Frequency Domain Design of Control Systems I: Bode Plots

While the Nyquist plot is a very powerful too to analyze stability, it is not convenient for designing controllers, since it is difficult to see how the addition of a controller can change the shape of the Nyquist plot. So classical control design is typically carried out using Bode plots.

A rational transfer function that has no poles or zeros in $\mathrm{Re}\, s \geq 0$, except possibly for poles at 0, can be written in the form

$$G(s) = \frac{K}{s^l} \frac{\prod_m (1+T_m s) \prod_i \left(1 + 2\frac{\zeta_i}{\omega_{n_i}} s + \frac{s^2}{\omega_{n_i}^2}\right)}{\prod_n (1+T_n s) \prod_k \left(1 + \frac{2\zeta_k}{\omega_{n_k}} s + \frac{s^2}{\omega_{n_k}^2}\right)}$$

$l \geq 0$ is an integer, $T_m, T_n > 0$ $(T_m \neq T_n)$ are the constants associated with the linear factors, $\omega_{n_i}, \omega_{n_k} > 0$, $0 < \zeta_i, \zeta_k < 1$ are the constants associated with the distinct quadratic factors, representing complex conjugate pairs of zeros or poles.

The Bode plot of G is a plot of the frequency response $G(j\omega)$, $\omega \geq 0$ using 2 plots, the magnitude plot and the phase plot, in the form of semilog plots.

We represent $G(j\omega)$ in terms of its magnitude in db and its phase in degrees.

$$20\log_{10}|G(j\omega)| = 20\log|K| + 20\sum \log|1 + j\omega T_m|$$

$$+ 20\sum \log\left|1 + j2\zeta_i\frac{\omega}{\omega_{n_i}} - \frac{\omega^2}{\omega_{n_i}^2}\right|$$

$$- 20\, l \log|\omega| - 20\sum \log|1 + j\omega T_n| - 20\sum \log\left|1 + 2j\zeta_k\frac{\omega}{\omega_{n_k}} - \frac{\omega^2}{\omega_{n_k}^2}\right|$$

$$\angle G(j\omega) = \angle K + \sum \angle(1+j\omega T_m) + \sum \angle\left(1 + 2\zeta_i \frac{\omega}{\omega_{n_i}} - \frac{\omega^2}{\omega_{n_i}^2}\right)$$

$$-l\angle(j\omega) - \sum \angle(1+j\omega T_n) - \sum \angle\left(1+2\zeta_k \frac{\omega}{\omega_{n_k}} - \frac{\omega^2}{\omega_{n_k}^2}\right)$$

All factors add.

(i) Constant K: Gain plot is a straight line at $20\log_{10}|K|$. The phase plot is 0 for $K>0$ and $-180°$ if $K<0$

(ii) Poles at 0: For $\omega > 0$, $-20l\log|\omega| =$

$-20l\log_{10}\omega$ is a straight line at a slope of $-20l$ db/decade. The phase is $-90l$ degrees.

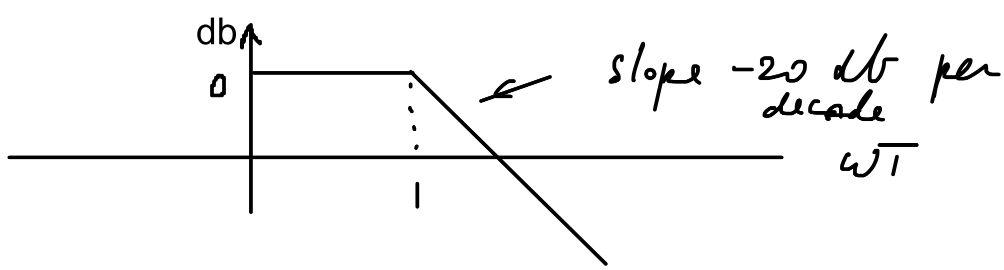
(iii) Linear pole factors: (zero factors similar)

Gain $= -20\log_{10}|1+j\omega T| = -20\log\sqrt{1+\omega^2T^2} \xrightarrow[\omega\to 0]{} 0$

$\longrightarrow -20\log_{10}\omega T$

i.e., slope of -20 db per decade increase in ωT.

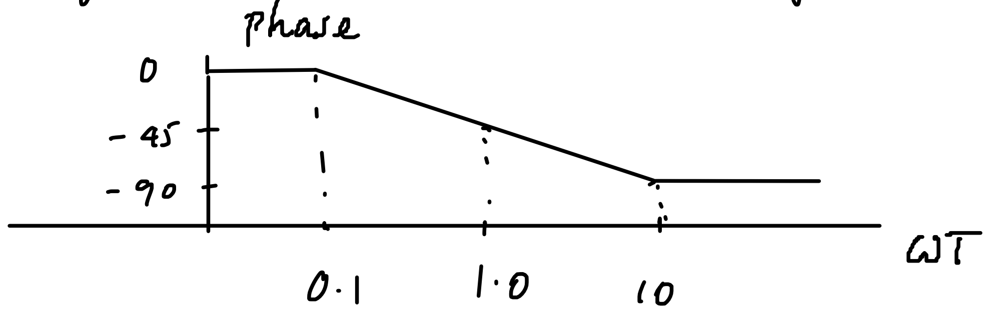
Straight line approximation for the gain plot:



phase $= -\angle(1+j\omega T) = -\tan^{-1}\omega T$

$\tan^{-1}\omega T = 45°$ for $\omega T = 1$

Straight line approx. of $\angle \frac{1}{1+j\omega T} = -\tan^{-1}\omega T$



Complex poles (zeros similar):

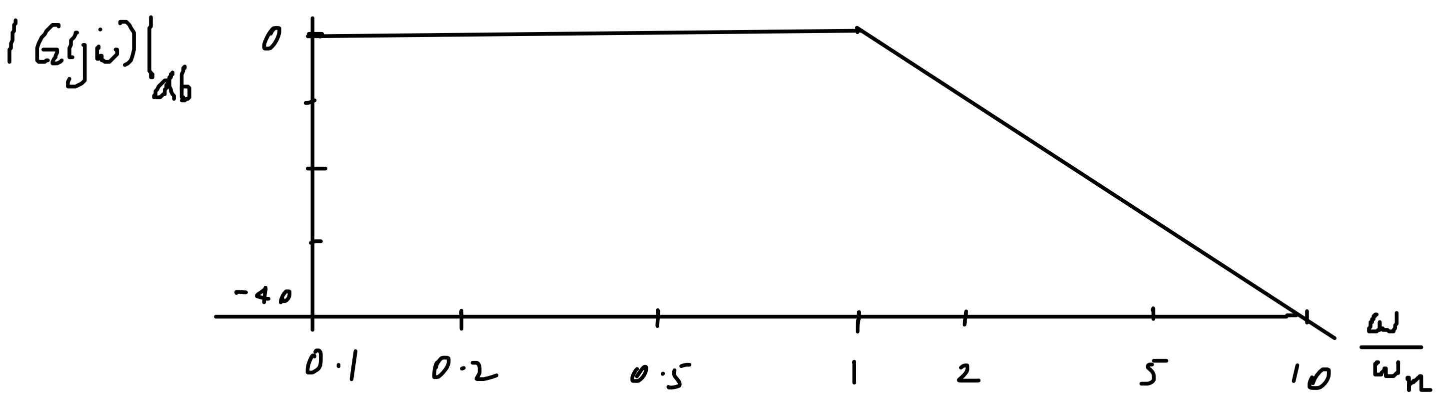
$$-20\log_{10}\left|1 + j2\zeta\frac{\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right|$$

$$= -20\log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}$$

$$\simeq -20\log\left(\frac{\omega}{\omega_n}\right)^2 \qquad \omega \to \infty$$

$$= -40\log\frac{\omega}{\omega_n}$$

Corner freq. $= \omega_n$



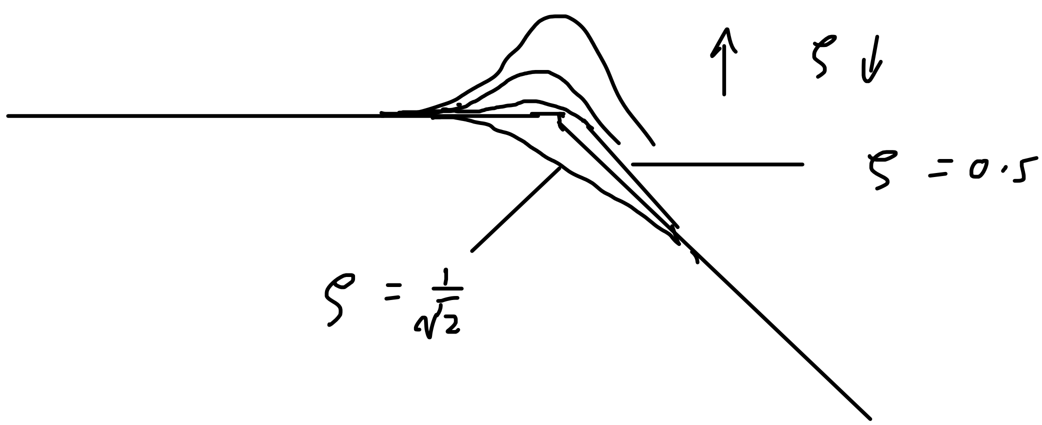
Not very accurate, but note at $\omega = \omega_n$

$$|G(j\omega)| = \frac{1}{2\zeta} \quad \text{or} \quad -20\log_{10}2\zeta$$

$\zeta = 0.5$, close approx. to straight line

$\zeta = \frac{1}{\sqrt{2}}$, $\quad -20 \log_{10} \sqrt{2} < 0$

$\zeta < 0.5$, $\quad -20 \log 2\zeta > 0$



One additional point can be determined by noting that with

$$G(j\omega) = \frac{1}{1 + j2\zeta\frac{\omega}{\omega_n} - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$\frac{d}{d\omega}|G(j\omega)|^2 = \frac{4\frac{\omega}{\omega_n^2}\left(1 - \frac{\omega^2}{\omega_n^2}\right) - 8\zeta^2\frac{\omega}{\omega_n^2}}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\left(\frac{\zeta\omega}{\omega_n}\right)^2\right]^2}$$

If $\zeta^2 \geq \frac{1}{2}$, $\frac{d}{d\omega}|G(j\omega)|^2 \leq 0$, all ω so that $|G(j\omega)|$ has its maximum at $\omega = 0$, i.e.

$$\max_{\omega} |G(j\omega)| = 1 \quad \text{for } \zeta^2 \geq \frac{1}{2}$$

If $\zeta^2 < \frac{1}{2}$, $\frac{d}{d\omega}|G(j\omega)|^2 = 0$ at

$$\omega_r = \omega_n \sqrt{1-2\zeta^2}$$

and
$$M_r = \max_{\omega} |G(j\omega)|^2$$
$$= \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \text{for } \zeta^2 < \frac{1}{2}$$

For $\zeta < \frac{1}{\sqrt{2}}$, these formulas give you a second point on the magnitude plot to give you a bit better accuracy when sketching the magnitude plot.

For example,

for $\zeta = 0.5$,
$$\omega_r = \omega_n \frac{1}{\sqrt{2}}$$
$$M_r = \frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}} \sim 1.25\,db$$

The phase plot for
$$\frac{1}{1 + 2j\zeta\frac{\omega}{\omega_n} - \left(\frac{\omega}{\omega_n}\right)^2}$$

is given by

$$-\angle\left(1+j2\zeta\frac{\omega}{\omega_n}-\left(\frac{\omega}{\omega_n}\right)^2\right)$$

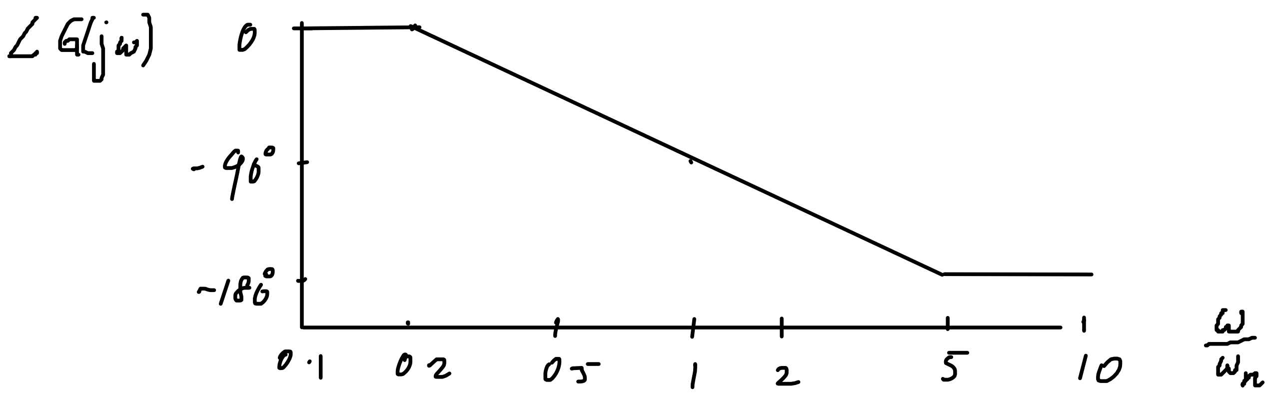
For $\frac{\omega}{\omega_n} < 1$, get $-\tan^{-1}\left(\dfrac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$

For $\frac{\omega}{\omega_n} > 1$, get $-\left(\pi - \tan^{-1}\dfrac{2\zeta\frac{\omega}{\omega_n}}{\frac{\omega^2}{\omega_n^2}-1}\right)$

$$= -\pi + \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{\frac{\omega^2}{\omega_n^2}-1}$$

$$\xrightarrow[\omega\to 0]{} 0, \quad \xrightarrow[\omega\to\infty]{} -\pi$$

Straight line approx. starts at $0.2 \le \frac{\omega}{\omega_n} \le 5$



Again, one can determine additional points on the phase curve to improve sketching accuracy, if desired. For example, for

$$\frac{\omega}{\omega_n} = \frac{1}{2}, \quad \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{1}{2}}{3/4}\right)$$

$$= -\tan^{-1}\left(\frac{4\zeta}{3}\right)$$

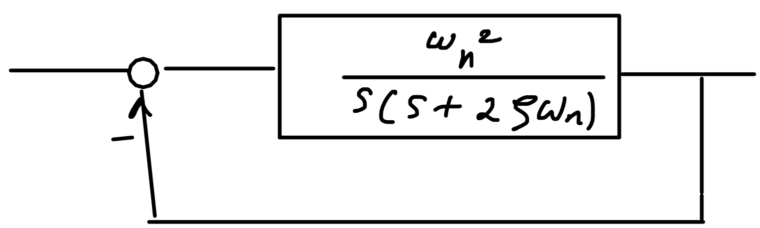
$$\frac{\omega}{\omega_n} = \frac{1}{3}, \quad \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{1}{3}}{1-\frac{1}{9}}\right)$$

$$= -\tan^{-1}\left(\frac{3\zeta}{4}\right)$$

For a specfic value of ζ, these give you 2 more points on the phase curve to get a more accurate approximation.

Designing simple controllers in the frequency domain:

Consider



$$L(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Set $|L(j\omega)| = 1 = \dfrac{\omega_n^2}{|\omega|\,(\omega^2+4\zeta^2\omega_n^2)^{1/2}}$

$$\omega_g = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

$$(\quad \omega^2(\omega^2+4\zeta^2\omega_n^2) - \omega_n^4 = 0$$

$$\omega^4 + 4\zeta^2\omega_n^2\omega^2 - \omega_n^4 = 0$$

$$\omega^2 = \frac{-4\zeta^2\omega_n^2 + \sqrt{16\zeta^4\omega_n^4 + 4\omega_n^4}}{2}$$

$$= \frac{2\omega_n^2\sqrt{4\zeta^4+1} - 4\zeta^2\omega_n^2}{2}$$

$$= \omega_n^2\left(\sqrt{1+4\zeta^4} - 2\zeta^2\right)$$

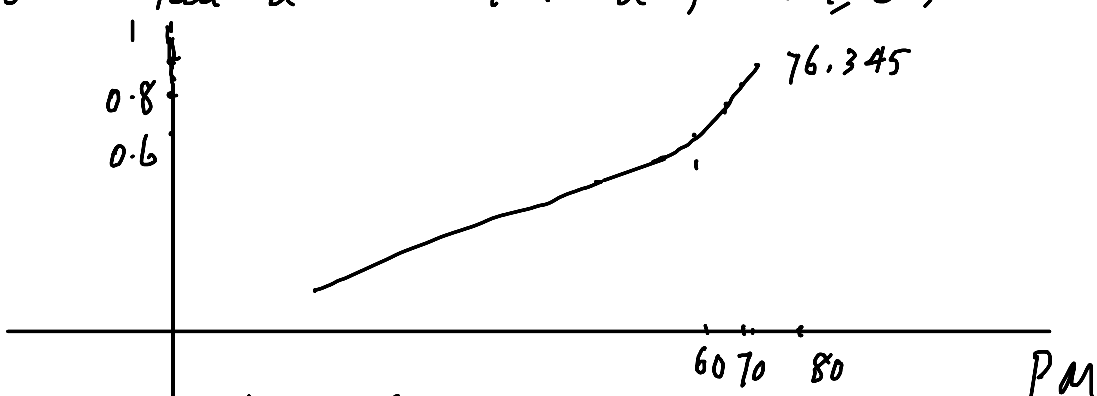
$$\omega_g = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

$$\angle L(j\omega_g) = -90^\circ - \tan^{-1}\frac{\omega_g}{2\zeta\omega_n}$$

$$\therefore \quad PM = 90^\circ - \tan^{-1}\frac{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}{2\zeta}$$

$$= \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}$$

$$\left(90^\circ - \tan^{-1} a = \tan^{-1}\frac{1}{a}, \quad a \geq 0\right)$$



Approximately linear up to $PM \simeq 60^\circ$

$\zeta < 0.6$ to give $\zeta \simeq \frac{PM}{100}$

Bandwidth ω_b is freq. where

$$|T(j\omega_b)| = \frac{1}{\sqrt{2}}$$

i.e.

$$\frac{1}{\left(1-\frac{\omega_b^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega_b^2}{\omega_n^2}} = \frac{1}{2}$$

$$\left(1-\frac{\omega_b^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega_b^2}{\omega_n^2} - 2 = 0$$

$$x = \frac{\omega_b^2}{\omega_n^2} \qquad (1-x)^2 + 4\zeta^2 x - 2 = 0$$

$$x^2 + (4\zeta^2 - 2)x - 1 = 0$$

$$x = \frac{2 - 4\zeta^2 + \sqrt{(2-4\zeta^2)^2 + 4}}{2}$$

$$= 1 - 2\zeta^2 + \frac{\sqrt{16\zeta^4 - 16\zeta^2 + 8}}{2}$$

$$= 1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}$$

$$\therefore \quad \omega_b = \omega_n\left[1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]^{\frac{1}{2}}$$

$$\therefore \quad T_s \simeq \frac{4}{\zeta\omega_n}$$

$$= \frac{4}{\zeta\omega_b}\left[1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]^{\frac{1}{2}}$$

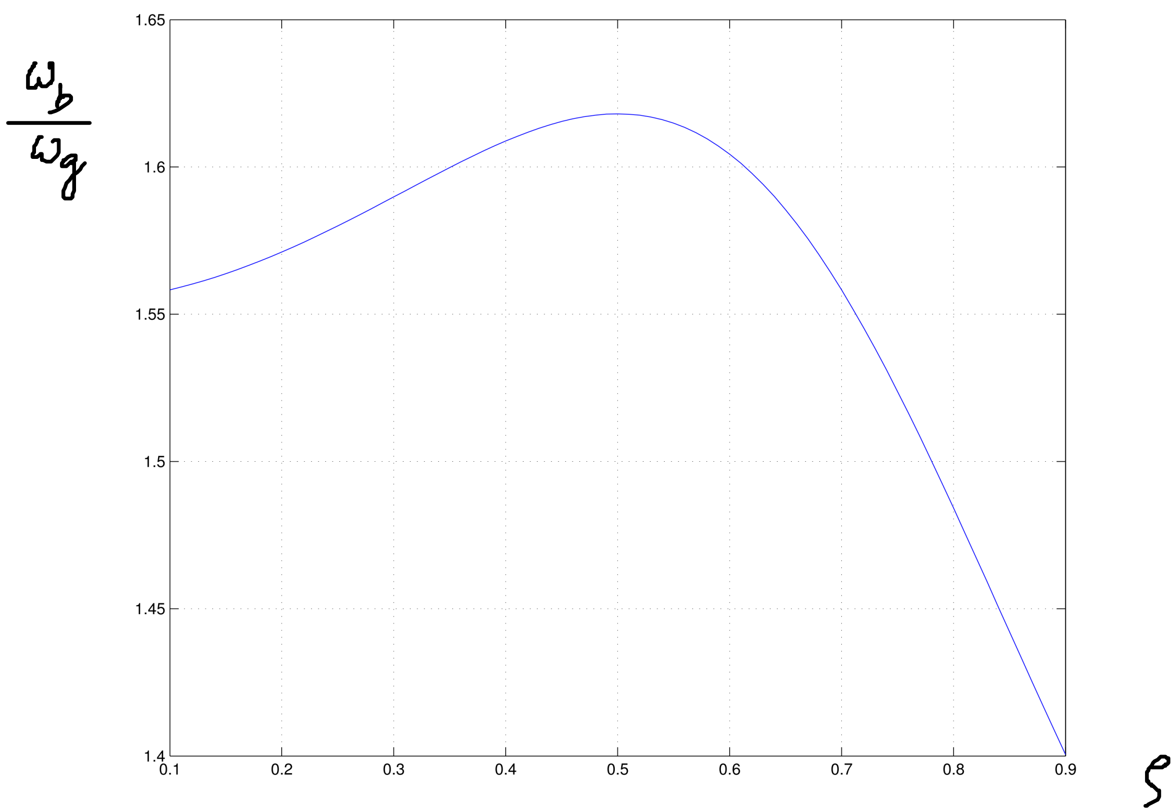
The following approximate relation is also valid.

$OS < 46\%$ $\quad PM > 27^\circ$

$OS(\%) + PM \simeq 75$

$\therefore$ larger $PM \Rightarrow$ smaller OS.

For the standard second order system, the ratio $\dfrac{\omega_b}{\omega_g}$ is a function of ζ only (see the expressions for ω_b and ω_g above). A plot of $\dfrac{\omega_b}{\omega_g}$ as a function of ζ is given below



For simplicity, sometimes we just use the estimate $\omega_b \simeq 2\omega_g$.

Suggests designing simple controllers based on PM (note GM here is ∞)
Then check, if necessary, closed loop stability using Nyquist.

Reading GM, PM, crossover frequencies from Bode plots: