Using Partial Fractions Expansions to Determine Impulse Response from Transfer Function

This example illustrates the use of partial fractions expansion to determine the impulse response sequence from a transfer function that may arise from an ARMAX model.

Let the transfer function be given by

$$H(z^{-1}) = \frac{1 + 0.3z^{-1} + 0.02z^{-2}}{(1 - 0.4z^{-1})(1 - 0.5z^{-1})}$$

Note that the poles of the transfer function are at 0.4 and 0.5, hence stable.

For computations, it is usually more convenient to express $H(z^{-1})$ in terms of the variable z rather than z^{-1}. Multiplying numerator and denominator by z^2, we get

$$H(z^{-1}) = \frac{z^2 + 0.3z + 0.02}{(z - 0.4)(z - 0.5)} = \frac{z^2 + 0.3z + 0.02}{z^2 - 0.9z + 0.2}$$

Note that this is a proper rational function with the degree of numerator the same as that of the denominator. The first step is to use long division to extract the constant term. This gives

$$\begin{aligned} H(z^{-1}) &= 1 + \frac{1.2z - 0.18}{z^2 - 0.9z + 0.2} \\ &= 1 - \frac{3}{z - 0.4} + \frac{4.2}{z - 0.5} \\ &= 1 - \frac{3z^{-1}}{1 - 0.4z^{-1}} + \frac{4.2z^{-1}}{1 - 0.5z^{-1}} \\ &= 1 - 3z^{-1}\sum_{k=0}^{\infty}(0.4)^k z^{-k} + 4.2z^{-1}\sum_{k=0}^{\infty}(0.5)^k z^{-k} \\ &= 1 - 3\sum_{k=1}^{\infty}(0.4)^{k-1} z^{-k} + 4.2\sum_{k=1}^{\infty}(0.5)^{k-1} z^{-k} \end{aligned}$$

We now recognize that the impulse response sequence corresponding tqo $H(z^{-1})$ is given by

$$\begin{aligned} h_k &= 1, \;\; k = 0 \\ &= -3(0.4)^{k-1} + 4.2(0.5)^{k-1} \;\; k \geq 1 \end{aligned}$$

Let $\mathbb{1}(k)$ be the unit step, i.e., $\mathbb{1}(k) = 1$, $k \geq 0$, $\mathbb{1}(k) = 0$, $k < 0$. We can then write

$$h_k = \delta_k + [4.2(0.5)^{k-1} - 3(0.4)^{k-1}]\mathbb{1}(k-1)$$

The impulse response is stable in the sense that

$$\sum_{k=0}^{\infty} |h_k| < \infty$$

The example can clearly be generalized to rational transfer functions with n stable poles. Hence, if we have a transfer function of the form $H(z^{-1}) = \frac{C(z^{-1})}{A(z^{-1})}$, arising in ARMAX models with $A(z^{-1})$ having stable poles, the output

$$y_k = \frac{C(z^{-1})}{A(z^{-1})} w_k$$

can be expressed also in the form

$$y_k = \sum_{k=0}^{\infty} h_j w_{k-j}$$

with h_j the stable impulse response associated with $H(z^{-1})$. The output process y_k is well-defined as a second order process by the stability of $A(z^{-1})$.