ECE356 – 1999
Solutions to the Midterm Test

1. We use the property

$$\frac{d}{dt}e^{At} = Ae^{At}$$

Hence $A = \frac{d}{dt}\, e^{At}\Big|_{t=0}$

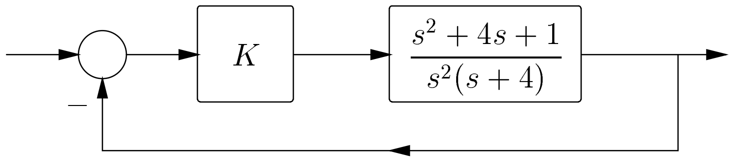
$$\frac{d}{dt}\begin{bmatrix} \frac{1}{2}(3e^{-t} - e^{-3t}) & \frac{1}{2}(e^{-t} - e^{-3t}) \\ -\frac{3}{2}(e^{-t} - e^{-3t}) & \frac{1}{2}(3e^{-t} - e^{-3t}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(-3e^{-t} + 3e^{-3t}) & \frac{1}{2}(-e^{-t} + 3e^{-3t}) \\ -\frac{3}{2}(-e^{-t} + 3e^{-3t}) & \frac{1}{2}(-9e^{-3t} + e^{-t}) \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

2. (i) The characteristic polynomial is given by

$$p(s) = (s + K)s(s + 4) + K = s^3 + 4s^2 + K(s^2 + 4s + 1)$$

Thus the root locus of the closed loop system has the same behavior as the root locus of the system



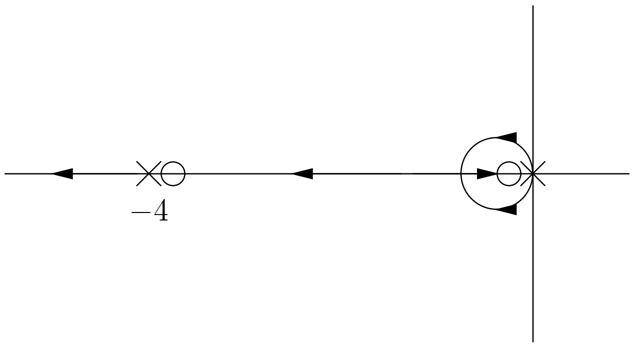
Roots of $s^2 + 4s + 1$ are at

$$\frac{-4 \pm \sqrt{16 - 4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

Note that these are not zeros of the original system. They serve only as limiting values for the root locus. Note also that $p(s)$ is stable for all $K > 0$ since its Routh table is

s^3:	1	$4K$
s^2:	$4 + K$	K
s^1:	$15K + 4K^2$	0
s^0:	K	

Thus the root locus sketch looks like



(The real axis break in point, which you don't need to determine, can be estimated from

$$\begin{aligned}
&(s^2+4s+1)(3s^2+8s)-(s^3+4s^2)(2s+4)\\
&= 3s^4+20s^3+35s^2+8s-(2s^4+12s^3+16s^2)\\
&= s^4+8s^3+19s^2+8s\\
&= s(s^3+8s^2+19s+8)=0
\end{aligned}$$

Since $s^3+8s^2+19s+8$ changes signs on $[-1,0]$, there is a real root in $[-1,0]$.)

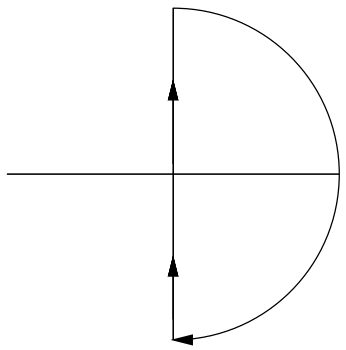
(ii) The normal behaviour of relative degree 3 systems, being unstable for K sufficiently large, is for a characteristic polynomial of the form $D(s)+KN(s)$ with D, N independent of K, and degree of D = degree of $N+3$. This is not the case for this system and there is no contradiction.

3. (i)

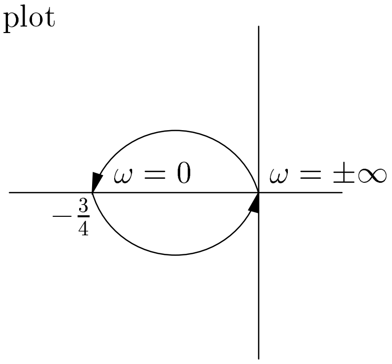
$$\begin{aligned}
P(i\omega) &= \frac{i\omega+3}{(i\omega+2)^2(i\omega-1)} = \frac{(i\omega+3)(-i\omega+2)^2(-i\omega-1)}{(4+\omega^2)^2(1+\omega^2)}\\
&= -\frac{12+9\omega^2+\omega^4}{(4+\omega^2)^2(1+\omega^2)} - i\frac{4\omega}{(4+\omega^2)^2(1+\omega^2)}
\end{aligned}$$

At $\omega=\pm\infty$, $P(i\omega)=0$. At $\omega=0$, $P(i\omega=-\frac{3}{4}$. For ω in $(-\infty,0)$, $P(i\omega)$ is in 2nd quadrant, and in $(0,\infty)$, $P(i\omega)$ is in the 3rd quadratnt.

Nyquist path



Nyquist plot



(ii) For stability, we need 1 counterclockwise encirclement of $-\frac{1}{K}$ since there is one unstable open loop pole at $s = 1$. Thus we require

$$-\frac{3}{4} < -\frac{1}{K} < 0 \qquad \text{or} \qquad K > \frac{4}{3}$$