SIGNALS AND SYSTEMS II
Assignment 5 Solutions

1. The root locus equation is given by

$$s(s+3)(s^2+2s+2)+K=0$$

Poles are at $0, 3, -1 \pm i$. Therefore the centroid is at $-\frac{5}{4}$. Asymptotes form angles $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Angle of departure at $-1+i$ may be found as follows:

$$\arg[(-1+i)-0] = 135^\circ$$

$$\arg[(-1+i)-(-1-i)] = 90^\circ$$

$$\arg[(-1+i)-(-3] = \tan^{-1}\frac{1}{2} \approx 26.6^\circ$$

Therefore

$$\text{angle of departure} = -(135^\circ + 90^\circ + 26.6^\circ) + 180^\circ = -71.6^\circ$$

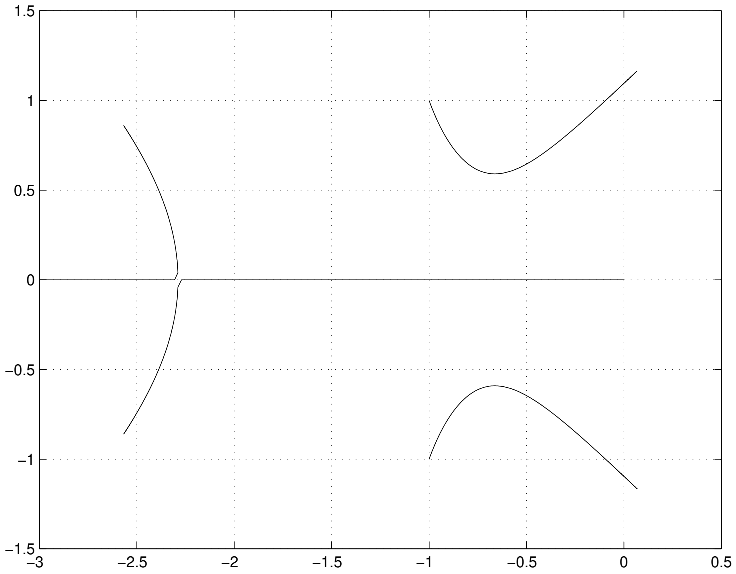
Breakaway point may be located from

$$\frac{d}{ds}[s(s+3)(s^2+2s+2)] = 0$$

or

$$4s^3 + 15s^2 + 16s + 6 = 0$$

The polynomial changes sign in $[-2.4, -2.2]$. The root locus can thus be sketched.

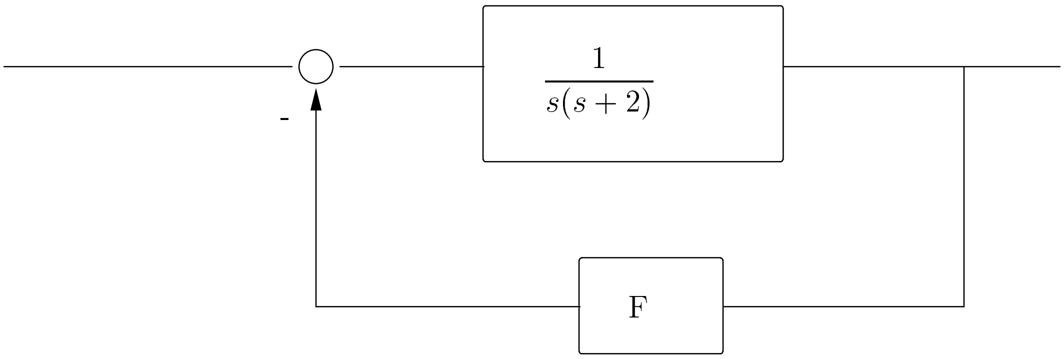


The value of K at which the root locus crosses the imaginary axis may be found through the Routh-Hurwitz criterion. We have for the polynomial $s^4+5s^3+8s^2+6s+K$, the following Routh table:

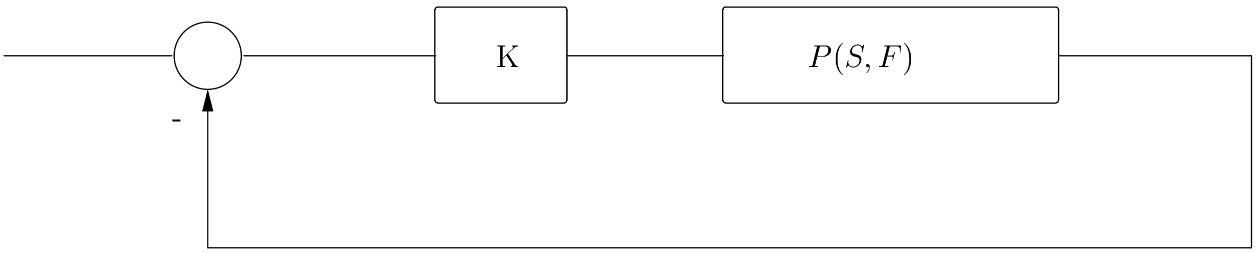
$$\begin{array}{llll}
s^4: & 1 & 8 & K \\
s^3: & 5 & 6 & 0 \\
s^2: & \frac{34}{5} & K & 0 \\
s^1: & \frac{40.8-5K}{\frac{34}{5}} & 0 & \\
s^0: & K & &
\end{array}$$

Hence at $K = 8.16$, there are 2 poles on the imaginary axis, at $\pm i1.095$ Since the asymptotes cross the $i\omega$-axis at $\pm i\frac{5}{4}$, the root locus must cross the imaginary axis below the asymptotes.

2. (i) The subsystem



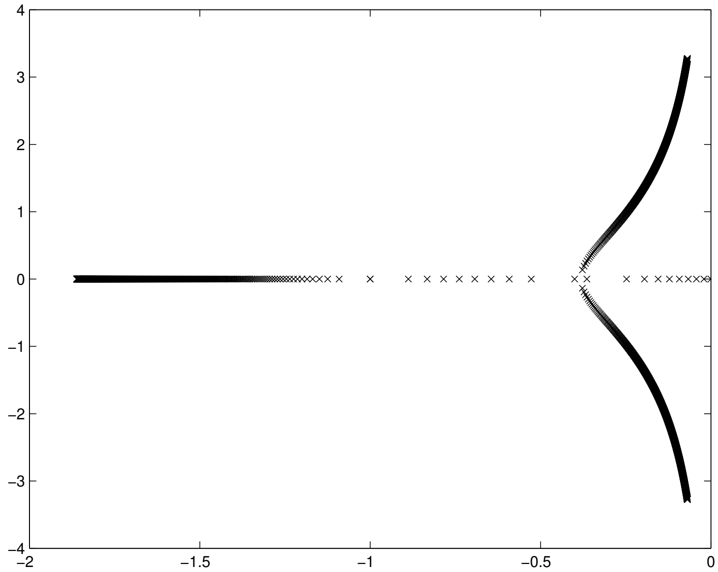
has a closed loop transfer function$=\frac{1}{s^2+2s+F}$. Thus the original system can be written as



with

$$P(s,F) = \frac{1+2/s}{s^2+2s+F} = \frac{s+2}{s(s^2+2s+F)}$$

(ii) For $F = 1$, we have the following root locus



(iii) To study what happens to open loop poles as a function of F, we consider the root locus of $s(s^2+2s+F)$ as a function of F. It is easily seen that for $F > 1$, the open loop poles has one real pole and a complex conjugate pair of poles. The 2 possible scenarios in this case would be that the complex conjugate pair of poles may enter the real axis, break away later to tend to the asymptotes, or they can tend to the asymptotes without generating a breakin point.

(iv) Any breakin point, if exists, is a solution of the equation

$$DN' - ND' = 0 \tag{1}$$

Now the root locus sketch indicates that if there is a real axis breakin point, there will be a breakaway point, and hence 2 real values of s at which multiple roots of the root locus occur. Hence if (1) has only 1 real solution, there cannot be a real axis breakin point. Now

$$DN' - ND' = -2s^3 - 8s^2 - 8s - 2F$$

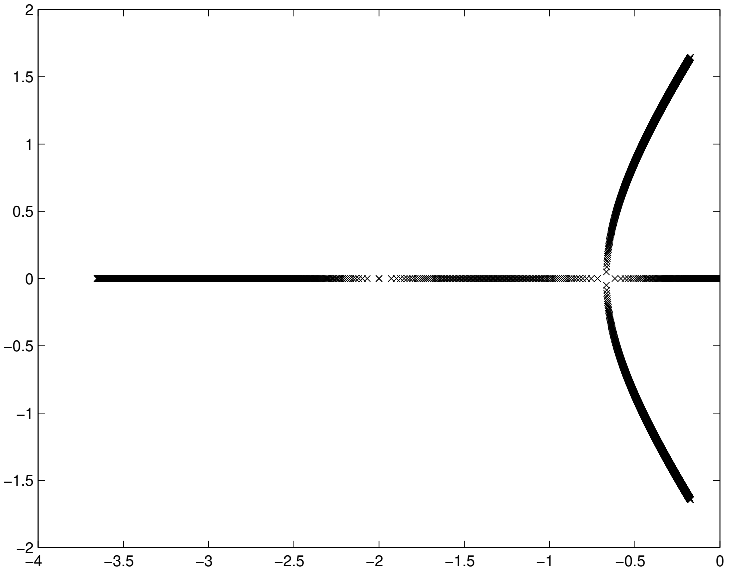
so that the root locus equation

$$s^3 + 4s^2 + 4s + F = 0 \tag{2}$$

as a function of F will reveal whether or not there is a real axis breakin point. The root locus of (2) is given by

To determine the range of values of F for which we get repeated real roots for the root locus of CLS, we need to determine where the breakaway point for the root locus of (2) is. Think of $s^3 + 4s^2 + 4s$ as $\bar{D}$ and $\bar{N} = 1$. The breakaway point of (2) is determined by the equation

$$\bar{N}\bar{D}' = 0$$



or

$$3s^2 + 8s + 4 = 0$$

yielding $s = -\frac{2}{3}$. The corresponding value of F is

$$F = (-s^3 - 4s^2 - 4s)|_{s=-\frac{2}{3}} = \frac{32}{27} = 1.185$$

Thus for $F < \frac{32}{27}$, all the multiple roots of the root locus for CLS are real. Hence there is a real axis breakin point for the root locus for $F = 1.1$. For $F > \frac{32}{27}$, there is only one multiple root on the real axis, and there cannot be a real axis breakin point.

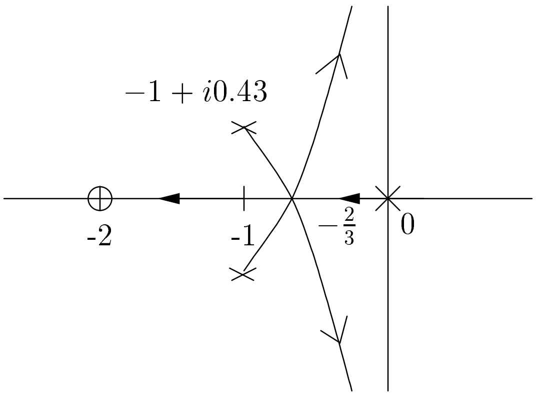
(v) To find out what happens at $F = \frac{32}{27}$, we know that for this value, $s = -\frac{2}{3}$ is a point on the root locus of CLS. The corresponding value for the gain K is:

$$K = \frac{1}{|L(s)|} = \frac{4}{27}$$

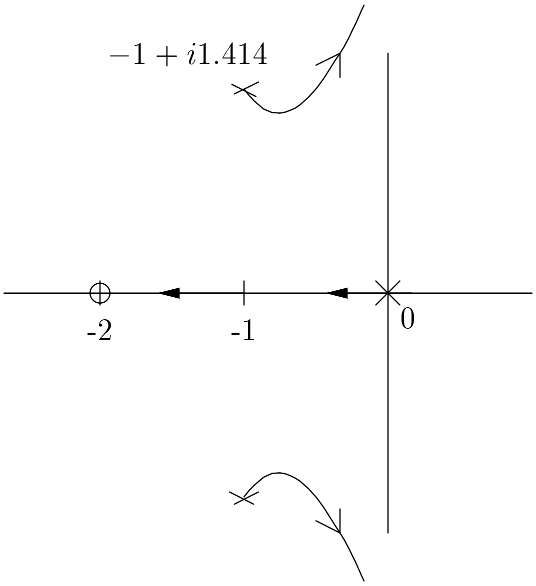
The characteristic polynomial at this value of K is given by

$$\begin{aligned} D(s) + KN(s) &= s^3 + 2s^2 + \frac{36}{27}s + \frac{8}{27} \\ &= (s + \frac{2}{3})^3 \end{aligned}$$

The root locus plot is given by



For $F=4$, we know from (iv) that there are no real axis breakin points. Thus the root locus would look like



3.

$$
\begin{aligned}
PCF(i\omega) &= \frac{1+2i\omega}{(1+i\omega)(1+i4\omega)(1+5\omega)} \\
&= \frac{(1+2i\omega)(1-i\omega)(1-i4\omega)(1-i5\omega)}{(1+\omega^2)(1+16\omega^2)(1+25\omega^2)} \\
&= \frac{1-9\omega^2-40\omega^4}{(1+\omega^2)(1+16\omega^2)(1+25\omega^2)} - \frac{i\omega(8+38\omega^2)}{(1+\omega^2)(1+16\omega^2)(1+25\omega^2)}
\end{aligned}
$$

$$PCF(i\omega) = \arctan 2\omega - \arctan\omega - \arctan 4\omega - \arctan 5\omega$$

At $\omega = 0$, $PCF(i\omega) = 1$.

$$\text{Im } PCF(i\omega) < 0 \text{ for all } \omega > 0$$

$$\text{Re } PCF(i\omega) = 0 \text{ at zeros of } 1 - 9\omega^2 - 40\omega^4, \text{ i.e. :}$$

$$\begin{aligned}
\omega^2 &= \tfrac{9 \pm \sqrt{81+160}}{2\times 40} \\
&= \tfrac{9+\sqrt{241}}{2\times 40} \qquad \text{(positive roots chosen since } \omega \text{ is real)} \\
&= \tfrac{12.46}{40}
\end{aligned}$$

Re $PCF(i\omega) = 0$ at $\omega = \pm 3.5$.

The plot is shown in Figure 1. The closed loop system is hence stable for all $K > 0$.

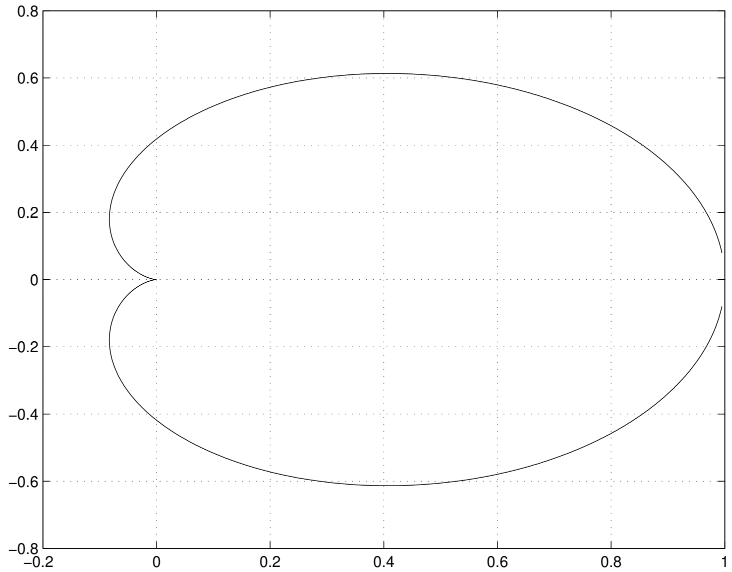


Figure 1: Nyquist plot for Problem 3.

4.

$$\begin{aligned}
PCF(i\omega) &= \frac{2+i\omega}{i\omega(i\omega-1)} \\
&= \frac{-i(-1-i\omega)(3+i\omega)}{\omega(1+\omega^2)} \\
&= -\frac{4}{1+\omega^2} + i\frac{3-\omega^2}{\omega(1+\omega^2)}
\end{aligned}$$

Nyquist path is shown in Figure 2.

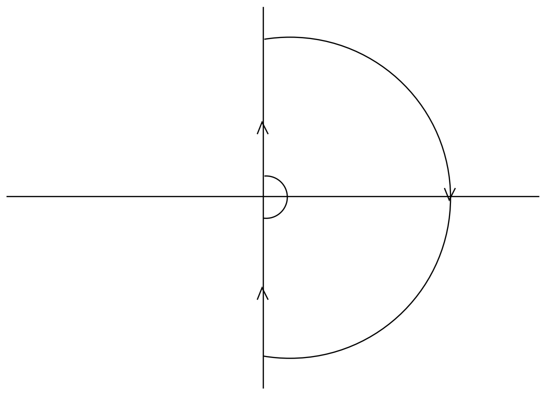


Figure 2: Nyquist path for Problem 4.

As $\omega \to 0^+$, $Re\ PCF \to -4$ and $Im\ PCF \to -\infty$.

At $\omega = \pm 3$, $Im\ PCF = 0$, $Re\ PCF = -1$.

Nyquist plot is shown in Figure 3.

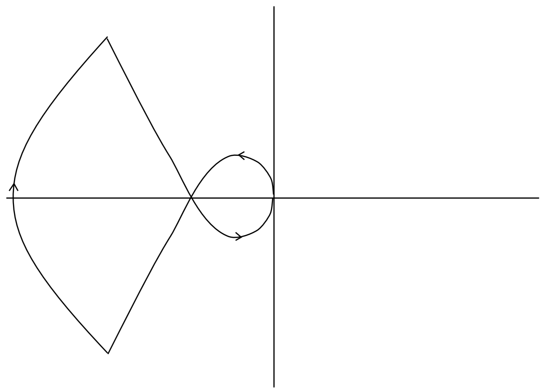


Figure 3: Nyquist plot for Problem 4.

Number of loops of the open-loop system in the RHP is 1. Therefore we need one counterclockwise encirclement of $-1/K$. Hence $-1/K > -1$ or $K > 1$ for stability.