SIGNALS AND SYSTEMS II
Assignment 2 Solutions

1

We have:

$$\begin{aligned}(sI-A)(sI-A)^{-1} &= I \\ (sI-A)\frac{adj(sI-A)}{det(sI-A)} &= I \\ (sI-A)adj(sI-A) &= det(sI-A)I\end{aligned}$$

Write $adj(sI-A) = \sum_{k=1}^{n} B_k s^{n-k}$ and $det(sI-A) = s^n + \sum_{k=1}^{n} a_k s^{n-k}$ to get:

$$\begin{aligned}(sI-A)\sum_{k=1}^{n} B_k s^{n-k} &= Is^n + I\sum_{k=1}^{n} a_k s^{n-k} \\ \sum_{k=1}^{n} B_k s^{n-k+1} - \sum_{k=1}^{n} AB_k s^{n-k} &= Is^n + \sum_{k=1}^{n}(a_k I)s^{n-k} \\ \sum_{k=0}^{n-1} B_{k+1} s^{n-k} - \sum_{k=1}^{n} AB_k s^{n-k} &= Is^n + \sum_{k=1}^{n}(a_k I)s^{n-k} \\ B_1 s^n + \sum_{k=1}^{n-1}(B_{k+1} - AB_k)s^{n-k} - AB_n &= Is^n + \sum_{k=1}^{n-1}(a_k I)s^{n-k} + a_n I\end{aligned}$$

Equating powers of s, we will obtain:

$$\left|\begin{array}{l} B_1 = I \\ B_{k+1} = AB_k + a_k I \quad ; \quad 1 \le k \le n-1 \\ AB_n + a_n I = 0 \end{array}\right.$$

The first two equations can be solved recursively to determine $B_k,\ k = 1, \ldots, n$. The matrix B_n thus obtained will satisfy the third equation by Cayley-Hamilton theorem:

$$A^n + \sum_{k=1}^{n} a_k A^{n-k} = 0$$

For the given matrix $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$ we have:

$$det(sI - A) = det \begin{bmatrix} s-4 & -2 \\ -3 & s-3 \end{bmatrix} = s^2 - 7s + 6$$

Therefore $a_1 = -7$ and $a_2 = 6$.

$$\left| \begin{array}{l} B_1 = I \\ B_2 = AB_1 + a_1 I = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} - 7I = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} \end{array} \right.$$

Thus,

$$adj(sI - A) = B_1 s + B_2 = Is + \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} s-3 & 2 \\ 3 & s-4 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 - 7s + 6} \begin{bmatrix} s-3 & 2 \\ 3 & s-4 \end{bmatrix}$$

2

$$H(s) = c(sI - A)^{-1} b$$

a)

$$\begin{aligned} H(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & b \\ -1 & s+a \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ c \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2 + as + b} \begin{bmatrix} s+a & -b \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ c \end{bmatrix} \\ &= \frac{1}{s^2 + as + b} \begin{bmatrix} 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ c \end{bmatrix} \\ &= \frac{cs + 1}{s^2 + as + b} \end{aligned}$$

b)

$$\begin{aligned} H(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ b & s+a \end{bmatrix}^{-1} \begin{bmatrix} c \\ 1-ac \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2+as+b} \begin{bmatrix} s+a & 1 \\ -b & s \end{bmatrix} \begin{bmatrix} c \\ 1-ac \end{bmatrix} \\ &= \frac{1}{s^2+as+b} \begin{bmatrix} s+a & 1 \end{bmatrix} \begin{bmatrix} c \\ 1-ac \end{bmatrix} \\ &= \frac{cs+1}{s^2+as+b} \end{aligned}$$

So (a) and (b) describe the same input/output behavior.

c)

Part (a): By the output equation, $y = x_2$. Take the derivatives to get:

$$\dot{y} = \dot{x_2} = x_1 - ax_2 + cu = x_1 - ay + cu$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{y} + ay - cu \\ y \end{bmatrix}$$

Part (b): By the output equation, $y = x_1$. Take the derivatives to get:

$$\dot{y} = \dot{x_1} = x_2 + cu$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} - cu \end{bmatrix}$$

3

a)

$$\begin{aligned} H(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2-1} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2-1} \begin{bmatrix} 1 & s \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{s-1}{s^2-1} = \frac{1}{s+1} \end{aligned}$$

Observe that $s = -1$ is the only pole of $H(s)$ and hence it is stable.

b) Take $u = 0$, which is obviously bounded. We have $x(t) = e^{At}x_0$.

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}\left[\begin{array}{cc} \frac{s}{s^2-1} & \frac{1}{s^2-1} \\ \frac{1}{s^2-1} & \frac{s}{s^2-1} \end{array}\right] \\ &= \frac{1}{2}\left[\begin{array}{cc} \mathcal{L}^{-1}(\frac{1}{s-1}+\frac{1}{s+1}) & \mathcal{L}^{-1}(\frac{1}{s-1}-\frac{1}{s+1}) \\ \mathcal{L}^{-1}(\frac{1}{s-1}-\frac{1}{s+1}) & \mathcal{L}^{-1}(\frac{1}{s-1}+\frac{1}{s+1}) \end{array}\right] \\ &= \frac{1}{2}\left[\begin{array}{cc} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t} \end{array}\right] \end{aligned}$$

Choosing $x_0 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ will give us

$$y(t) = ce^{At}x_0 = \frac{1}{2}(e^t + e^{-t})$$

which grows exponentially without bound as $t \to \infty$.