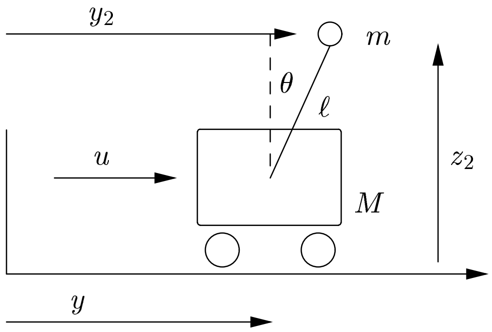
# Chapter 1

# Mathematical Modelling of Physical Systems

**Example:** Inverted pendulum on a cart




To derive the differential equation for this system, it is convenient to use Lagrange's equation. Define a set of generalized coordinates $q_i$ with kinetic and potential energy given by $T(q,\dot{q})$ and $V(q)$. The Lagrangian $L(q,\dot{q})$ is given by

$$L = T - V$$

Lagrange's equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i$$

where $F_i$ are the generalized forces (these are mostly external forces, forces not derived from a potential). For inverted pendulum:

$$\begin{aligned} T_1 &= \frac{1}{2}M\dot{y}^2 \\ T_2 &= \frac{1}{2}m(\dot{y}_2^2 + \dot{z}_2^2) \end{aligned}$$

$$\begin{aligned} y_2 = y + l\sin\theta \qquad & \dot{y}_2 = \dot{y} + l\cos\theta\dot{\theta} \\ z_2 = l\cos\theta \qquad & \dot{z}_2 = -l\dot{\theta}\sin\theta \end{aligned}$$

$$\begin{aligned} T &= T_1 + T_2 = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m(\dot{y} + l\dot{\theta}\cos\theta)^2 \\ &\quad + \frac{1}{2}m(l^2\dot{\theta}^2\sin^2\theta) \\ &= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 + ml\dot{y}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 \end{aligned}$$

Potential energy $V = mgl\cos\theta$.

Hence Lagrange's equations give

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = u$$

$$(M+m)\ddot{y} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u$$

Now

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= ml\dot{y}\cos\theta + ml^2\dot{\theta} \\
\frac{\partial L}{\partial \theta} &= -ml\dot{y}\dot{\theta}\sin\theta + mgl\sin\theta
\end{aligned}$$

$$\begin{aligned}
\therefore\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= ml\ddot{y}\cos\theta - ml\dot{y}\dot{\theta}\sin\theta + ml^2\ddot{\theta} \\
&\quad +ml\dot{y}\dot{\theta}\sin\theta - mgl\sin\theta \\
&= ml\ddot{y}\cos\theta + ml^2\ddot{\theta} - mgl\sin\theta = 0
\end{aligned}$$

or

$$\ddot{y}\cos\theta + l\ddot{\theta} - g\sin\theta = 0$$

In terms of matrix notation:

$$\begin{bmatrix} M+m & ml\cos\theta \\ \cos\theta & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + ml\dot{\theta}^2\sin\theta \\ g\sin\theta \end{bmatrix}$$

Explicit expression for $\ddot{y}$ and $\ddot{\theta}$ are

$$\begin{aligned}
\ddot{y} &= \frac{\det\begin{bmatrix} u + ml\dot{\theta}^2\sin\theta & ml\cos\theta \\ g\sin\theta & l \end{bmatrix}}{\det\begin{bmatrix} M+m & ml\cos\theta \\ \cos\theta & l \end{bmatrix}} \\
&= \frac{u + ml\dot{\theta}^2\sin\theta - mg\cos\theta\sin\theta}{M+m-m\cos^2\theta} \\
&= \frac{u + ml\dot{\theta}^2\sin\theta - mg\cos\theta\sin\theta}{M+m\sin^2\theta} \\
\ddot{\theta} &= \frac{\det\begin{bmatrix} M+m & u + ml\dot{\theta}^2\sin\theta \\ \cos\theta & g\sin\theta \end{bmatrix}}{l(M+m\sin^2\theta)} \\
&= \frac{(M+m)g\sin\theta - u\cos\theta - ml\dot{\theta}^2\sin\theta\cos\theta}{l(M+m\sin^2\theta)}
\end{aligned}$$

A highly nonlinear set of differential equations!

Let

$$x_1 = y \qquad x_2 = \theta \qquad x_3 = \dot{y} \qquad x_4 = \dot{\theta}$$

We then get a first order vector-valued equation in state form:

$$\dot{x} = f(x,u)$$

where

$$\begin{aligned}
f_1(x,u) &= x_3 \\
f_2(x,u) &= x_4 \\
f_3(x,u) &= \frac{mlx_4^2\sin x_2 - mg\cos x_2\sin x_2 + u}{M+m\sin^2 x_2} \\
f_4(x,u) &= \frac{(M+m)g\sin x_2 - mlx_4^2\sin x_2\cos x_2 - u\cos x_2}{l(M+m\sin^2 x_2)}
\end{aligned}$$

Nonlinear differential equations are difficult to analyze. If we are interested in small perturbations about an equilibrium point, we can linearize the nonlinear differential equation.

## Linearization of nonlinear o.d.e.'s

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

Assume there is an equilibrium point $(x_0, u_0)$

$$f(x_0, u_0) = 0$$

$(x_0, u_0)$ is then an equilibrium solution to the o.d.e.

Think of a small perturbation about this equilibrium solution. Define

$$\begin{aligned} \Delta x &= x - x_0 \\ \Delta y &= y - h(x_0, u_0) \\ x(t) &= x_0 + \Delta x(t) \\ u(t) &= u_0 + \Delta u(t) \end{aligned}$$

Performing Taylor's series expansion about $(x_0, u_0)$ and keeping only the first order terms,

$$\begin{aligned} \dot{x} = \Delta\dot{x} &\cong f(x_0,u_0) + \frac{\partial f}{\partial x}(x_0,u_0)\Delta x \\ &\quad + \frac{\partial f}{\partial u}(x_0,u_0)\Delta u \end{aligned}$$

where $\frac{\partial f}{\partial x}(x_0, u_0)$ is the Jacobian matrix with entries

$$\left[\frac{\partial f}{\partial x}(x_0,u_0)\right]_{ij} = \frac{\partial f_i}{\partial x_j}(x_0,u_0)$$

Similarly for $\frac{\partial f}{\partial u}(x_0, u_0)$. Hence,

$$\begin{aligned} \Delta\dot{x} &= \frac{\partial f}{\partial x}(x_0,u_0)\Delta x + \frac{\partial f}{\partial u}(x_0,u_0)\Delta u \\ &= A\Delta x + B\Delta u \\ \Delta y &= \frac{\partial h}{\partial x}(x_0,u_0)\Delta x + \frac{\partial h}{\partial u}(x_0,u_0)\Delta x \\ &= C\Delta x + D\Delta u \end{aligned}$$

**Example:** Inverted pendulum on cart.

$$\begin{aligned} f(x_0,u_0) &= 0 \qquad \text{corresponds to} \\ x_{30} &= 0 \\ x_{40} &= 0 \end{aligned}$$

$$u_0 + mlx_{40}^2 \sin x_{20} - mg\cos x_{20}\sin x_{20} = 0 \tag{1.1}$$

$$(M+m)g\sin x_{20} - mlx_{40}^2 \sin x_{20}\cos x_{20} - u_0\cos x_{20} = 0 \tag{1.2}$$

(1.1) $\star$ $\cos x_{20}$ and adding to (1.2) gives

$$(M+m)g\sin x_{20} - mg\cos^2 x_{20}\sin x_{20} = 0$$

or

$$\sin x_{20}(M + m\sin^2 x_{20}) = 0$$

$$\therefore \quad \sin x_{20} = 0 \qquad \text{or} \qquad x_{20} = 0 \;\text{ or }\; \pi$$

From (1.1), we have $u_0 = 0$.

Equilibrium points are $x_0 = \begin{bmatrix} \text{arbitrary} \\ \pi \\ 0 \\ 0 \end{bmatrix}$, $u_0 = 0$ or

$$x_0 = \begin{bmatrix} \text{arbitrary} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad u_0 = 0$$

Physically $x_{20} = 0$ (pendulum up), of interest in stabilization. On the other hand, $x_{20} = \pi$ (pendulum down), stable but not interesting.

Take $x_0 = \begin{bmatrix} \text{arbitrary} \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $u_0 = 0$. All entries in $\frac{\partial f}{\partial x}(x_0, u_0)$ and $\frac{\partial f}{\partial u}(x_0, u_0)$ are zero except:

$$\begin{aligned}
\frac{\partial f_1}{\partial x_3} &= 1 \\
\frac{\partial f_2}{\partial x_4} &= 1 \\
\frac{\partial f_3}{\partial x_2} &= \frac{\begin{array}{c}(M + m\sin^2 x_2)(mlx_4^2 \cos x_2 - mg\cos^2 x_2 + mg\sin^2 x_2) \\ -(u + mlx_4^2 \sin x_2 - mg\cos x_2 \sin x_2)(2m\sin x_2 \cos x_2)\end{array}}{(M + m\sin^2 x_2)^2} \\
\frac{\partial f_3}{\partial x_2}(x_0, u_0) &= \frac{-Mmg}{M^2} = -\frac{mg}{M}
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial f_4}{\partial x_2} &= \frac{\begin{array}{c}l(M + m\sin^2 x_2)[(M+m)g\cos x_2 - mlx_4^2 \cos x_2^2 + mlx_4^2 \sin^2 x_2 + u\sin x_2] \\ -[(M+m)g\sin x_2 - mlx_4^2 \sin x_2 \cos x_2 - u\cos x_2][2lm\sin x_2 \cos x_2]\end{array}}{l^2(M + m\sin^2 x_2)^2} \\
\frac{\partial f_4}{\partial x_2}(x_0, u_0) &= \frac{lM(M+m)g}{l^2M^2} = \frac{(M+m)g}{Ml} \\
\frac{\partial f_3}{\partial u}(x_0, u_0) &= \frac{1}{M} \qquad \frac{\partial f_4}{\partial u}(x_0, u_0) = -\frac{1}{Ml} \\
\Delta \dot{x} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{Ml} & 0 & 0 \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{Ml} \end{bmatrix} \Delta u
\end{aligned}$$

The linearized system tells us quite a bit though not everything about the nonlinear system.

Assume from now on, we either have a linear system or we have linearized a nonllinear system to give

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}$$