SIGNALS AND SYSTEMS II
Assignment 6 Solutions

1. (a)

$$\begin{aligned}
X(e^{i\omega}) &= \sum_{k=-\infty}^{\infty} x_k e^{-i\omega k} \\
&= \sum_{k=-\infty}^{-1} (\frac{1}{3})^{-k} e^{-i\omega k} + \sum_{k=0}^{\infty} (\frac{1}{2})^{k} e^{-i\omega k} \\
&= \sum_{k=1}^{\infty} (\frac{1}{3} e^{i\omega})^{k} + \sum_{k=0}^{\infty} (\frac{1}{2} e^{-i\omega})^{k} \\
&= \frac{\frac{1}{3}e^{i\omega}}{1-\frac{1}{3}e^{i\omega}} + \frac{1}{1-\frac{1}{2}e^{-i\omega}} \\
&= -\frac{5e^{i\omega}}{2(e^{i\omega}-3)(e^{i\omega}-\frac{1}{2})}
\end{aligned}$$

(b) Fourier inversion is given by

$$x_k = \frac{1}{2\pi}\int_{2\pi} X(e^{i\omega})e^{i\omega k} d\omega \tag{1}$$

$$= \frac{1}{2\pi}\int_{2\pi} -\frac{5e^{i\omega(k+1)}}{2(e^{i\omega}-3)(e^{i\omega}-\frac{1}{2})} d\omega \tag{2}$$

Let $z = e^{i\omega}, dz = izd\omega$. Equation 2 can be rewritten as

$$\begin{aligned}
x_k &= \frac{1}{2\pi i}\oint_c X(z)z^{k-1}dz \\
&= \frac{1}{2\pi i}\oint_c -\frac{5z^k}{2(z-3)(z-\frac{1}{2})}dz \\
&:= \frac{1}{2\pi i}\oint_c f(z)dz
\end{aligned}$$

where c is the counterclockwise unit circle. Now we can use Residue Theorem to carry out the above integral.

(i). $k \geq 0$

Inside the unit circle, there is only one pole at $z_o = \frac{1}{2}$. Thus

$$\begin{aligned}
x_k &= \left[-\frac{5z^k}{2(z-3)}\right]_{z=\frac{1}{2}} \\
&= (\frac{1}{2})^k, \quad k \geq 0
\end{aligned}$$

(ii).$k < 0$

Now there are a pole at $z_o = \frac{1}{2}$ and a pole of order $-k$ at the origin, inside the

unit circle. It is not easy to calculate the residue at the origin.
Let $v = \frac{1}{z}$, $dv = -\frac{1}{z^2}dz = -v^2 dz$. Notice that the integral path c' in v-plane is the *clockwise* unit circle [1].

$$\begin{aligned}
x_k &= \frac{1}{2\pi i}\oint_{c'} f(\frac{1}{v})(-v^2)dv \\
&= -\frac{1}{2\pi i}\oint_{c'} f(\frac{1}{v})v^2 dv \\
&= \frac{1}{2\pi i}\oint_{c} f(\frac{1}{v})v^2 dv \\
&= \frac{1}{2\pi i}\oint_{c} -\frac{5v^{-k}}{3(v-\frac{1}{3})(v-2)}dv
\end{aligned}$$

Then by Residue Theorem, we have

$$x_k = \left[-\frac{5v^{-k}}{3(v-2)}\right]_{v=\frac{1}{3}} = (\frac{1}{3})^{-k}, \quad k < 0,$$

since there is only one pole of order 1 at $v_o = \frac{1}{3}$ inside the unit circle.

2. (a)

$$\begin{aligned}
x_k &= \frac{1}{2\pi i}\oint X(z)z^{k-1}dz \\
&= \frac{1}{2\pi i}\oint \frac{z^{k+1}}{(z-a)(z-b)}dz \\
&= \left[\frac{z^{k+1}}{z-b}\right]_{z=a} + \left[\frac{z^{k+1}}{z-a}\right]_{z=b} \\
&= \frac{a^{k+1}-b^{k+1}}{a-b}, \quad k \geq 0
\end{aligned}$$

(b)

$$\begin{aligned}
X(z) &= \alpha_1\frac{1}{z^{-1}-a^{-1}} + \alpha_2\frac{1}{z^{-1}-b^{-1}} \\
&= \frac{-a\cdot\alpha_1}{1-az^{-1}} + \frac{-b\cdot\alpha_2}{1-bz^{-1}} \\
&= \frac{(-a\alpha_1 - b\alpha_2) + (ab\alpha_1 + ab\alpha_2)z^{-1}}{(1-az^{-1})(1-bz^{-1})} \\
&= \frac{1}{(1-az^{-1})(1-bz^{-1})}
\end{aligned}$$

[1] At path c, $z = e^{j\omega}$ goes counterclockwise when ω goes from 0 to 2π. Now at path c' in v-plane, $v = \frac{1}{z} = e^{-j\omega}$ must go clockwise along the unit circle.

Thus, $\alpha_1 = \frac{1}{b-a}, \alpha_2 = \frac{1}{a-b}$. And

$$\begin{aligned}
x_1(k) &= Z^{-1}\left[\alpha_1 \frac{1}{z^{-1}-a^{-1}}\right] \\
&= Z^{-1}\left[\frac{a}{a-b} \cdot \frac{1}{1-az^{-}1}\right] \\
&= \frac{a}{a-b} \cdot a^k \\
&= \frac{a^{k+1}}{a-b}, \quad k \geq 0 \\
x_2(k) &= Z^{-1}\left[\alpha_2 \frac{1}{z^{-1}-b^{-1}}\right] \\
&= \frac{b^{k+1}}{b-a}, \quad k \geq 0
\end{aligned}$$

Therefore, $x_k = x_1(k) + x_2(k) = \frac{a^{k+1}-b^{k+1}}{a-b}, \quad k \geq 0$

3. (a) i. $\{kx_k\}$

$$\begin{aligned}
X(z) &= \sum_{k=0}^{\infty} x_k z^{-k} \\
\frac{dX(z)}{dz} &= -\sum_{k=0}^{\infty} k x_k z^{-k-1} \\
-z \cdot \frac{dX(z)}{dz} &= \sum_{k=0}^{\infty} k x_k z^{-k} \\
&= Z[kx_k]
\end{aligned}$$

ii. $\{k^2 x_k\}$

Already know

$$-z \cdot \frac{dX(z)}{dz} = \sum_{k=0}^{\infty} k x_k z^{-k}$$

Continue the $(-z) \cdot \frac{d}{dz}$ operation on both sides, we have

$$\begin{aligned}
-\frac{dX(z)}{dz} - z\frac{d^2X(z)}{dz^2} &= -\sum_{k=0}^{\infty} k^2 x_k z^{-k-1} \\
z\frac{dX(z)}{dz} + z^2\frac{d^2X(z)}{dz^2} &= \sum_{k=0}^{\infty} k^2 x_k z^{-k} \\
&= Z[k^2 x_k]
\end{aligned}$$

(b) By definition, $Z[x_k y_k] = \sum_{k=0}^{\infty} x_k y_k z^{-k}$. If $|z| > R_y$, then $Y(z)$ exists. Rewrite the above equation by substituting the inverse z-transform of y_k for y_k, we get

$$Z[x_k y_k] = \sum_{k=0}^{\infty} x_k \left[\frac{1}{2\pi i} \oint Y(v) v^{k-1} dv\right] z^{-k}$$

$$\begin{aligned} &= \frac{1}{2\pi i}\oint Y(v)\sum_{k=0}^{\infty}(x_k z^{-k} v^{k-1})dv \\ &= \frac{1}{2\pi i}\oint \frac{Y(v)}{v}\cdot\sum_{k=0}^{\infty} x_k(\frac{z}{v})^{-k}dv \end{aligned}$$

If $|\frac{z}{v}| > R_x$, then $X(\frac{z}{v})$ exists. We have

$$Z[x_k y_k] = \frac{1}{2\pi i}\oint X(\frac{z}{v})Y(v)\frac{dv}{v}.$$

For the existence of $Y(v)$, we need $|v| > R_y$. For the existence of $X(\frac{z}{v})$, we need $|\frac{z}{v}| > R_x$. Thus the ROC at z-plane is $|z| > R_x R_y$.

(c) The ROC for $Z[k]$ is $|z| > 1$. By the result in 3(b), the ROC for $Z[kx_k]$ is $|z| > R_x \cdot 1 = R_x$.
The ROC for $Z[k^2]$ is $|z| > 1$. By the result in 3(b), the ROC for $Z[k^2 x_k]$ is $|z| > R_x \cdot 1 = R_x$.

4. (a)

$$\begin{aligned} X(z) &= \sum_{k=-\infty}^{\infty} x_k z^{-k} \\ &= \sum_{k=-\infty}^{-1} x_k z^{-k} + \sum_{k=0}^{\infty} x_k z^{-k} \\ &= \sum_{k=-\infty}^{-1} 3^k z^{-k} + \sum_{k=0}^{\infty} 2^k z^{-k} \\ &= \sum_{k=-\infty}^{-1} (\frac{z}{3})^{-k} + \sum_{k=0}^{\infty} (2z^{-1})^k \\ &= \sum_{k=1}^{\infty} (\frac{z}{3})^{k} + \sum_{k=0}^{\infty} (2z^{-1})^k \\ &= \frac{\frac{1}{3}z}{1-\frac{1}{3}z} + \frac{1}{1-2z^{-1}}, \quad if\ |\frac{z}{3}| < 1\ \&\ |2z^{-1}| < 1 \\ &= \frac{z}{3-z} + \frac{z}{z-2} \end{aligned}$$

The ROC is $2 < |z| < 3$ from $|\frac{z}{3}| < 1$ & $|2z^{-1}| < 1$.

(b) i. $|z| < 4$.

$$\begin{aligned} X(z) &= \frac{1}{z-4} \\ &= -\frac{1}{4}\cdot\frac{1}{1-\frac{z}{4}} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}\cdot\sum_{k=0}^{\infty}(\frac{z}{4})^k \\
&= -\sum_{k=0}^{\infty}(\frac{1}{4})^{k+1}z^k \\
&= -\sum_{k=-\infty}^{0}(\frac{1}{4})^{-k+1}z^{-k} \\
&= \sum_{k=-\infty}^{0}(-4^{k-1})z^{-k}
\end{aligned}
$$

Therefore, from the definition of bilateral z-transform,

$$
x_k = \begin{cases} -4^{k-1}, & k \leq 0 \\ 0 & k > 0 \end{cases}
$$

ii. $|z| > 4$.

$$
\begin{aligned}
X(z) &= \frac{1}{z-4} \\
&= \frac{\frac{1}{z}}{1-\frac{4}{z}} \\
&= \sum_{k=0}^{\infty}\frac{1}{z}(\frac{4}{z})^k \\
&= \sum_{k=0}^{\infty}4^k z^{-k-1} \\
&= \sum_{k=1}^{\infty}4^{k-1}z^{-k}
\end{aligned}
$$

Therefore, from the definition of bilateral z-transform,

$$
x_k = \begin{cases} 0 & k < 1 \\ 4^{k-1} & k \geq 1 \end{cases}
$$