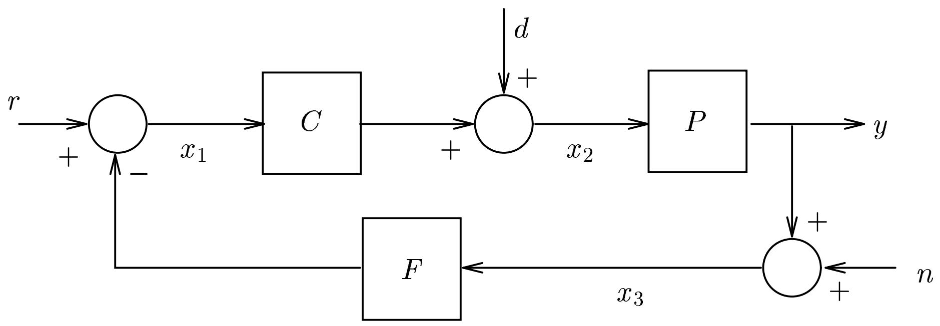
Chapter 3

Feedback Systems

3.1 Stability

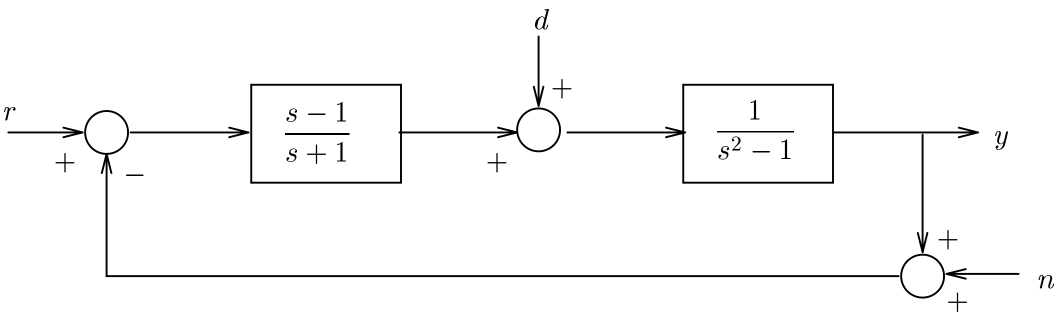
Consider the feedback system



Definition: This feedback system is <u>BIBO stable</u> (or just <u>stable</u>, for short) if x_1, x_2, x_3 are bounded whenever r, d, n are bounded.

Note that all signals in this system are bounded if x_1, x_2, x_3, r, d, n are bounded; for example, $y = x_3 - n$, so it's bounded.

Example:



TF from r to y is stable,

TF from d to y is not stable,

Hence feedback system is <u>not</u> stable.

We want to develop a test for stability. Note that, by definition, the feedback system is stable iff all 9 transfer functions from r, d, n to x_1, x_2, x_3 are proper and stable. Derivation of TFs:

$$\left\{\begin{array}{rcl} \hat{x}_1 & = & \hat{r} - F\hat{x}_3 \\ \hat{x}_2 & = & \hat{d} + C\hat{x}_1 \\ \hat{x}_3 & = & \hat{n} + P\hat{x}_2 \end{array}\right.$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & F \\ -C & 1 & 0 \\ 0 & -P & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} \hat{r} \\ \hat{d} \\ \hat{n} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & F \\ -C & 1 & 0 \\ 0 & -P & 1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{r} \\ \hat{d} \\ \hat{n} \end{bmatrix}$$

$$= \tfrac{1}{1+PCF} \begin{bmatrix} 1 & -PF & -F \\ C & 1 & -FC \\ PC & P & 1 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{d} \\ \hat{n} \end{bmatrix}$$

Theorem 1: *Assume P, C, F are proper and PCF is strictly proper. Then the 9 TFs are all proper.*

(Proof is easy.)

Definition: Write $P = \dfrac{N_P}{D_P}$, $C = \dfrac{N_C}{D_C}$, $F = \dfrac{N_F}{D_F}$ as ratios of coprime polynomials (i.e. N_P, D_P have no common factors, etc.). The characteristic polynomial (ch.p.) of the system is

$$D_P D_C D_F + N_P N_C N_F.$$

Theorem 2: *Assume P, C, F are proper and PCF is strictly proper. Then the feedback system is stable if and only if the ch.p. has no roots in Re $s \geq 0$.*

Example: (continued)

$$\begin{array}{rcl} \text{ch.p.} & = & (s+1)(s^2-1) + s - 1 \\ & = & (s-1)(s^2+2s+2); \text{ root at } s = 1 \end{array}$$

Therefore feedback system is not stable (as we saw before).

Proof of theorem

For simplicity assume $F = 1$ (the general case is similar but a bit messier). Then

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \frac{1}{1+PC} \begin{bmatrix} 1 & -P & -1 \\ C & 1 & -C \\ PC & P & 1 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{d} \\ \hat{n} \end{bmatrix}$$

$$= \frac{1}{D_P D_C + N_P N_C} \begin{bmatrix} D_P D_C & -N_P D_C & -D_P D_C \\ D_P N_C & D_P D_C & -D_P N_C \\ N_P N_C & N_P D_C & D_P D_C \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{d} \\ \hat{n} \end{bmatrix} \tag{3.1}$$

Note that ch.p. $= D_P D_C + N_P N_C$. Thus the feedback system is stable if ch.p. has no roots in Re $s \geq 0$.

Conversely, assume the feedback system is stable. Suppose, for a proof by contradiction, that ch.p. has a root s_0 in Re $s \geq 0$. Then each of the 9 numerator polynomials in (3.1) must also have a root at s_0 (because all 9 TFs must be stable). Thus

$$\begin{aligned} (D_P D_C + N_P N_C)(s_0) &= 0 \qquad \text{(ch.p.)} \\ (D_P D_C)(s_0) &= 0 \qquad \text{((1,1)-element in (1))} \qquad (3.2) \\ (N_P N_C)(s_0) &= 0 \qquad \text{((3,1)-element in (1)).} \qquad (3.3) \end{aligned}$$

Then (3.2) implies that either $D_P(s_0) = 0$ or $D_C(s_0) = 0$ (or both).

Suppose $D_P(s_0) = 0$. Then $N_P(s_0) \neq 0$ (by coprimeness), so (3.3) implies that $N_C(s_0) = 0$. But also

$$(N_P D_C)(s_0) = 0 \qquad \text{((1,2)-element in (3.1))}$$

so $D_C(s_0) = 0$. This implies that N_C, D_C are not coprime, which is a contradiction.

Similarly, the condition $D_C(s_0) = 0$ leads to a contradiction. ■

The final result is another test for feedback stability.

Theorem 3: *Assume P, C, F are proper and PCF is strictly proper. Then the feedback system is stable if and only if*

$$1 + PCF \neq 0$$

in Re $s \geq 0$, and the product PCF has no pole-zero cancellations in Re $s \geq 0$.

Example: (continued)

$$P(s) = \frac{1}{s^2 - 1}, \qquad C(s) = \frac{s-1}{s+1}, \qquad F(s) = 1$$

Thus PCF has a pole-zero cancellation at $s = 1$; hence the feedback system is not stable.

(Proof of Theorem is left as an exercise.)

Summary

Theorem 1 gives a sufficient condition for all closed-loop TFs to be proper (the open-loop TFs P, C, F should be proper and at least one should be strictly proper). Theorem 2 gives a test for stability (the ch.p. must have no roots in Re $s \geq 0$). Finally, Theorem 3 gives an equivalent test for stability (the TF $1 + PCF$ must have no zeros in Re $s \geq 0$ and there must be no pole-zero cancellations in Re $s \geq 0$ in forming the product).

There is an algebraic test, called the *Routh-Hurwitz criterion*, for a polynomial to have all its roots in Re $s < 0$. To check feedback stability we apply this criterion to the ch.p.

Let the polynomial be

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0, \qquad a_i \text{ real numbers.}$$

We apply an algorithm to generate an array with $n+1$ rows; the array looks like

$$\begin{array}{c|cccc} n & a_n & a_{n-2} & a_{n-4} & \cdots \\ n-1 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ n-2 & b_{n-2} & b_{n-4} & \cdots & \\ n-3 & c_{n-3} & c_{n-5} & \cdots & \\ \vdots & \vdots & & & \\ 0 & & & & \end{array}$$

Note that the first two rows are written down immediately from the polynomial. The 3rd row is defined as follows:

$$\begin{aligned} b_{n-2} &:= -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \\ b_{n-4} &:= -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \\ \text{etc.} \end{aligned}$$

The fourth row is generated in the same way, but using rows 2 and 3.

Theorem 4: *The roots of the polynomial all lie in Re $s < 0 \Leftrightarrow$ all elements in the first column of the array are nonzero and have the same sign. A necessary condition is that $a_n, \ldots, a_0$ all be nonzero and have the same sign.*

Example 1: $s^2 + a_1 s + a_0$

$$\begin{array}{c|cc} 2 & 1 & a_0 \\ 1 & a_1 & 0 \\ 0 & a_0 & 0 \end{array}$$

all roots in Re $s < 0 \Leftrightarrow a_1 > 0$ & $a_0 > 0$

Example 2: $s^3 + a_2 s^2 + a_1 s + a_0$

$$\begin{array}{c|cc} 3 & 1 & a_1 \\ 2 & a_2 & a_0 \\ 1 & b_1 & 0 \\ 0 & a_0 & 0 \end{array} \qquad b_1 = \frac{a_2 a_1 - a_0}{a_2}$$

all roots in Re $s < 0 \Leftrightarrow a_2 > 0$, $a_1 > 0$, $a_0 > 0$, $a_2 a_1 > a_0$.

Example 3: $s^4 + s^3 + s^2 + s + K$

$$\begin{array}{c|ccc} 4 & 1 & 1 & K \\ 3 & 1 & 1 & \\ 2 & 0 & & \\ 1 & & & \\ 0 & & & \end{array}$$

for every K there is some root in Re $s \geq 0$.

Additional Information:

A row vanishes identically. Preceding row gives a factor of $p(s)$.

Example:

$$p(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$$

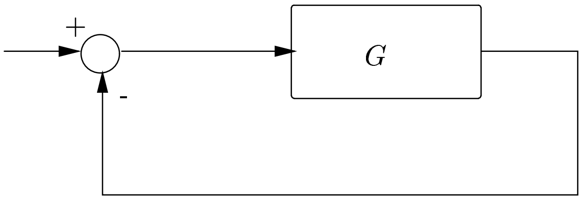
$$\begin{array}{cccc} s^5: & 1 & 4 & 3 \\ s^4: & 1 & 24 & 63 \\ s^3: & -20 & -60 & 0 \\ s^2: & 21 & 63 & 0 \\ s^1: & 0 & 0 & 0 \end{array}$$

$$21s^2 + 63 = 0 \text{ is a factor } s = \pm i\sqrt{3}$$

$$
\begin{array}{rrrrrrr}
 & & & s^3 & +s^2 & +s & +21 \\
s^2+3 & s^5 & +s^4 & +4s^3 & +24s^2 & +3s & +63 \\
 & s^5 & & +3s^3 & & & \\
 & & s^4 & +s^3 & +24s^2 & & \\
 & & s^4 & & +3s^2 & & \\
 & & & s^3 & +21s^2 & & \\
 & & & s^3 & & +3s & \\
 & & & & 21s^2 & + & 63
\end{array}
$$

3.2 Steady state performance

Define the trading error $e(t) = r(t) - y(t)$. Then the steady state error $= \lim_{t\to\infty} e(t) = e_\infty$, if it exists. Consider a unity feedback system



In terms of unit step, we have

$$e(s) = \frac{1}{1+G(s)} r(s)$$

To get e_∞, use the final value theorem of Laplace transform:

Let $f \epsilon PS[0,\infty)$. Suppose f_∞ exists and is finite. Then $f_\infty = \lim_{\substack{s\to 0 \\ s \text{ real}}} s\hat{f}(s)$

(Alternatively, suppose $\hat{f}(s)$, strictly proper rational, has no poles in Re $s \geq 0$ except possibly for simple pole at $s = 0$. Then $\lim_{t\to\infty} f(t)$ exists and equals $\lim_{s\to 0} s\hat{f}(s)$.)

Warning: Laplace transform of $\cos\omega t = \frac{s}{s^2+\omega^2}$.

Apply to our problem to give (assuming closed-loop system is stable so that y_∞ exists)

$$e_\infty = \lim_{s\to 0} s \frac{1}{1+G(s)} \frac{1}{s} = \frac{1}{1+G(0)},$$

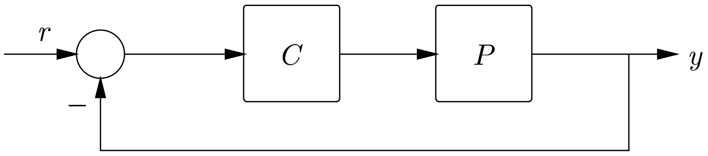
$G(0)$ often referred to as the dc gain of the system.

If $G(s) = \frac{q(s)}{sp(s)}$ can interpret $G(0) = \infty$ or

$$\lim_{s\to 0} \frac{1}{1+G(s)} = \frac{sp(s)}{sp(s)+q(s)} = 0$$

If $G(0) \uparrow, e_\infty \downarrow$ as long as system remains stable. If $e_\infty = 0$, we say asymptotic tracking is achieved. We can combine the above ideas to give a controller structure that will provide asymptotic tracking of steps.

Example:



Let $P(s) = \frac{1}{s^2+4s-5}$. We would like to design a controller $C(s)$ to achieve asymptotic tracking of step reference inputs. The two ingredients we need are

(i) An integrator at 0 in $C(s)$.

(ii) Stability of the closed loop system.

Consider the controller $C(s) = \frac{K(s+1)}{s}$. K is to be chosen to give closed loop stability. The ch. p. is given by $s^3 + 4s^2 + (K-5)s + K$. Applying Routh-Hurwitz, we find that the necessary and sufficient condition for stability is $K > \frac{20}{3}$. Thus for

$$C(s) = \frac{7(s+1)}{s}$$

asymptotic tracking of steps is achieved.