}$$

Direct calculation shows

$$\begin{aligned} A^k &= (V\Lambda V^{-1})^k \\ &= V\Lambda^k V^{-1} \\ &= V \begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \lambda_n^k \end{bmatrix} V^{-1} \end{aligned} \tag{2.24}$$

It is known from linear algebra that if $A$ has distinct eigenvalues, or if $A$ is symmetric, then $A$ has $n$ linearly independent eigenvectors $\{v_1, v_2, \ldots, v_n\}$. The diagonalizing matrix $V$ is then given by

$$V = [v_1 \; v_2 \; \cdots \; v_n]$$

The above results show that in these diagonalizable cases, determining $A^k$ amounts to solving the eigenvalue problem associated with the $A$ matrix.

**Example 2.4.1:**

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\det(zI - A) = \det \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix}$$

$$= z^2 + 3z + 2 = (z+2)(z+1)$$

Since $A$ has distinct eigenvalues, the matrix $V$ consisting of the linearly independent eigenvectors of $A$ as its columns will diagonalize $A$. We next determine the eigenvectors.

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Solving for $v_1$ and $v_2$ yields

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2 \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Similarly,

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We can verify that

$$V = \begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}$$

does diagonalize $A$:

$$\begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -8 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Using (2.24), we obtain

$$A^k = -\frac{1}{2}\begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4(-1)^k & 2(-1)^k \\ -(-2)^k & -(-2)^k \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -4(-1)^k + 2(-2)^k & -2(1)^k + 2(-2)^k \\ 4(-1)^k - 4(-2)^k & 2(1)^k - 4(-2)^k \end{bmatrix}$$

$$\begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix}$$

**Inverse Z-Transform**

The z-transform of a sequence $x_k$ on $Z_+$ is defined by

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

If we denote the z-tranform operator by $\mathcal{Z}$, the z-transform of the time-shifted sequence $x_{k+1}$ is given by

$$\begin{aligned} \mathcal{Z}(x_{k+1}) &= z\sum_{k=0}^{\infty} x_{k+1} z^{-(k+1)} \\ &= z(X(z) - x_0) \end{aligned}$$

Now $A^k$ can be interpreted as the solution of the difference equation

$$\begin{aligned} G_{k+1} &= AG_k \\ G_0 &= I \end{aligned}$$

Taking transforms of both sides gives

$$z[G(z) - I] = AG(z)$$

Solving for $G(z)$ gives

$$G(z) = (I - z^{-1}A)^{-1} = z(zI - A)^{-1}$$

Thus $A^k$ is given by

$$A^k = \mathcal{Z}^{-1}(I - z^{-1}A)^{-1} = \mathcal{Z}^{-1}(z(zI - A)^{-1})$$

**Example 2.4.2:**

Let us determine $A^k$ for the matrix $A$ in Example 2.4.1 using the z-transform method.

$$A^k = (I - z^{-1}A)^{-1} = \begin{bmatrix} 1 & -z^{-1} \\ 2z^{-1} & 1+3z^{-1} \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} 1+3z^{-1} & z^{-1} \\ 2z^{-1} & 1 \end{bmatrix}}{1+3z^{-1}+2z^{-2}} = \frac{\begin{bmatrix} 1-3z^{-1} & z^{-1} \\ -2z^{-1} & 1 \end{bmatrix}}{(1+z^{-1})(1+2z^{-1})}$$

$$= \begin{bmatrix} \frac{2}{1+z^{-1}} + \frac{-1}{1+2z^{-1}} & \frac{1}{1+z^{-1}} + \frac{-1}{1+2z^{-1}} \\ \\ \frac{-2}{1+z^{-1}} + \frac{2}{1+2z^{-1}} & \frac{-1}{1+z^{-1}} + \frac{2}{1+2z^{-1}} \end{bmatrix}$$

Now we know from z-transform tables that the following inversion formula holds

$$\mathcal{Z}^{-1}[\frac{z}{(z-p)^{i+1}}] = \frac{k!}{i!(k-i)!}p^{k-i} \quad \text{for all } i \geq 0 \tag{2.25}$$

Applying the formula gives

$$= \begin{bmatrix} 2(-1)^k + (-1)(-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix}$$

which is the same result as before.

We now consider the behaviour of $m_k$ and $\Sigma_k$ as $k \to \infty$. For simplicity, we assume that $u_k = 0$. We also assume that $A$ is stable, i.e. all its eigenvalues lie inside the unit disk $\{z : |z| < 1\}$. For stable $A$, it is known that there exist positive constants $\alpha$ and $\rho$ such that $\|A^k\| \leq \alpha\rho^k$ where $\alpha > 0$ and $0 < \rho < 1$.

From the explicit solution of $m_k$ (2.21), we immediately see that $m_k \to 0$ as $k \to \infty$. The interpretation is that the effects of the mean of the random initial condition $x_0$ should vanish as $k \to \infty$. Since the random disturbance $w_k$ has zero mean, $x_k$ should itself have zero mean in the steady state.

By making a change of variable $l = k - j - 1$, the solution for $\Sigma_k$ from (2.23) can be written as

$$\Sigma_k = A^k\Sigma_0(A^T)^k + \sum_{l=0}^{k-1} A^l GQG^T(A^T)^l \tag{2.26}$$

The first term in (2.26) tends to 0. The second term in (2.26) is nondecreasing as $k$ increases. Since $\|A^l GQG^T(A^T)^l\| \leq \beta\rho^{2l}$, the second term is bounded by $\beta\frac{1}{1-\rho^2}$ for all $k$. Hence we conclude that the second term has a limit as $k$ tends to $\infty$, so that

$$\lim_{k\to\infty} \Sigma_k = \Sigma_\infty = \sum_{l=0}^{\infty} A^l GQG^T(A^T)^l \tag{2.27}$$

Now that we know that $\Sigma_\infty$ exists, we can also determine it by taking the limit on both sides of (2.22) to get

$$\Sigma_\infty = A\Sigma_\infty A^T + GQG^T \tag{2.28}$$

Equation (2.28) is called the discrete-time Lyapunov equation. It is a linear algebraic equation in $\Sigma_\infty$. We have already shown in (2.27), that when $A$ is stable, there exists a solution to (2.28) given by $\sum_{l=0}^{\infty} A^l GQG^T(A^T)^l$. We now show that the solution is unique. Suppose there exist 2 solutions, $\Sigma_1$ and $\Sigma_2$, to (2.28). Let $\Delta = \Sigma_1 - \Sigma_2$. Then $\Delta$ satisfies the equation

$$\Delta = A\Delta A^T$$

Iterating yields

$$\Delta = A^k \Delta (A^k)^T \quad \text{for all } k$$

Letting $k \to \infty$ and using the stability of $A$, we see that $\Delta = 0$, proving uniqueness.

Combining these results, we can state the following important result on the discrete-time Lyapunov equation.

**Theorem 2.1.** Assume that $A$ is stable. The discrete-time Lyapunov equation (2.28) has a unique solution given by $\sum_{l=0}^{\infty} A^l GQG^T (A^T)^l$.

In practice, we rarely use (2.27) to determine $\Sigma_\infty$. We would determine $\Sigma_\infty$ by directly solving the linear equation (2.28).

We can now draw the following conclusion:

If $A$ is stable, and we allow the system to settle down to its steady state behaviour, the system will then have zero mean and a constant covariance for all $k$. This can be seen by noting that if $m_0 = 0$, $m_k = 0$ for all $k \geq 0$. Also if the initial covariance $\Sigma_0 = \Sigma_\infty$ for (2.22), the solution can readily be verified to be $\Sigma_k = \Sigma_\infty$, for all $k \geq 0$.

**Outputs:**

We often have an output equation of the form

$$y_k = Cx_k + Hv_k \tag{2.29}$$

In order to analyze the properties of $y$, we add the following assumptions.

**Assumption IV:** $Ev_k = 0$, and $Ev_j v_k^T = R\delta_{jk}$

**Assumption V:** $Ew_j v_k^T = 0$ for all $j < k$, and $Ex_0 v_k^T = 0$ for all $k$.

Under Assumption V, it is readily seen that $v_k$ is uncorrelated with $x_k$. A direct calculation then gives

$$Ey_k = Cm_k \tag{2.30}$$

and

$$\text{cov}(y_k) = C\Sigma_k C^T + HRH^T \tag{2.31}$$

Again, if $A$ is stable, $cov(y_k) \underset{k\to\infty}{\longrightarrow} C\Sigma_\infty C^T + HRH^T$.

## 2.6 ARMAX Models

So far, we have concentrated on state space models of linear stochastic systems. In this section, we discuss difference equation models of linear stochastic systems. For simplicity, we shall limit our discussion to scalar-valued processes, although much of the analysis goes through for vector-valued processes as well.

Consider the following difference equation in the process $y$:

$$y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n} = b_1 u_{k-1} + b_2 u_{k-2} + \cdots + b_n u_{k-n} + c_0 w_k + c_1 w_{k-1} + \cdots + c_n w_{k-n} \tag{2.32}$$

Here, $u_k$ is taken to be a known deterministic sequence, $w_k$ is a zero mean uncorrelated sequence with variance $\sigma_w^2$. The above description, with $u_k$ and $w_k$ as inputs and $y_k$ as the output, is called an ARMAX process. AR stands for autoregressive and describes the linear combination of $y_k$ and the past values $y_{k-j}$. MA stands for moving average and describes the linear combination of $w_k$ and the past values $w_{k-j}$. The "X" part stands for exogenous inputs, which describes the linear combination of $u_{k-1}$ and the past values of $u$. Note that we assume that there is a one-step delay between the input $u$ and the output $y$. This corresponds to having no direct feedthrough from $u$ to $y$, which is often the case. The extension to the case with a $b_0u_k$ term included is not difficult but increases the notational complexity. If there is no exogenous input $u_k$, (2.32) reduces to

$$y_k + a_1y_{k-1} + \cdots + a_ny_{k-n} = c_0w_k + c_1w_{k-1} + \cdots + c_nw_{k-n} \tag{2.33}$$

This is referred to as an ARMA model and is very popular in time series modelling and analysis.

Define the backward shift operator $z^{-1}$ as follows:

$$z^{-1}y_k = y_{k-1}$$

Define also the polynomials

$$\begin{aligned} A(z^{-1}) &= \sum_{j=0}^{n} a_jz^{-j} \quad \text{with } a_0 = 1 \\ B(z^{-1}) &= \sum_{j=1}^{n} b_jz^{-j} \\ C(z^{-1}) &= \sum_{j=0}^{n} c_jz^{-j} \end{aligned}$$

Equation (2.32) can then be written as

$$A(z^{-1})y_k = B(z^{-1})u_k + C(z^{-1})w_k \tag{2.34}$$

Note that there is no loss of generality in assuming that the degrees of the polynomials $A(z^{-1})$, $B(z^{-1})$, and $C(z^{-1})$ are as given, since they can always be made the same by padding with zeros.

To solve the equation for the $y_k$ process, we need either to provide initial conditions for the difference equation, or to solve the equation using the infinite past. Providing initial conditions for the equation amounts to providing information about the quantities $y_{-1}, y_{-2}, \cdots, y_{-n}$, $u_{-1}, u_{-2}, \cdots, u_{-n}$, and $w_{-1}, w_{-2}, \cdots, w_{-n}$. If the initial conditions are assumed known, the ARMAX equation can then be solved forward in time by recursion so that the output process $y_k$ can be written as

$$y_k = \phi(k, u_0^{k-1}, w_0^k)$$

where we have used the notation $w_0^k$ to denote $w_j, 0 \leq j \leq k$, and $\phi(k, u_0^{k-1}, w_0^k)$ is some linear function of $u_0^{k-1}, w_0^k$. Explicit determination of the solution for $y_k$ amounts to solving an nth order difference equation. This is usually more conveniently done using a state space representation of the ARMAX equation, as we shall see later on.

To interpret the solution of the ARMAX equation using the infinite past, we first need to introduce processes which are defined on the set of all integers. We shall focus primarily on the class of (wide-sense) stationary processes, which we shall now define and discuss.

A process $y_k$ is said to be second-order if $Ey_k^2 < \infty$ for all $k$. A second-order process $y_k$ is said to be (wide-sense) stationary if $Ey_k = m$, a constant, and the correlation $Ey_{n+k}y_n$ is a function of $k$ only. This implies that the second moment $Ey_k^2$ is also a constant, which, to avoid triviality, is assumed to be strictly positive. Hence a stationary process has second-order probabilistic properties which are invariant with respect to time shift. Such a process is therefore assumed to be defined on the set of all integers (negative as well as positive) $\mathcal{Z}$. For a stationary process $y$, we denote the correlation $Ey_{n+k}y_n$ as a function of $k$ by $R_y(k)$. We refer to $R_y(k)$ as the correlation function.

Now consider the following simple first order AR system

$$y_{k+1} - ay_k = w_{k+1} \tag{2.35}$$

If we solve this equation starting at time $k_0$, the solution, for $k > k_0$, is given by

$$y_k = a^{k-k_0} y_{k_0} + \sum_{j=k_0}^{k-1} a^{k-j-1} w_{j+1} = a^{k-k_0} y_{k_0} + \sum_{m=0}^{k-k_0-1} a^m w_{k-m}$$

Let $\eta(k;k_0) = \sum_{m=0}^{k-k_0-1} a^m w_{k-m}$. Note that $y_{k_0}$ is uncorrelated with $\eta(k;k_0)$ since $\eta(k;k_0)$ involves linear combination of $w_j$, $j > k_0$. Letting $k_0 \to -\infty$, we see that formally we would expect the second term in the solution to have the limit

$$\lim_{k_0 \to -\infty} \eta(k;k_0) = \sum_{m=0}^{\infty} a^m w_{k-m}$$

provided that the infinite sum converges. Fix k and denote the Nth partial sum $S_N$ by

$$S_N = \sum_{m=0}^{N} a^m w_{k-m}$$

We can then ask under what conditions will $S_N$ converge and what type of convergence it will be, as $S_N$ is a random sequence. The type of convergence we shall use, which particularly fits the assumption of the processes being stationary, is that of mean square convergence.

Let $X$ and $X_n, n = 1, 2, \cdots$ be random variables with finite second moments.

**Definition:** The random sequence $X_n$ is said to converge to $X$ in mean square if $\lim_{n\to\infty} E|X_n - X|^2 = 0$.

Note that $E|X_n - X|^2$ is a nonnegative number for each $n$. Mean square convergence is based therefore on whether this sequence of nonnegative numbers converges or not.

Given a random sequence $X_n$, one may ask whether or not it converges to some random variable $X$ in mean square. Since we generally do not know to which random variable $X$ the sequence may converge, we cannot directly apply the definition to check convergence. There is, however, an effective criterion to test convergence, called Cauchy's criterion.

**Cauchy's criterion:** A random sequence $X_n$ converges to some random variable $X$ in mean square if and only if $\lim_{n,m\to\infty} E|X_n - X_m|^2 = 0$.

We can now return to the question of convergence of the Nth partial sums $S_N$. By Cauchy's criterion, $S_N$ converges to some random variable in mean square if and only if $\lim_{n,m\to\infty} E|S_n - S_m|^2 = 0$. It is not difficult to verify that if $|a| < 1$, and $S_N = \sum_{j=0}^{N} a^j w_{k-j}$, that $\lim_{N,M\to\infty} E|S_N - S_M|^2 = 0$. As is natural, we shall denote the random variable to which $S_N$ converges by $\sum_{j=0}^{\infty} a^j w_{k-j}$.

Now consider the solution of the simple AR equation (2.35). The above results show that if $|a| < 1$, i.e. if the system is stable, the term $a^{k-k_0} y_{k_0}$ converges to 0 in mean square as $k_0 \to -\infty$, and the solution $y_k$ converges in mean square as $k_0 \to -\infty$ to $\sum_{j=0}^{\infty} a^j w_{k-j}$. The solution is in the form of the convolution of the sequence $a^k$ with the stochastic sequence $w_k$, and can therefore be interpreted as the response of a linear system with impulse response sequence $a^k$ to the input $w_k$.

Further ties between the ARMAX equation description and the transfer function description of a stochastic system driven by a stationary process can again be seen using the simple example (2.35). Using the backward shift operator $z^{-1}$, we can write

$$y_k = \frac{1}{1 - az^{-1}} w_k \tag{2.36}$$

If we expand $\frac{1}{1-az^{-1}}$ as a power series in $z^{-1}$, we find

$$\frac{1}{1-az^{-1}} = \sum_{j=0}^{\infty} a^j z^{-j}$$

Substituting into (2.36) yields

$$y_k = \sum_{j=0}^{\infty} a^j z^{-j} w_k = \sum_{j=0}^{\infty} a^j w_{k-j}$$

Observe that $\frac{1}{1-az^{-1}}$ is also the transfer function from $w$ to $y$. Hence we may interpret $\frac{1}{1-az^{-1}}$ as either the transfer function representing the input-output relationship in the z-domain, or as a power series in the backward shift operator $z^{-1}$ in the time domain.

In general, the ARMAX equation (2.32) can be written in input-output form as

$$y_k = \frac{B(z^{-1})}{A(z^{-1})} u_k + \frac{C(z^{-1})}{A(z^{-1})} w_k \tag{2.37}$$

whenever the polynomial $A(z^{-1})$ is stable (i.e., having all roots lie in $\{z : |z| < 1\}$). $y_k$ is then expressible as the sum of the convolution of $u$ with the impulse response sequence corresponding to the transfer function $\frac{B(z^{-1})}{A(z^{-1})}$, and the convolution of $w$ with the impulse response sequence corresponding to the transfer function $\frac{C(z^{-1})}{A(z^{-1})}$. In the next section, we shall analyze the second order properties of the output $y_k$.

## 2.7 Analysis of Linear Systems Driven by Stationary Processes

Consider a linear time invariant system described by its impulse response sequence $h_j,\ j = 0, 1, \cdots$. We assume that $h_j$ is bounded-input bounded-output stable in the sense that $\sum_{j=0}^{\infty} |h_j| < \infty$. If we denote the transfer function corresponding to $h_j$ by $H(z^{-1})$, it is well-known that bounded-input bounded-output stability of $h_j$ is equivalent to the poles of the rational function $H(z^{-1})$ all lying in $\{z : |z| < 1\}$. Suppose the input to the linear system is a stationary process $w_k$, with mean $m_w$ and correlation function $R_w(k)$. From the results of the previous section, we can therefore write the output of the linear system $y_k$ as

$$y_k = \sum_{j=0}^{\infty} h_j w_{k-j}$$

It can be shown that whenever we have processes which are mean square convergent, we can interchange the summation and expectation operation. Hence, the mean of $y_k$, denoted by $m_y(k)$, is given by

$$\begin{aligned} m_y(k) &= E \sum_{j=0}^{\infty} h_j w_{k-j} \\ &= \sum_{j=0}^{\infty} h_j E w_{k-j} \\ &= \sum_{j=0}^{\infty} h_j m_w \end{aligned} \tag{2.38}$$

Note that the right hand side of (2.38) is a constant. This means that whenever the input $w$ has a constant mean, the output $y$ also has a constant mean, given by (2.38). To determine the correlation function of $y$, we write

$$\begin{aligned} Ey_{n+k}y_n &= E\sum_{j=0}^{\infty} h_j w_{n+k-j} \sum_{l=0}^{\infty} h_l w_{n-l} \\ &= \sum_{j=0}^{\infty} h_j \sum_{l=0}^{\infty} h_l E w_{n+k-j} w_{n-l} \\ &= \sum_{j=0}^{\infty} h_j \sum_{l=0}^{\infty} h_l R_w(k+l-j) \end{aligned} \tag{2.39}$$

Since the right hand side of (2.39) is a function of $k$ only, we see that the correlation function $Ey_{n+k}y_n$ is a function of $k$ only. Combining, we conclude that if the input $w$ is a stationary process, the output $y$ is a stationary process also.

If we examine the expression for the correlation function $R_y(k)$ given by (2.39), we see that it contains essentially a double convolution. Since convolutions are more readily analyzed in the frequency domain using transforms, we introduce the concept of spectral density of a stationary process.

**Spectral Density:** The spectral density $\Phi_y(\omega)$ of a stationary process $y$ is the Fourier transform of the correlation function $R_y(k)$, whenever the Fourier transform exists. We can then write

$$\Phi_y(\omega) = \sum_{k=-\infty}^{\infty} R_y(k) e^{-ik\omega}$$

The properties of $\Phi_y(\omega)$ may be summarized as being a real, even, nonnegative function. Since $R_y(k)$ and $\Phi_y(\omega)$ are Fourier transform pairs, we can equivalently determine $\Phi_y(\omega)$ from the input-output equation.

Taking the Fourier transform of both sides of (2.39), we obtain

$$\begin{aligned} \Phi_y(\omega) &= \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} h_j \sum_{l=0}^{\infty} h_l R_w(k+l-j) e^{-ik\omega} \\ &= \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} h_j \sum_{l=0}^{\infty} h_l R_w(k+l-j) e^{-i(k+l-j)\omega} e^{-ij\omega} e^{il\omega} \end{aligned} \tag{2.40}$$

Making the change of variable $m = k + l - j$ and summing over $m$ first in (2.40), we find

$$\begin{aligned} \Phi_y(\omega) &= \sum_{j=0}^{\infty} h_j \sum_{l=0}^{\infty} h_l e^{-ij\omega} e^{il\omega} \sum_{m=-\infty}^{\infty} R_w(m) e^{-im\omega} \\ &= \sum_{j=0}^{\infty} h_j e^{-ij\omega} \sum_{l=0}^{\infty} h_l e^{il\omega} \Phi_w(\omega) \\ &= H(e^{-i\omega}) H(e^{i\omega}) \Phi_w(\omega) \qquad (2.41) \\ &= |H(e^{-i\omega})|^2 \Phi_w(\omega) \qquad (2.42) \end{aligned}$$

where we have used $H(e^{-i\omega})$ to denote the frequency response corresponding to the impulse response sequence $h_j$:

$$H(e^{-i\omega}) = \sum_{j=0}^{\infty} h_j e^{-ij\omega}$$

Thus given the input spectral density $\Phi_w(\omega)$ and the frequency response of the linear system $H(e^{-i\omega})$, we can determine the output spectral density $\Phi_y(\omega)$ very easily using either (2.41) or (2.42).

To obtain the correlation function $R_y(k)$ from the spectral density $\Phi_y(\omega)$, we use the Fourier inversion formula

$$R_y(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_y(\omega)e^{ik\omega}d\omega \tag{2.43}$$

The integral over the real variable $\omega$ is usually difficult to evaluate. Instead, we use contour integration to carry out the inversion. Put $z = e^{i\omega}$. Observe that $dz = ie^{i\omega}d\omega$ or $\frac{dz}{iz} = d\omega$. Expressing the spectral density as a function of $z$ rather than $\omega$, we can re-write (2.43) as

$$R_y(k) = \frac{1}{2\pi i}\oint \Phi_y(z)z^{k-1}dz \tag{2.44}$$

where the contour of integration is counterclockwise over the unit circle $C_1$. By Cauchy's integral theorem, we can therefore determine the output correlation function using residue calculus

$$R_y(k) = \sum_{res\in C_1} \Phi_y(z)z^{k-1} \tag{2.45}$$

where $\sum_{res\in C_1}$ denotes summation over the residues inside the unit circle.

**Example 2.6.1:**

As a simple example, again consider the AR system (2.35). If we assume that the input process $w$ is a zero mean orthogonal sequence (white noise) with variance $\sigma_w^2$, $w$ has spectral density $\Phi_w(\omega) = \sigma_w^2$. The output spectral density is given by

$$\Phi_y(\omega) = \frac{1}{(1 - ae^{-i\omega})(1 - ae^{i\omega})}\sigma_w^2$$

The output correlation function is then given by

$$\begin{aligned} R_y(k) &= \frac{1}{2\pi i}\oint \frac{\sigma_w^2}{(1 - az^{-1})(1 - az)}z^{k-1}dz \\ &= \frac{1}{2\pi i}\oint \frac{\sigma_w^2}{(z - a)(1 - az)}z^{k}dz \end{aligned} \tag{2.46}$$

For $k \geq 0$, the only pole inside the unit circle for the integrand of (2.46) is at $a$, since $|a| < 1$. Hence

$$R_y(k) = \frac{\sigma_w^2}{1 - a^2}a^k \quad \text{for } k \geq 0$$

For $k < 0$, there will be additional poles on the right hand side of (2.46) at $z = 0$. This complication can be avoided by making the change of variable $p = z^{-1}$ in (2.46). In terms of the integral with $p$ as the complex variable, the integrand will no longer have repeated poles at $p = 0$. Combining, we find

$$R_y(k) = \frac{\sigma_w^2}{1 - a^2}a^{|k|} \quad \text{for all } k$$

**Example 2.6.2:**

A more interesting example is the first order ARMA model for a stationary process $y_k$ given by

$$y_k - ay_{k-1} = w_k + cw_{k-1}$$

We assume, as usual, that $0 < |a| < 1$. For simplicity, we assume that $Ew_k^2 = 1$. The spectral density for $y$ is then given by

$$\Phi_y(\omega) = \frac{(1+ce^{-i\omega})(1+ce^{i\omega})}{(1-ae^{-i\omega})(1-ae^{i\omega})}$$

The output correlation function is given by

$$\begin{aligned} R_y(k) &= \frac{1}{2\pi i}\oint \frac{(1+cz^{-1})(1+cz)}{(1-az^{-1})(1-az)} z^{k-1} dz \\ &= \frac{1}{2\pi i}\oint \frac{(z+c)(1+cz)}{(z-a)(1-az)} z^{k-1} dz \end{aligned} \tag{2.47}$$

For $k = 0$, (2.47) becomes

$$R_y(k) = \frac{1}{2\pi i}\oint \frac{(z+c)(1+cz)}{z(z-a)(1-az)} dz$$

Evaluating the residues at $z = 0$ and $z = a$, we obtain

$$\begin{aligned} R_y(0) &= Ey_k^2 \\ &= -\frac{c}{a} + \frac{(a+c)(1+ca)}{a(1-a^2)} \\ &= \frac{a + c + ca^2 + c^2a - c(1-a^2)}{a(1-a^2)} \\ &= \frac{1 + 2ac + c^2}{1-a^2} \end{aligned}$$

The values of $R_y(k)$ for $k \geq 1$ is straightforward and left as an exercise.

## 2.8 From State Space to ARMAX

We have seen how we can analyze linear stochastic systems either in state space form or input-output form. We shall now show how one description can be transformed to the other.

Consider again the linear stochastic system in state space form given by

$$\begin{aligned} x_{k+1} &= Ax_k + Gw_k \\ y_k &= Cx_k + Hw_k \end{aligned} \tag{2.48}$$

Note that in (2.48), we have use the same process $w_k$ in both the dynamice and the observation equation. This is because our ARMAX model only has one independent noise process, which will be $w_k$. We have also taken $u_k = 0$ for simplicity. We can derive the input-output representation from the state space model as follows. First note that the transfer function from $w$ to $y$ is given by

$$H(z^{-1}) = C(zI - A)^{-1}G + H \tag{2.49}$$

Write

$$(zI - A)^{-1} = \frac{adj(zI - A)}{det(zI - A)}$$

Let

$$adj(zI - A) = B_1 z^{n-1} + B_2 z^{n-2} + \cdots + B_n \tag{2.50}$$

$$det(zI - A) = z^n + p_1 z^{n-1} + \cdots + p_n \tag{2.51}$$

Combining, we get the equation

$$y_k = (C\frac{B_1 z^{n-1} + B_2 z^{n-2} + \cdots + B_n}{z^n + p_1 z^{n-1} + \cdots + p_n} G + H)w_k \tag{2.52}$$

Multiplying (2.52) throughout by $det(zI - A)$, we get the following ARMA equation

$$y_k + p_1 y_{k-1} + \cdots + p_n y_{k-n} = Hw_k + (CB_1G + p_1H)w_{k-1} + \cdots + (CB_nG + p_nH)w_{k-n} \tag{2.53}$$

These results can clearly be extended to the case where there is also an exogenous input $u_k$, resulting in an ARMAX model.

In deriving the ARMAX model from the state space model, we have focused on the input-output relationship. If the state space and ARMAX equations are solved starting at 0, we need to consider initial conditions as well. Of course, if we assume that the initial conditions are 0, the output process $y_k$ from the two models will be the same.

Alternatively, we can consider the processes to be stationary processes defined on $\mathcal{Z}$. Recall that if we assume, for the state space description (2.48), the matrix $A$ to be stable, $m_k = 0$ and $\Sigma_k = \Sigma_\infty$, we will get a zero mean process with a constant covariance matrix. We now determine the correlation functions.

Write for $k > 0$,

$$Ex_{k+n}x_n^T = E[A^k x_n + \sum_{j=n}^{n+k-1} A^{n+k-j-1} G w_j] x_n^T \tag{2.54}$$

Since $x_n$ depends only on $w_j,\ j \le n-1$, we see that (2.54) simplifies to

$$Ex_{k+n}x_n^T = A^k \Sigma_\infty \tag{2.55}$$

Hence the correlation function of $x_k$ depends only on the time separation

$$R_x(k) = Ex_{k+n}x_n^T = A^k \Sigma_\infty \tag{2.56}$$

(2.56) shows that $x_k$ is a stationary process. The output correlation function can also be computed.

$$\begin{aligned} R_y(k) &= E[Cx_{n+k} + Hw_{n+k}][x_n^T C^T + w_n^T H^T] \\ &= CA^k \Sigma_\infty C^T + CA^{k-1} GQH^T \quad \text{for} \quad k > 0 \end{aligned} \tag{2.57}$$

For $k < 0$, we can write

$$Ex_{n+k}x_n^T = E(x_n x_{n+k}^T)^T = (A^{-k}\Sigma_\infty)^T = \Sigma_\infty (A^T)^{|k|}$$

Assume that the conditions for stationarity are satisfied for (2.48). Interpreted as stationary processes on $\mathcal{Z}$, the state space model and the ARMAX model give rise to the same output process $y_k$.

## 2.9 From ARMAX to State Space

While deriving an ARMAX model from a state space model is straightforward, deriving state space model from an ARMAX description is nontrivial. It amounts to finding a state space realization of a transfer function. For simplicity, we shall only study scalar-valued processes. We proceed as follows.

Given an ARMAX model

$$A(z^{-1})y_k = B(z^{-1})u_k + C(z^{-1})w_k \tag{2.58}$$

with

$$
\begin{aligned}
A(z^{-1}) &= \sum_{j=0}^{n} a_j z^{-j} \quad \text{with } a_0 = 1 \\
B(z^{-1}) &= \sum_{j=1}^{n} b_j z^{-j} \\
C(z^{-1}) &= \sum_{j=0}^{n} c_j z^{-j}
\end{aligned}
$$

define the following state variables

$$
x_{n-j}(k) = -\sum_{i=j+1}^{n} a_i z^{-(i-j)} y_k + \sum_{i=j+1}^{n} b_i z^{-(i-j)} u_k + \sum_{i=j+1}^{n} c_i z^{-(i-j)} w_k \tag{2.59}
$$

From the ARMAX equation (2.58), we immediately find

$$
x_n(k) = y_k - c_0 w_k \tag{2.60}
$$

Now

$$
\begin{aligned}
x_{n-j}(k+1) &= -\sum_{i=j+1}^{n} a_i z^{-(i-j-1)} y_k + \sum_{i=j+1}^{n} b_i z^{-(i-j-1)} u_k + \sum_{i=j+1}^{n} c_i z^{-(i-j-1)} w_k \\
&= -a_{j+1} y_k + b_{j+1} u_k + c_{j+1} w_k + x_{n-j-1}(k) \\
&= x_{n-j-1}(k) - a_{j+1} x_n(k) + b_{j+1} u_k + (c_{j+1} - a_{j+1} c_0) w_k
\end{aligned} \tag{2.61}
$$

Combining (2.61) and (2.60), we get the following matrices for the state space description

$$
A = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & \ddots & 0 & \cdots & -a_{n-2} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & 1 & -a_1 \end{bmatrix} \tag{2.62}
$$

$$
B = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ \vdots \\ b_1 \end{bmatrix} \tag{2.63}
$$

$$
G = \begin{bmatrix} c_n - a_n c_0 \\ c_{n-1} - a_{n-1} c_0 \\ \vdots \\ \vdots \\ c_1 - a_1 c_0 \end{bmatrix} \tag{2.64}
$$

$$
C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2.65}
$$

$$H = c_0 \tag{2.66}$$

**Definition:** Let $A$ be an $n \times n$ matrix, $C$ a $p \times n$ matrix. The pair $(C, A)$ is called observable if $Rank[C^T \;\; A^T C^T \cdots (A^T)^{n-1} C^T] = n$.

This state space representation of the ARMAX equation (2.58) is called the observable representation, since the pair $(C, A)$ is always observable. Other representations are possible, but we shall not go into details.

**Example 2.8.1:**
Let the process $y_k$ be described by the ARMAX equation

$$y_k + 0.7y_{k-1} + 0.01y_{k-2} = 2u_{k-1} + u_{k-2} + w_k + 1.7w_{k-1} + 0.72w_{k-2}$$

The corresponding observable state space representation is given by

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & -0.01 \\ 1 & -0.7 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_k + \begin{bmatrix} 0.71 \\ 1 \end{bmatrix} w_k \\ y_k &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + w_k \end{aligned}$$

With the above results, one can go easily from one representation to another, and use whichever representation is easiest to work with in a specific situation. For systems defined on $\mathcal{Z}_+$ with initial conditions, usually the state space representation is easiest to work with.

## Exercises

1. Consider the following system

$$x_{k+1} = Ax_k + Gw_k \tag{$ex1.1$}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0.4 & 0.5 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$cov(x_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $w_k$ is an orthogonal sequence with zero mean and variance 1.

(a). Determine explicitly $\Sigma_k = cov(x_k)$.

(b). Associated with (ex1.1) is the Lyapunov equation

$$\Sigma = A\Sigma A^T + GG^T \tag{ex1.2}$$

Determine the solutions of (ex1.2). Is there a unique solution?

(c). Does $\lim_{k\to\infty} \Sigma_k$ exist? If so, does it correspond to a solution of the Lyapunov equation (ex1.2)?

(d). What is the difference between this problem and the standard results on the Lyapunov equation?

2. Consider the first order ARMA system

$$y_k - ay_{k-1} = w_k + cw_{k-1}$$

where $0 < |a| < 1$, $Ew_k = 0$, and $Ew_k^2 = 1$. Write down the observable state space representation of the system. Solve the resulting Lyapunov equation for the steady state covariance of $x_k$. Hence determine the steady state mean square value of $y_k$, and verify that the result is the same as the direct calculation in Section 2.6.

3. Consider the linear stochastic system

$$\begin{aligned} x_{k+1} &= Ax_k + Gw_k \\ y_k &= Cx_k \end{aligned}$$

where

$$A = \begin{bmatrix} 0.4 & 0 \\ -0.6 & 0.2 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and $w_k$ is a sequence of zero mean orthogonal random variables with $Ew_k^2 = 1$, all $k$. Let the initial covariance $cov(x_0) = I$, the identity matrix. Find the covariance matrix $\Sigma_k = E(x_k - m_k)(x_k - m_k)^T$ and determine its limit as $k \to \infty$. Verify that it is identical to the solution of the Lyapunov equation

$$\Sigma = A\Sigma A^T + GG^T$$

For the resulting stationary process, determine the output correlation function $R_y(k)$.

4. Consider again the linear stochastic system described in problem 3. Determine the ARMA model relating $w$ and $y$. Write down the spectral density of $y$, and determine $R_y(k)$ using the inversion formula for the spectral density. Verify that the result is the same as that of problem 3.

5. Suppose the stationary process $y_k$ satisfies the equation

$$y_k + 1.1y_{k-1} + 0.24y_{k-2} = w_k$$

where $w_k$ is an i.i.d. sequence with zero mean and variance 1.

(i) Represent the process in the observable representation. Determine the covariance matrix of the resulting state process $x_k$, and use it to find the correlation function of $y_k$.

(ii) In this part, we look at an alternative state space representation of the system. Let $x_k = \begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix}$. Write the state equation for $x_k$. Determine the covariance matrix of $x_k$, and find the correlation function $r_k = Ey_{k+l}y_l$. Verify that the result is the same as that in (i).

(iii) Now find the correlation function $r_k$ using contour integration, and once again show that it gives the same result as that obtained in (i) and (ii).