SIGNALS AND SYSTEMS II
Assignment 3 Solutions

1

a) When the input is zero, $y(t) = Ce^{At}x(0)$. Therefore, $y(t)$ is linear in $x(0)$.

Write:

$$\begin{bmatrix} 2 \\ 0.5 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solve for k_1 and k_2:

$$\begin{bmatrix} 1 & -1 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

$$k_1 = \frac{2.5}{1.5} = \frac{5}{3}; \quad k_2 = \frac{-0.5}{1.5} = -\frac{1}{3}$$

We have:

$$\begin{aligned} y(t) := Ce^{At} \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} &= Ce^{At}(\frac{5}{3} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}) \\ &= \frac{5}{3}Ce^{At} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} - \frac{1}{3}Ce^{At} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{5}{3}(e^{-t} - 0.5e^{-2t}) - \frac{1}{3}(-0.5e^{-t} - e^{-2t}) \\ &= \frac{11}{6}e^{-t} - \frac{1}{2}e^{-2t} \end{aligned}$$

b) The system response to input $U(s)$ and initial condition x_0 is:

$$Y'(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}x_0$$

Changing the initial condition to $2x_0$, the output would become:

$$Y''(s) = C(sI - A)^{-1}BU(s) + 2C(sI - A)^{-1}x_0$$

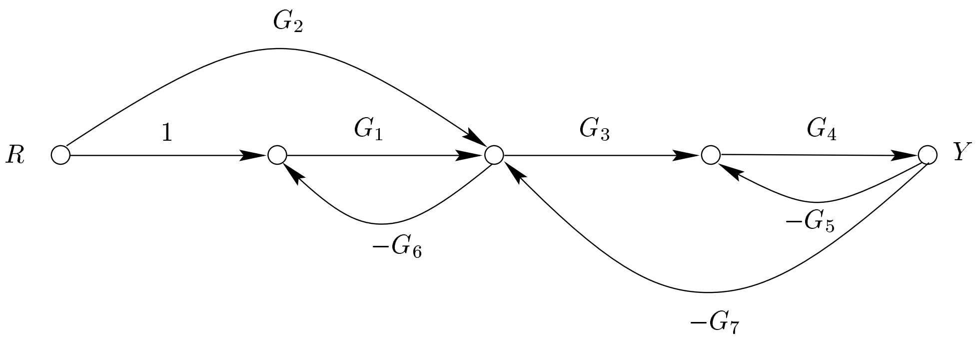
Therefore,

$$Y(s) := 2Y'(s) - Y''(s) = C(sI - A)^{-1}BU(s)$$

is the zero-state response to input $U(s)$, desired in part (b).

$$y(t) = 2y'(t) - y''(t) = 0.5(1 + e^{-2t})$$

2



The signal-flow graph of the system is shown above. Apply Mason's law to obtain:

$$\frac{Y}{R} = \frac{G_1G_3G_4 + G_2G_3G_4}{1 + G_1G_6 + G_4G_5 + G_3G_4G_7 + G_1G_6G_4G_5}$$

3

For this problem we need to assume that the polynomials in pairs (N_c, D_c) and (N_p, D_p) are coprime.

Denote the output of the first and second summing points by x_1 and x_2, respectively. As shown in the class, the transfer functions from the inputs to x_1 and x_2 can be written in matrix form as:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{1+PC}\begin{bmatrix} 1 & -P \\ C & 1 \end{bmatrix}\begin{bmatrix} R \\ D \end{bmatrix}$$

Thus,

$$\begin{aligned} \frac{X_1}{R} = \frac{X_2}{D} &= \frac{1}{1+PC} = \frac{D_cD_p}{D_cD_p + N_cN_p} \\ \frac{X_1}{D} &= -\frac{P}{1+PC} = -\frac{D_cN_p}{D_cD_p + N_cN_p} \\ \frac{X_2}{R} &= \frac{C}{1+PC} = \frac{N_cD_p}{D_cD_p + N_cN_p} \end{aligned}$$

We are to prove that if all four transfer functions are stable, i.e. the system is internally stable, then $D_cD_p + N_cN_p$ has all its roots in the left half plane. We prove this result by contradiction, i.e. suppose for some root s_0 of $D_cD_p + N_cN_p$ we have $Re(s_0) \geq 0$. Since all the transfer functions are stable, they all must have a zero at s_0. Consider the last two transfer functions. We must have:

$$D_c(s_0)D_p(s_0) + N_c(s_0)N_p(s_0) = 0 \quad (1)$$

$$D_c(s_0)N_p(s_0) = 0 \quad (2)$$

$$N_c(s_0)D_p(s_0) = 0 \quad (3)$$

From (2) it follows that either $D_c(s_0) = 0$ or $N_p(s_0) = 0$. Suppose $N_p(s_0) = 0$. Since (D_p, N_p) are coprime, $D_p(s_0) \neq 0$. It follows from (3) that $N_c(s_0) = 0$. But then (1) implies $D_c(s_0) = 0$, which contradicts the coprimeness of (D_c, N_c). The case of $D_c(s_0) = 0$ is similar.

4

As before, we denote the outputs of the summing points, clockwise and starting from the input, by x_1, x_2 and x_3, respectively. Then we find them to be related to the inputs by nine transfer functions, given below in the matrix form:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{1+PCF} \begin{bmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{bmatrix} \begin{bmatrix} R \\ D \\ V \end{bmatrix}$$

Let $P := \frac{N_p}{D_p}$, $C := \frac{N_c}{D_c}$ and $F := \frac{N_f}{D_f}$. Then the internal stability theorem states that all the transfer functions are stable if and only if $D_c D_p D_f + N_c N_p N_f$ has all its roots in $Re(s) < 0$.