Chapter 2

SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

In this chapter, we discuss the solution of the linear differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ x(t) &\in \mathbb{R}^n \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

Very often, for simplicity, we shall suppress the time dependence in the notation. We give a systematic method for solving (2.1) and discuss properties of the solution. Finally, we discuss linear systems with inputs and outputs.

2.1 Existence, Uniqueness and the Transition Matrix

It is known from the theory of ordinary differential equations that under certain regularity assumptions, a (nonlinear) differential equation of the form

$$\begin{aligned} \dot{x}(t) &= f(x,t) \\ t &\in [t_0, t_1] \\ x(t_0) &= x_0 \end{aligned} \tag{2.2}$$

has a unique solution passing through x_0 at $t = t_0$. The regularity conditions are

(i) $f(\cdot,\cdot)$ is a continuous function from $R^n \times [t_0, t_1]$ to R^n,

(ii) f satisfies a global Lipschitz condition

$$||f(x_1,t) - f(x_2,t)|| \leq k(t)||x_1 - x_2|| \qquad \forall\, x_1, x_2 \in R^n$$

where $k(t)$ is continuous on $[t_0, t_1]$.

Furthermore, the solution to (2.2) can be obtained by the so-called Picard's iteration scheme:
Define:

$$\begin{aligned} x_0(t) &= x_0 \\ x_{n+1}(t) &= x_0 + \int_{t_0}^t f[x_n(s), s]ds \end{aligned} \tag{2.3}$$

Then

$$\lim_{n\to\infty} \sup_{t\in[t_0,t_1]} ||x_n(t) - x(t)|| = 0$$

where $x(t)$ is the unique solution of (2.2). Let us apply the above results to (2.1). The R.H.S. of (2.1) clearly satisfies the regularity conditions (i) and (ii). Hence there exists a unique solution to (2.1). Picard's iteration scheme then gives

$$\begin{aligned} x_0(t) &= x_0 \\ x_{n+1}(t) &= x_0 + \int_0^t Ax_n(s)ds \end{aligned} \tag{2.4}$$

From (2.4), we see that $x(t)$ is given by an infinite series of the form

$$\begin{aligned} x(t) &= x_0 + \int_0^t A ds x_0 + \int_0^t \int_0^{s_1} A^2 ds_2 ds_1 \, x_0 \\ &+ ... + \int_0^t \int_0^{s_1} ... \int_0^{s_{n-1}} A^n ds_n ds_{n-1} ... ds_1 x_0 \\ &+ ... \end{aligned}$$

The integrals can easily be evaluated, the nth term of the series being given by

$$\frac{A^n t^n}{n!} x_0$$

In analogy with the scalar exponential function, define the matrix exponential

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

where A^0 is defined to be I, the identity matrix. We can then define a function e^{At}, which we call the *transition matrix* of (2.1), by

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \tag{2.5}$$

The infinite sum is uniformly convergent, and Picard's iteration suggests that the solution of (2.1) can be written as

$$x(t) = e^{At} x_0 \tag{2.6}$$

2.2 Properties of the Transition Matrix

We now prove several useful properties of the transition matrix

(1)
$$\begin{aligned} \frac{d}{dt} e^{At} &= Ae^{At} \\ e^{At}|_{t=0} &= I \end{aligned} \tag{2.7}$$

This can be considered the defining equations of the transition matrix. To see this, note that (2.7) is a linear differential equation satisfying the regularity conditions for existence and uniqueness. Since the infinite series defining e^{At} is uniformly convergent, its derivative can be determined by differentiating term by term. It is easily verified that the results corresponds to (2.7). Uniqueness then shows that (2.7) defines e^{At}.

(2)
$$e^{A(t+s)} = e^{At} e^{As} \qquad \forall \; t, s$$

This is often referred to as the semigroup property.

Proof: Let

$$F(t) = e^{At} e^{As}$$

Then

$$\begin{aligned} F(0) &= e^{As} \\ \frac{d}{dt} F(t) &= AF(t) \end{aligned}$$

so that $F(t)$ satisfies the same differential equation and initial condition (at $t = s$) as $e^{A(t+s)}$. By uniqueness, they must be equal.

(3)
$$(e^{At})^{-1} = e^{-At}$$

Proof: We have
$$e^{-At}e^{At} = I$$

This shows that (3) is true. In particular, e^{At} is always nonsingular for all t.

Using property (1), we immediately verify that

$$x(t) = e^{At}x_0 \tag{2.8}$$

satisfies the differential equation

$$\dot{x}(t) = Ax(t) \tag{2.9}$$

and the initial condition $x(0) = x_0$. Hence it is the unique solution.

To study the structure of e^{At}, let us first analyze (2.9) and its solution (2.8) in the special case where

$$A = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & 0 \\ 0 & & 0 & \lambda_n \end{bmatrix}$$

The λ_i's are not necessarily distinct and they may be complex, provided we interpret (2.9) as a differential equation in C^n. In this case (2.9) is completely decoupled into n differential equations

$$\dot{x}_i(t) = \lambda_i x_i(t) \qquad i = 1, ..., n \tag{2.10}$$

so that

$$x_i(t) = e^{\lambda_i t}x_{0i} \tag{2.11}$$

where x_{0i} is the ith component of x_0. It follows that

$$x(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & & 0 \\ 0 & e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ 0 & & 0 & e^{\lambda_n t} \end{bmatrix} x_0 \tag{2.12}$$

and the matrix exponential e^{At} can simply be read off from the R.H.S. of (2.12):

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & & 0 \\ 0 & e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ 0 & & 0 & e^{\lambda_n t} \end{bmatrix} \tag{2.13}$$

To extend the above results to more general situations, assume that

(i) *A is diagonalizable*, i.e., there exists a nonsingular matrix P (in general complex) such that $P^{-1}AP$ is a diagonal matrix, say

$$P^{-1}AP = \Lambda = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & 0 \\ 0 & & 0 & \lambda_n \end{bmatrix} \tag{2.14}$$

Then

$$e^{At} = e^{P\Lambda P^{-1}t} \tag{2.15}$$

Claim:

$$e^{PBP^{-1}t} = Pe^{Bt}P^{-1} \qquad \text{for any } n \times n \text{ matrix } B \tag{2.16}$$

Proof:

$$e^{PBP^{-1}t} = \sum_{k=0}^{\infty} \frac{(PBP^{-1})^k t^k}{k!}$$

The kth term of this series is

$$\frac{t^k}{k!}(PBP^{-1})^k = \frac{t^k}{k!}(PBP^{-1})(PBP^{-1})...(PBP^{-1})\text{ , k terms}$$

But it is now seen that other than the P on the left most side and the P^{-1} on the right most side, all the other P and P^{-1} multiply to give the identify matrix I. Hence

$$\frac{(PBP^{-1})^k t^k}{k!} = \frac{PB^kP^{-1}t^k}{k!}$$

which is precisely the kth term of the infinite series for $Pe^{Bt}P^{-1}$.

From (2.13), (2.14) and (2.16), we find that whenever A is diagonalizable,

$$e^{At} = P\begin{bmatrix} e^{\lambda_1 t} & 0 & & 0 \\ 0 & e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ 0 & & 0 & e^{\lambda_n t} \end{bmatrix}P^{-1} \tag{2.17}$$

From linear algebra, we know that each of the following two conditions will guarantee diagonalizability:

(i) A has distinct eigenvalues

(ii) A is a symmetric matrix

In these two cases, (2.17) completely characterizes the structure of e^{At}. The diagonalizing matrix P can be taken to be one whose columns are the independent eigenvectors of A.

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Due to the form of A, we see right away that the eigenvalues are 1, 2 and -1 so that

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The eigenvector corresponding to the eigenvalue 2 is given by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, while that of -1 is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. To find the eigenvector for the eigenvalue 1, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

so that $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector. The diagonalizing matrix P is then

$$\begin{aligned} P &= \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ P^{-1} &= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \end{aligned}$$

$$e^{At} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 & 0 \\ -e^t + e^{2t} & e^{2t} & 0 \\ \frac{1}{2}(e^t - e^{-t}) & 0 & e^{-t} \end{bmatrix}$$

We can give a dynamical interpretation to the above results. Suppose A has distinct eigenvalues $\lambda_1, ..., \lambda_n$ so that it has a set of linearly independent eigenvectors $v_1, ..., v_n$. If $x_0 = v_j$, then

$$\begin{aligned} x(t) &= e^{At}x_0 = e^{At}v_j \\ &= e^{\lambda_j t}v_j \end{aligned}$$

so that the solution is just the stretching or shrinking of the eigenvector v_j. In general,

$$x_0 = \sum_{j=1}^{n} \xi_j v_j = P\xi \tag{2.18}$$

where the jth column of $P = v_j$ and $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$

$$\begin{aligned} x(t) &= e^{At}x_0 = e^{At} \sum_{j=1}^{n} \xi_j v_j \\ &= \sum_{j=1}^{n} \xi_j e^{\lambda_j t} v_j \end{aligned} \tag{2.19}$$

so that $x(t)$ is expressible as a (time-varying) linear combination of the eigenvectors of A. We now connect this representation of $x(t)$ with the representation of e^{At} in (2.17). Note that the P in (2.18) is the diagonalizing matrix for A, i.e.

$$P^{-1}AP = \Lambda$$

and

$$\begin{aligned} e^{At}x_0 &= Pe^{\Lambda t}P^{-1}x_0 = Pe^{\Lambda t}\xi \\ &= P \begin{bmatrix} \xi_1 e^{\lambda_1 t} & 0 & & 0 \\ 0 & \xi_2 e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ 0 & & 0 & \xi_n e^{\lambda_n t} \end{bmatrix} \\ &= \sum_j \xi_j e^{\lambda_j t} v_j \end{aligned}$$

the same result as in (2.19). The representation (2.19) of the solution $x(t)$ in terms of the eigenvectors of A is often called the modal decomposition of $x(t)$.

If A is not diagonalizable, then the above procedure cannot be carried out. A is not in general diagonalizable if it does not have distinct eigenvalues. Let us consider the following $l \times l$ matrix A which has the eigenvalue λ with multiplicity l:

$$A = \begin{bmatrix} \lambda & 1 & 0 & & 0 \\ & & & \ddots & \\ & & & & 0 \\ & & & & 1 \\ 0 & & 0 & & \lambda \end{bmatrix} \tag{2.20}$$

Write $A = \lambda I + N$ where

$$N = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ & & & \ddots & \\ & & & & 0 \\ & & & & 1 \\ 0 & & & & 0 \end{bmatrix} \tag{2.21}$$

Direct calculation shows that

$$\begin{aligned} [N^k]_{j,j+k} &= 1 \quad j = 1, 2, ..., l \\ &= 0 \quad \text{in all other entries} \end{aligned}$$

so that $N^l = 0$. N is thus a nilpotent matrix. But then the matrix exponential e^{Nt} is easily evaluated to be

$$e^{Nt} = \sum_{j=0}^{l-1} \frac{N^j t^j}{j!} = \begin{bmatrix} 1 & t & \cdots & \frac{t^{l-1}}{(l-1)!} \\ 0 & & & \vdots \\ & & & t \\ 0 & 0 & & 1 \end{bmatrix} \tag{2.22}$$

To compute e^{At} when A is of the form (2.20), we use the following important property of matrix exponentials:

If A and B commute, i.e. $AB = BA$, then $e^{(A+B)t} = e^{At}e^{Bt}$.

Proof: Consider $e^{At}e^{Bt}$

$$\frac{d}{dt}\left(e^{At}e^{Bt}\right) = Ae^{At}e^{Bt} + e^{At}Be^{Bt} \tag{2.23}$$

If A and B commutes, $e^{At}B = Be^{At}$ so that the R.H.S. of (2.23) becomes $(A+B)e^{At}e^{Bt}$. Furthermore, at $t = 0$, $e^{At}e^{Bt} = I$. Hence $e^{At}e^{Bt}$ satisfies the same differential equation as well as initial condition as $e^{(A+B)t}$. By uniqueness, they must be equal.

If A is of the form (2.20), then since λI commutes with N, we can write

$$\begin{aligned} e^{At} &= e^{(\lambda I)t}e^{Nt} = (e^{\lambda t}I)(e^{Nt}) \\ &= e^{\lambda t} \begin{bmatrix} 1 & t & \cdots & \frac{t^{l-1}}{(l-1)!} \\ 0 & & & \vdots \\ & & & t \\ 0 & 0 & & 1 \end{bmatrix} \end{aligned} \tag{2.24}$$

using (2.22).

We are now in a position to show the general structure of the matrix exponential. From linear algebra, we know that there always exists a matrix P such that

$$P^{-1}AP = J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix}$$

i.e., the Jordan canonical form of A with

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ & \lambda_i & 1 & & 0 \\ & & & & \vdots \\ & & & & 1 \\ 0 & & & & \lambda_i \end{bmatrix} \quad \text{, an } n_i \times n_i \text{ matrix}$$

$$\sum_{i=1}^{k} n_i = n$$

But

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & & & 0 \\ & e^{J_2 t} & & \\ & & \ddots & \\ 0 & & & e^{J_k t} \end{bmatrix} \tag{2.25}$$

and each $e^{J_i t}$ is of the form

$$e^{J_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \cdots & \frac{t^{n_i - 1}}{(n_i - 1)!} \\ & & \ddots & \\ & & \ddots & t \\ 0 & & & 1 \end{bmatrix} \tag{2.26}$$

Finally,

$$e^{At} = P e^{Jt} P^{-1} \tag{2.27}$$

Equations (2.25), (2.26) and (2.27) give the complete form for the matrix exponential e^{At}.

The above procedure in principle enables us to evaluate e^{At}, the only problem being to find the matrix P which transforms A either to diagonal or Jordan canonical form. In general, this can be a very tedious task. The above formula is most useful as a means for studying the qualitative dependence of e^{At} on t, and will be particularly important in our discussion of stability.

We remark that the above results hold regardless of whether the λ_i's are real or complex. In the latter case, we shall take the underlying vector space to be complex and the results then go through without modification.

2.3 Laplace Transform Method of Evaluating e^{At}

Another method of evaluating e^{At} analytically is to use Laplace transforms.

If we let $G(t) = e^{At}$, then the Laplace transform of $G(t)$, denoted by $\hat{G}(s)$, satisfies

$$s\hat{G}(s) = A\hat{G}(s) + I$$

or

$$\hat{G}(s) = (sI - A)^{-1} \tag{2.28}$$

Applying the inversion integral, we find

$$e^{At} = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\sigma - i\beta}^{\sigma + i\beta} (sI - A)^{-1} e^{st} ds \tag{2.29}$$

Each term in the inversion integral may be evaluated by residue calculus.

$(sI - A)^{-1}$ may be evaluated in a recursive way: Let $\det(sI - A) = s^n + p_1 s^{n-1} + ... + p_n = p(s)$. Write

$$(sI - A)^{-1} = \frac{B(s)}{p(s)} = \frac{s^{n-1}B_1 + s^{n-2}B_2 + ... + B_n}{p(s)}$$

In problem set 2, you are asked to use the identity $(sI - A)B(s) = p(s)I$ to derive recursive equations for the B_k matrices.

The above procedure is particularly easy to use if A has distinct eigenvalues $\lambda_1, ..., \lambda_n$. For then

$$(sI - A)^{-1} = \frac{B(s)}{\sum_{i=1}^{n} (s - \lambda_i)} = \sum_{i=1}^{n} \frac{R_i}{(s - \lambda_i)}$$

by partial fractions expansion with

$$R_i = \lim_{s \to \lambda_i} (s - \lambda_i) \frac{B(s)}{p(s)} = \frac{B(\lambda_i)}{\sum_{k=1, k \neq i}^{n} (\lambda_i - \lambda_k)}$$

Hence,

$$e^{At} = \sum_{i=1}^{n} R_i e^{\lambda_i t} \tag{2.30}$$

(since inverse Laplace transform of $\frac{1}{s - \lambda_i} = e^{\lambda_i t}$).

To illustrate this procedure for evaluating e^{At}, again consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} p(s) &= \det \begin{bmatrix} s-1 & 0 & 0 \\ -1 & s-2 & 0 \\ -1 & 0 & s+1 \end{bmatrix} = (s-1)(s-2)(s+1) \\ &= s^3 - 2s^2 - s + 2 \end{aligned}$$

The matrix polynomial $B(s)$ can then be determined using the recursive procedure as

$$\begin{aligned} B(s) &= \begin{bmatrix} s^2 - s - 2 & 0 & 0 \\ s+1 & s^2 - 1 & 0 \\ s-2 & 0 & s^2 - 3s + 2 \end{bmatrix} \\ (sI - A)^{-1} &= \frac{1}{(s-1)(-2)} \begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \frac{1}{(s-2)(3)} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad + \frac{1}{(s+1)(6)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 6 \end{bmatrix} \\ e^{At} &= \begin{bmatrix} e^t & 0 & 0 \\ -e^t & 0 & 0 \\ \frac{1}{2}e^t & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ e^{2t} & e^{2t} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2}e^{-t} & 0 & e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} e^t & 0 & 0 \\ -e^t + e^{2t} & e^{2t} & 0 \\ \frac{1}{2}(e^t - e^{-t}) & 0 & e^{-t} \end{bmatrix} \end{aligned}$$

the same result as before.

As a second example, let $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$. Then

$$A = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} .$$

But

$$e^{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t} = \mathbf{L}^{-1} \left\{ \begin{bmatrix} s & -\omega \\ \omega & s \end{bmatrix}^{-1} \right\}$$

where $\mathbf{L}^{-1}$ is the inverse Laplace transform operator

$$
\begin{aligned}
&= \mathbf{L}^{-1} \begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{\omega}{s^2+\omega^2} \\ \frac{-\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \\
e^{At} &= e^{(\sigma I)t} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} = \begin{bmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}
\end{aligned}
$$

Of course, the above procedures of evaluating e^{At} analytically would be virtually impossible to carry out when the dimension n is say ≥ 5. In general, we must resort to numerical techniques. Numerically stable and efficient methods of evaluating the matrix exponential can be found in the research literature.

2.4 Differential Equations with Inputs

The solution of the homogeneous equation (2.1) can be easily generalized to differential equations with inputs. Consider the equation

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
x(0) &= x_0
\end{aligned}
\tag{2.31}
$$

where u is a piecewise continuous $\mathbb{R}^m$-valued function. Define a function $z(t) = e^{-At}x(t)$. Then $z(t)$ satisfies

$$
\begin{aligned}
\dot{z}(t) &= -e^{-At}Ax(t) + e^{-At}Ax(t) + e^{-At}Bu(t) \\
&= e^{-At}Bu(t)
\end{aligned}
$$

Since the above equation does not depend on $z(t)$ on the right hand side, it can be directly integrated to give

$$
z(t) = z(0) + \int_0^t e^{-As}Bu(s)ds = x(0) + \int_0^t e^{-As}Bu(s)ds
$$

Hence

$$
\begin{aligned}
x(t) &= e^{At}z(t) \\
x(t) &= e^{At}x_0 + \textstyle\int_0^t e^{A(t-s)}Bu(s)ds
\end{aligned}
\tag{2.32}
$$

This is the variation of constants formula for solving (2.31). Uniqueness follows from uniqueness of the homogeneous equation.

We now consider linear systems with inputs and outputs.

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
x(0) &= x_0 \qquad (2.33) \\
y(t) &= Cx(t) + Du(t) \qquad (2.34)
\end{aligned}
$$

The above results give the solutions immediately

$$
y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t) \tag{2.35}
$$

In the case $x_0 = 0$ and $D = 0$, we find

$$
y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds
$$

which is of the form $y(t) = \int_0^t H(t-s)u(s)ds$, a convolution integral. $H(t) = Ce^{At}B$ is called the impulse response of the system. If we allow generalized functions, we can incorporate a nonzero D term as

$$
H(t) = Ce^{At}B + D\delta(t) \tag{2.36}
$$

where $\delta(t)$ is the Dirac δ-function. Noting that the Laplace transform of e^{At} is $(sI - A)^{-1}$, the transfer function of the linear system from u to y, which is the Laplace trasnform of the impulse response, is given by

$$
H(s) = C(sI - A)^{-1}B + D \tag{2.37}
$$

2.5 Stability

Let us consider the unforced state equation first:

$$\dot{x} = Ax, \qquad x(0) = x_0 \ .$$

The vector $x(t) \to 0$ as $t \to \infty$ for every $x_0 \Leftrightarrow$ all the eigenvalues of A lie in the open left half-plane.

Idea of proof:

$$\hat{x}(s) = (sI - A)^{-1} x_0$$

Thus

$$\begin{aligned} & x(t) \to 0 \quad \forall x_0 \\ \Leftrightarrow\ & \text{all poles of } (sI-A)^{-1} \text{ lie in } \{s : \mathrm{Re} s < 0\} \\ \Leftrightarrow\ & \text{all eigenvalues of } A \text{ lie in } \{\lambda : \mathrm{Re}\lambda < 0\} \qquad \blacksquare \end{aligned}$$

Now let us look at the full system model:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \ . \end{aligned}$$

The transfer matrix from u to y is

$$H(s) = C(sI - A)^{-1} B + D \ .$$

We can write this as

$$H(s) = \frac{1}{\det(sI - A)} C \cdot \mathrm{adj}(sI - A) \cdot B + D \ .$$

Notice that the elements of the matrix $\mathrm{adj}(sI - A)$ are all polynomials in s; consequently, they have no poles. Notice also that $\det(sI - A)$ is the characteristic polynomial of A. We can therefore conclude from the preceding equation that

$$\{\text{eigenvalues of } A\} \supset \{\text{poles of } H(s)\} \ .$$

Hence, if all the eigenvalues of A are in the open left half-plane, then $H(s)$ is a stable transfer matrix. The converse is not necessarily true.

Example:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & -2 & 0 \\ 2 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$C = [2 \ \ 2 \ \ 1], \qquad D = [0 \ \ 0]$$

$$\{\text{eigenvalues of } A\} = \{0, -1, -2\}$$

$$\begin{aligned} H(s) &= \frac{1}{\det(sI - A)} C \cdot \mathrm{adj}(sI - A) \cdot B \\ &= \frac{1}{s^2 + 3s^2 + 2s} \cdot [2 \ \ 2 \ \ 1] \cdot \begin{bmatrix} s^2+3s+2 & s+2 & s+2 \\ -2s-2 & s^2+s-2 & -2 \\ 2s+2 & s+2 & s^2+2s+2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \frac{1}{s^2 + 3s^2 + 2s} [2s^2+4s+2 \ \ 2s^2+5s+2 \ \ s^2+4s+2] \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \frac{1}{s^3 + 3s^2 + 2s} [s^2+2s \ \ s^2+s] \\ &= \begin{bmatrix} \frac{s+2}{s^2+3s+2} & \frac{s+1}{s^2+3s+2} \end{bmatrix} \end{aligned}$$

Thus {poles of $H(s)$} = {-1,-2}. Hence the eigenvalue of A at $\lambda = 0$ does not appear as a pole of $H(s)$. Actually, this shouldn't be surprising – a more obvious example is

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C &= [1 \;\; 0], & D &= 0 \\ H(s) &= \frac{1}{s-1} & & . \end{aligned}$$