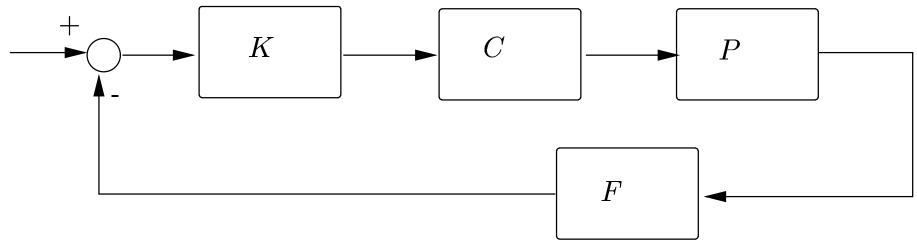
Chapter 4

Root Locus and Nyquist Plots

4.1 Root Locus

Consider the single loop feedback system



The dynamics of the closed loop system is intimately related to locations of closed loop poles.

$$\text{Closed loop transfer function} = \frac{KCP}{1+KCPF}, \text{ with } K \text{ an adjustable gain parameter}$$

Let

$$L(s) = C(s)P(s)F(s) = \frac{N(s)}{D(s)} \Rightarrow \text{ closed loop poles at } D(s) + KN(s)$$

Closed loop poles are roots of the characteristic polynomial $D(s) + KN(s)$. May assume no common roots in D, N, otherwise cancel common factors. For the rest of this chapter, take $F = 1$.

Since K is a freely adjustable parameter, and we often get better performance by increasing K, it is of interest to study variation of the closed loop poles as a function of K. Without loss of generality, may assume $N(s)$, $D(s)$ monic degree of $D = n$, degree of $N = m$, with $n \geq m$. The (positive) root locus is the locus of the roots of $D(s) + KN(s)$ as a function of K, for $0 \leq K < \infty$. The root locus equation

$$D(s) + KN(s) = 0$$

is equivalent to

$$L(s) = \frac{N(s)}{D(s)} = -\frac{1}{K}$$

This implies that a point s is on the root locus if and only if

$$\arg L(s) = (2k+1)\pi$$

i.e., $\sum$ angle at zeros - $\sum$ angle at poles $= (2k+1)\pi$ (including multiplicity). This is referred to as the angle condition. The corresponding gain K is given by $K = \frac{1}{|L(s)|}$.

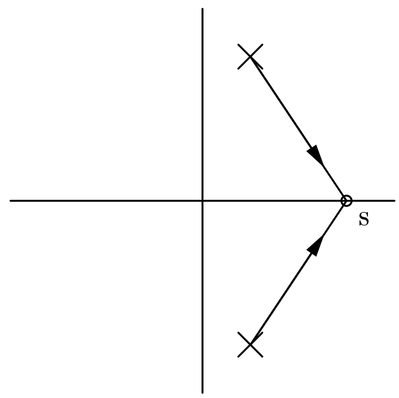
Properties:

1. There are n branches, each a continuous path.

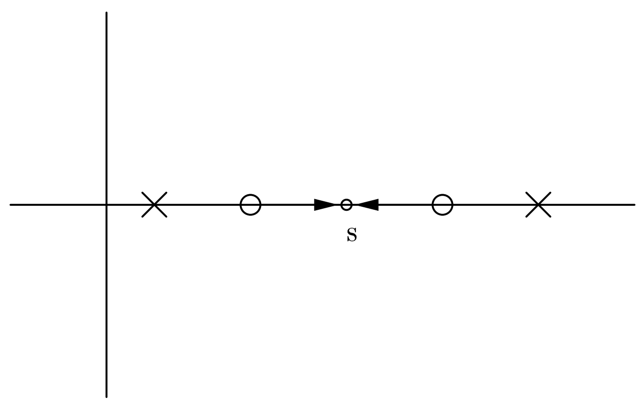
2. Begin at poles. m branches $\xrightarrow[K\to\infty]{}$ zeros, $n-m$ branches $\to \infty$. This is because each closed loop pole s_0 satisfies $\frac{1}{K}D(s_0) + N(s_0) = 0$. If $|s_0|$ bounded as $K \to \infty \Rightarrow s_0$ tends to a zero. This accounts for m branches. For the remaining $n-m$ branches, $|s_0|$ is unbounded.

3. Symmetry with respect to real axis.

4. A point s on the real axis that is not a pole or zero is on the root locus if and only if it lies to the left of odd number of poles and zeros.



contrib = 0



From the above figures, we see that the only poles and zeros which make a net nonzero contribution to the angle of $L(s)$ are those on the real axis to the right of the point s. Let a = # of zeros to the right of s, and b = # of poles to the right of s. By the angle condition, we must have $(a-b)\pi$ = odd multiple of π. Since $a-b$ is odd $\Leftrightarrow a+b$ is odd, this proves property (4).

5. For $n = m$, all branches are bounded. For $n > m$, $n-m$ branches tend to ∞ as $K \to \infty$ along asymptotes radiating from centroid σ_0 on the real axis

$$\sigma_0 = \frac{\sum \text{ poles} - \sum \text{ zeros}}{n-m}$$

with asymptotic angles $\frac{(2k+1)\pi}{n-m}$. $n-m$ is called the relative degree of $L(s)$.

Heuristic explanation:

$$\frac{s^n + a_{n-1}s^{n-1} ... + a_n}{s^m + b_{m-1}s^{m-1} + ... + b_m} + K = 0$$

$$\frac{s^n}{s^m}\left(\frac{1 + a_{n-1}s^{-1} + ...}{1 + b_{m-1}s^{-1}...}\right) + K$$

$$s^{n-m}\left(1+\frac{a_{n-1}-b_{m-1}}{s}+...\right)+K=0$$

$$s^{n-m}\left(1+\frac{a_{n-1}-b_{m-1}}{s}\right)\simeq -K$$

$$s\left(1+\frac{a_{n-1}-b_{m-1}}{s}\right)^{\frac{1}{n-m}}\simeq K^{\frac{1}{n-m}}e^{i(2k+1)\frac{\pi}{n-m}}$$

$$s\left(1+\frac{a_{n-1}-b_{m-1}}{s(n-m)}\right)\simeq K^{\frac{1}{n-m}}e^{i(2k+1)\frac{\pi}{n-m}}$$

Let $s=\sigma+i\omega$. We have

$$\sigma+i\omega-\sigma_0\simeq K^{\frac{1}{n-m}}\left[\cos\left((2k+1)\frac{\pi}{n-m}\right)+n\sin\left((2k+1)\frac{\pi}{n-m}\right)\right]$$

where

$$\sigma_0=-\frac{a_{n-1}-b_{m-1}}{(n-m)}=\frac{\sum\text{ poles}-\sum\text{ zeros}}{n-m}$$

Letting $\alpha=\frac{\pi}{n-m}$, we obtain

$$\sigma-\sigma_0=K^{\frac{1}{n-m}}\cos(2k+1)\alpha$$

$$\omega=K^{\frac{1}{n-m}}\sin(2k+1)\alpha$$

Eliminating $K^{\frac{1}{n-m}}$, we get

$$\omega=(\sigma-\sigma_0)\tan(2k+1)\alpha$$

6. If branches of root locus meet at s, it is necessary that

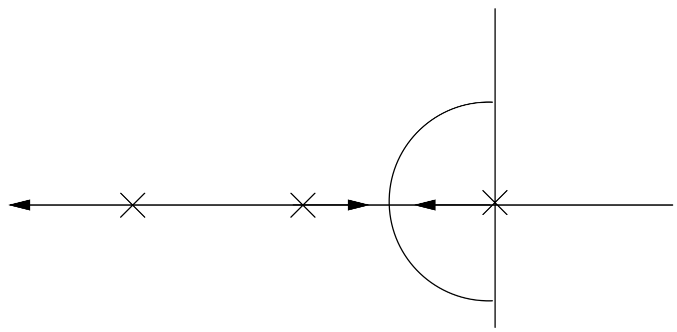
$$N(s)D'(s)-D(s)N(s)'=0 \tag{4.1}$$

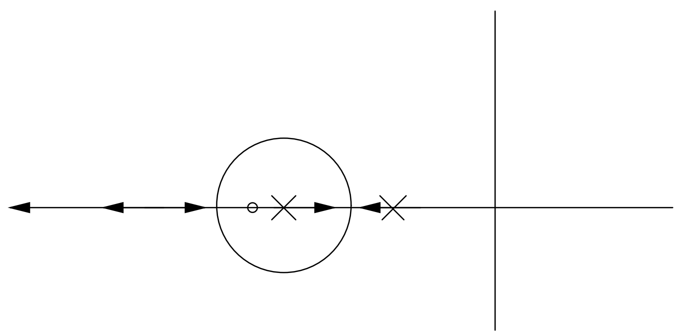
This is because at s, we must have

$$\begin{aligned} D+KN &= 0 \\ D'+KN' &= 0 \end{aligned}$$

Eliminating K gives the desired equation. Note that not all solutions of Eq. (4.1) will correspond to meeting points of root locus branches. Those meeting points on the real axis at which branches leave (enter) the real axis are called breakaway (breakin) points.

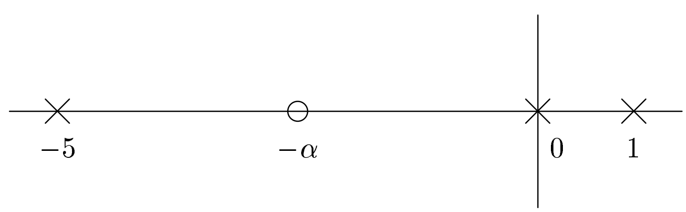
Simple examples of root locus sketches

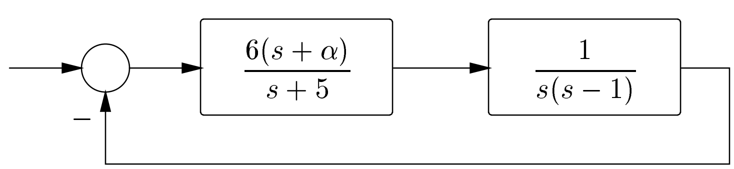




Root locus examines variation of roots of $D(s) + KN(s)$. Any polynomial dependent on a single parameter can be considered in the same way.

Example





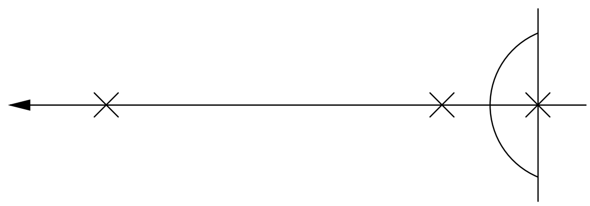
$$s^3 + 4s^2 - 5s + 6(s + \alpha) = s^3 + 4s^2 + s + 6\alpha$$

$$= s(s^2 + 4s + 1) + 6\alpha$$

$s + 4s + 1$ has roots

$$\frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$$

Considering 6α as $\bar{K}$, the root locus for $\bar{D}(s) = s(s^2 + 4s + 1)$ and $\bar{N}(s) = 1$ gives locus of the closed loop poles as a function of $\bar{K}$, hence of α.



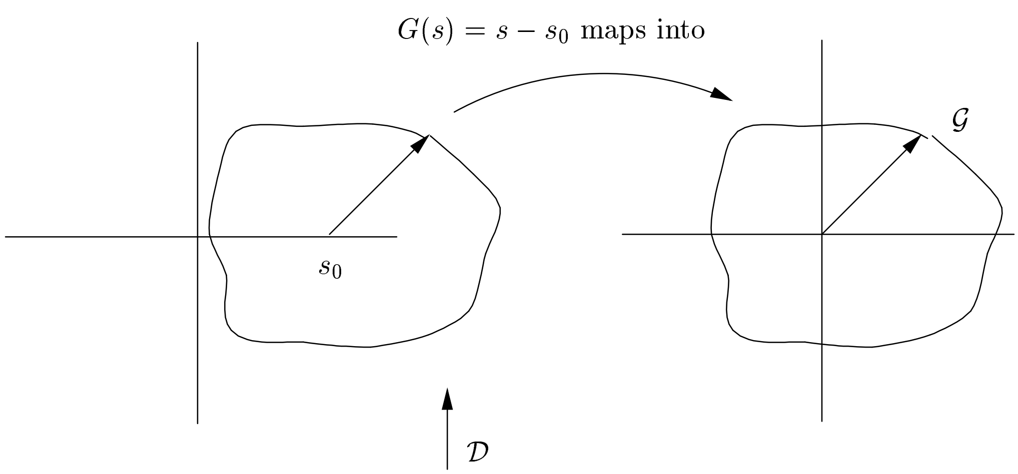
4.2 Nyquist stability criterion

(i) Root locus need complete knowledge of open loop poles and zeros (knowing number of poles in RHP not enough)

(ii) Does not really determine stability of feedback system (in root locus, stability determined by Routh Hurwitz)

Graphical method using frequency response.

Based on the following observations
Consider a closed path $\mathcal{D}$ in the s-plane with no self-intersections, in the clockwise direction. Suppose $\mathcal{D}$ enclosed a point s_0. Then as s traverses $\mathcal{D}$, net change in $\angle(s - s_0)$ is -2π (i.e. 1 CW rotation)



s_0 becomes new origin in $\mathcal{G}$

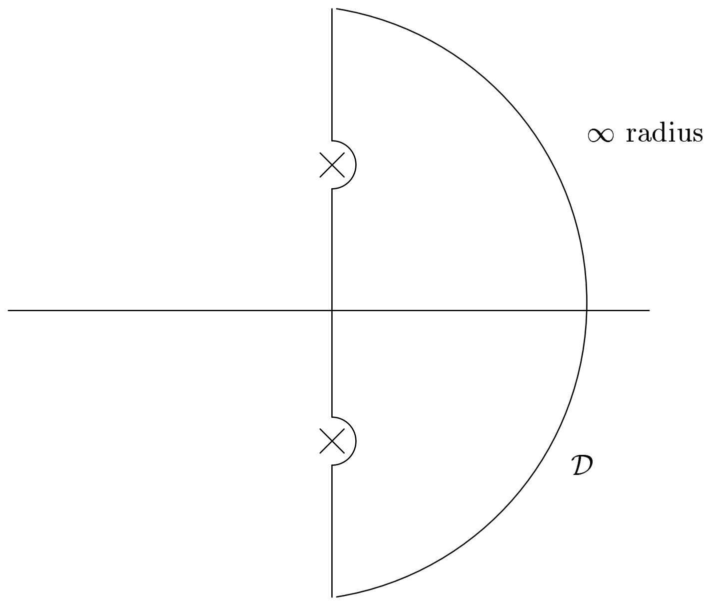
Consider a rational function $G(s) = K\dfrac{\prod(s - z_i)}{\prod(s - p_i)}$, with $K > 0$.

Principle of the argument for rational functions
Suppose G has no poles or zeros on $\mathcal{D}$, and $\mathcal{D}$ encloses n poles and m zeros of G. Then the image of $\mathcal{D}$ under G, $\mathcal{G}$, encircles the origin exactly $n - m$ times c.c.w.

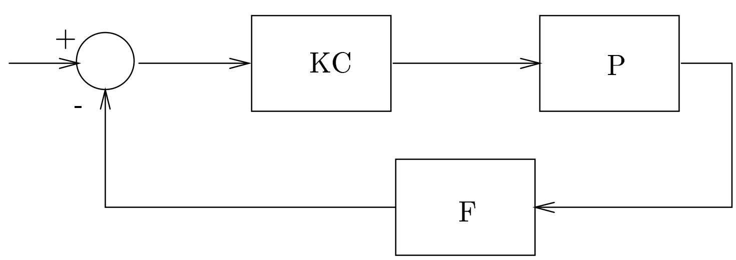
Proof: $\angle G(s) = \angle K + \sum_i \angle(s - z_i) - \sum_i \angle(s - p_i)$. By above observation, if z_i is enclosed by $\mathcal{D}$, then net change in $\angle(s - z_i) = -2\pi$.

Net change in $\angle G(s) = m(-2\pi) - n(-2\pi) = (n - m)2\pi \Rightarrow n - m$ c.c.w. encirclements.

Nyquist contour or path



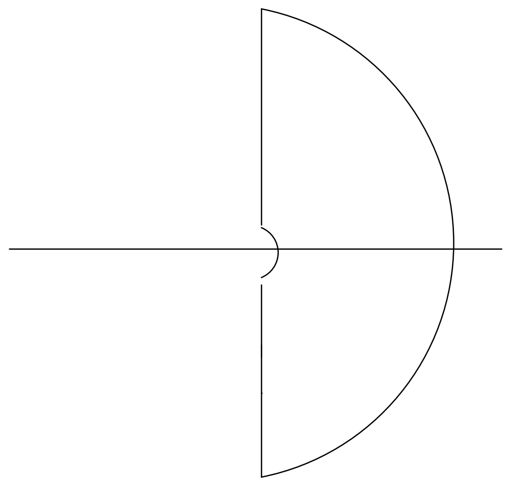
$\mathcal{G}$ is called the Nyquist plot. Assume no poles and zeros on $\mathcal{D}$ (can always be done)
$\Rightarrow$ Nyquist plot encircles 0 $n - m$ times c.c.w.
Now consider the feedback system



Assume PCF has no pole-zero cancellations.
Then the feedback system is stable $\Leftrightarrow G(s) = 1 + KP(s)C(s)F(s) \neq 0$ in $Res \geq 0$.
Suppose there are n_0 poles of PCF in $Res \geq 0$. Since
Poles of G = poles of PCF
$\Rightarrow G$ has n_0 poles in $Res \geq 0$
Then there are no zeros enclosed by $\mathcal{D}$ in $Res \geq 0$
$\Leftrightarrow$ Nyquist plot of G does not pass through 0 and encircles it n_0 times c.c.w.
$\Leftrightarrow$ PCF does not pass through $-\dfrac{1}{K}$ and encircles it n_0 times c.c.w.

Example 1: 2nd order system
(i)

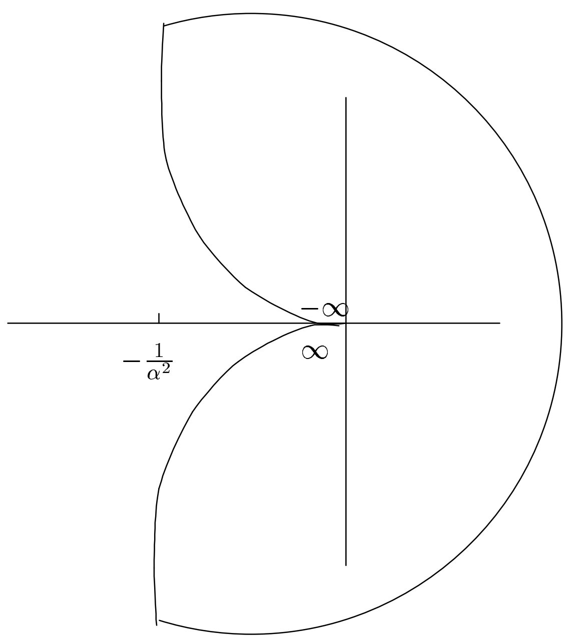
$$\frac{1}{s(s+\alpha)} = \frac{1}{\alpha s(1+\frac{s}{\alpha})} \quad \alpha > 0$$



$$G(i\omega) = \frac{1}{i\omega(i\omega+\alpha)} = \frac{1}{\omega(-\omega+i\alpha)}$$

$$= \frac{(-\omega-i\alpha)}{\omega(\omega^2+\alpha^2)} = \frac{-1}{\omega^2+\alpha^2} - i\frac{\alpha}{\omega(\omega^2+\alpha^2)}$$

$$\angle G(i\omega) = -90^o - \tan^{-1}\frac{\omega}{\alpha}$$



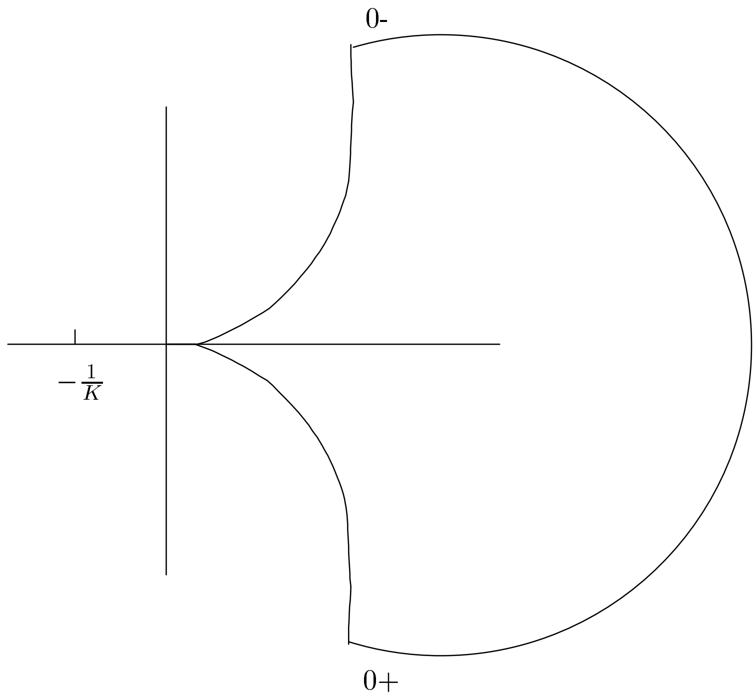
Always stable
for all $K > 0$

(ii)

$$P(s) = \frac{1}{s(1-sT)}$$

$$P(i\omega) = \frac{1}{i\omega(1-i\omega T)} = \frac{T}{1+\omega^2T^2} - i\frac{1}{\omega(1+\omega^2T^2)}$$

$$\angle P(i\omega) = -90^o + \tan^{-1}\omega T$$

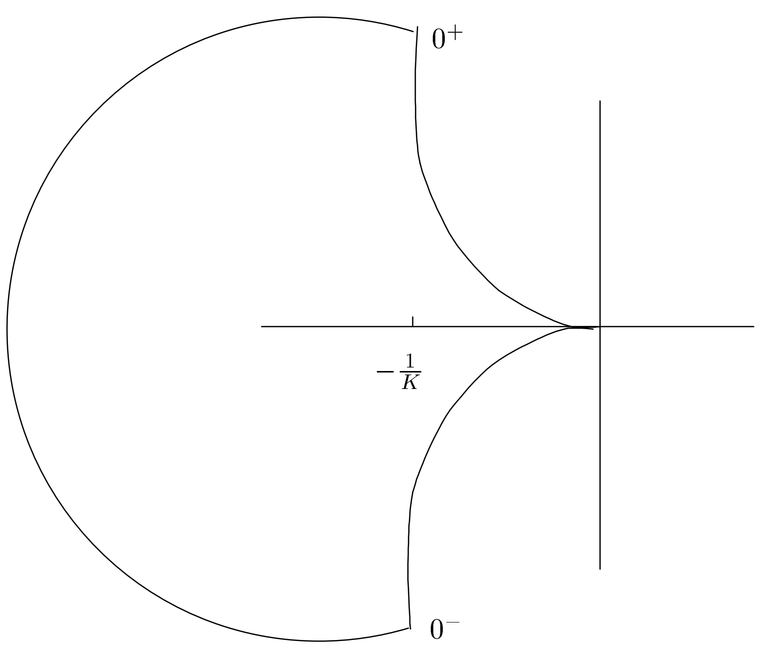


$$P(s) = -\frac{1}{Ts(s-\frac{1}{T})}$$

For $K > 0$ give the negative root locus for the system. Hence number of closed loop poles in RHP = 1.
(iii)

$$P(s) = \frac{1}{s(-1+Ts)}$$

$$P(i\omega) = \frac{1}{i\omega(-1+i\omega T)} = \frac{-i(-1-i\omega T)}{\omega(1+\omega^2T^2)} = -\frac{T}{1+\omega^2T^2} + i\frac{1}{\omega(1+\omega^2T^2)}$$

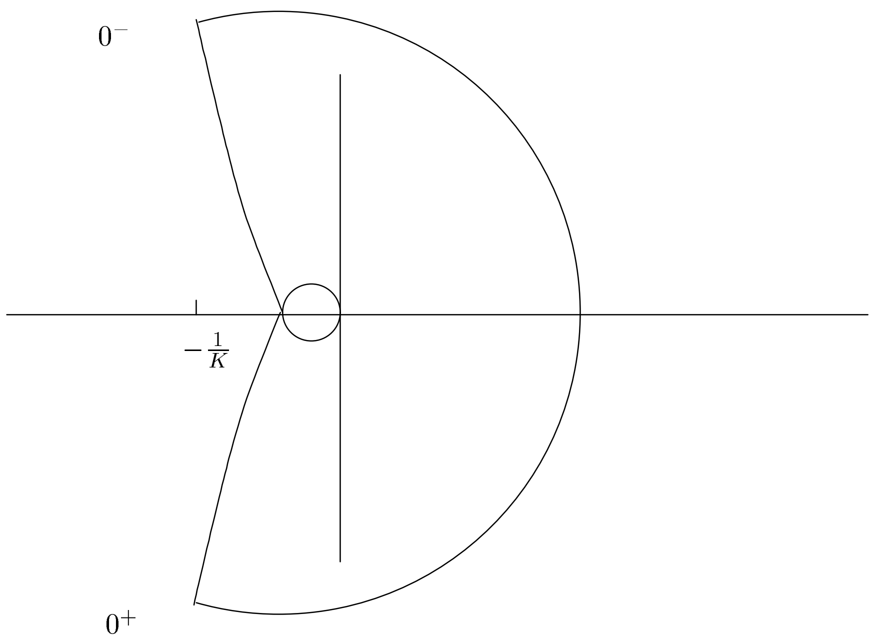


Example 2:

$$G(s) = \frac{K}{s(1+\tau_1 s)(1+\tau_2 s)} \quad \tau_1 > 0,\ \tau_2 > 0$$

$$
\begin{aligned}
G(i\omega) &= \frac{K}{i\omega(1+i\tau_1\omega)(1+i\tau_2\omega)} \\
&= \frac{-iK(1-i\tau_1\omega)(1-i\tau_2\omega)}{\omega(1+\tau_1^2\omega^2)(1+\tau_2^2\omega^2)} \\
&= \frac{-iK(1-\tau_1\tau_2\omega^2-i(\tau_1+\tau_2)\omega)}{\omega(1+\tau_1^2\omega^2)(1+\tau_2^2\omega^2)} \\
&= \frac{-K(\tau_1+\tau_2)}{(1+\tau_1^2\omega^2)(1+\tau_2^2\omega^2)} - i\frac{K(1-\tau_1\tau_2\omega^2)}{\omega(1+\tau_1^2\omega^2)(1+\tau_2^2\omega^2)}
\end{aligned}
$$

$$
G(i\omega) = -\frac{\tau_1+\tau_2}{(1+\tau_1^2\omega^2)(1+\tau_2^2\omega^2)} - i\frac{(1-\tau_1\tau_2\omega^2)}{\omega(1+\tau_1^2\omega^2)(1+\tau_2^2\omega^2)}
$$



$$
\omega^2 = \frac{1}{\tau_1\tau_2}
$$

$$
-\frac{1}{K} < -\frac{\tau_1+\tau_2}{(1+\frac{\tau_1^2}{\tau_1\tau_2})(1+\frac{\tau_2^2}{\tau_1\tau_2})} = -\frac{\tau_1+\tau_2}{(1+\frac{\tau_1}{\tau_2})(1+\frac{\tau_2}{\tau_1})}
$$

$$
\Rightarrow \frac{1}{K} > \frac{\tau_1+\tau_2}{(1+\frac{\tau_1}{\tau_2})(1+\frac{\tau_2}{\tau_1})}
$$

$$
\frac{(1+\frac{\tau_1}{\tau_2})(1+\frac{\tau_2}{\tau_1})}{\tau_1+\tau_2} > K > 0 \quad \text{for stability}
$$

Example 3:

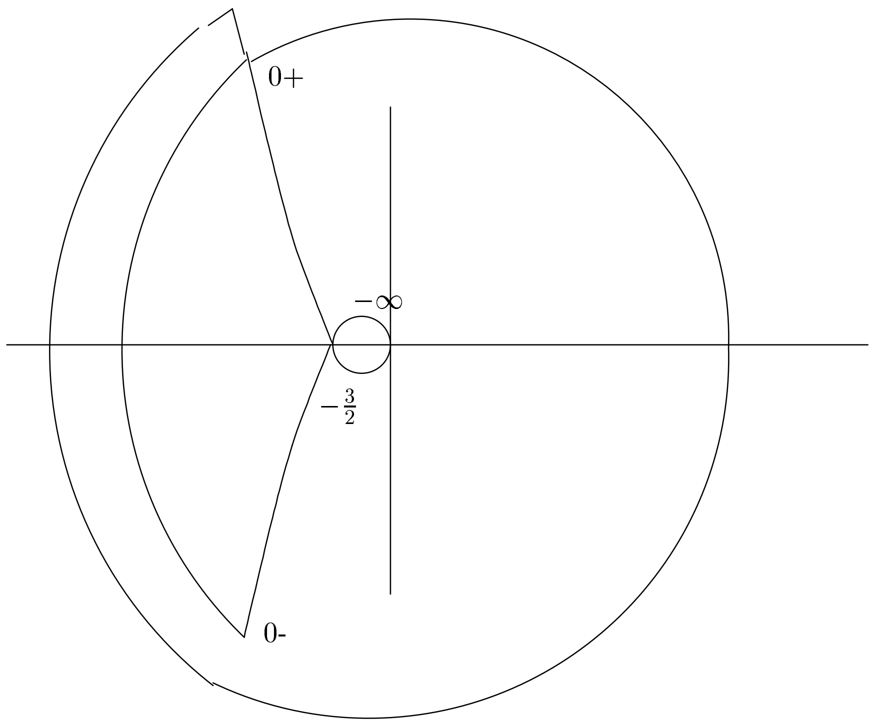
$$
G(s) = \frac{(s+1)(s+2)}{s^3}
$$

$$
G(i\omega) = -\frac{3}{\omega^2} + i\frac{2-\omega^2}{\omega^3}
$$

$$
\angle G(i\omega) = 90^o + \tan^{-1}\omega + \tan^{-1}\frac{\omega}{2}
$$

$$
ImG = 0 \qquad \omega = \pm\sqrt{2}
$$

$$
\text{At } \omega^2 = 2, \qquad ReG(i\omega) = -\frac{3}{2}
$$



No encirclement if $0 > -\frac{1}{K} > -\frac{3}{2}$, $\frac{1}{K} < \frac{3}{2}$ or $K > \frac{2}{3}$ for stability

4.3 Design Specifications

Design specifications for simple control systems are often developed based on the transient response of a standard second order system driven by a step input. For the standard second order system

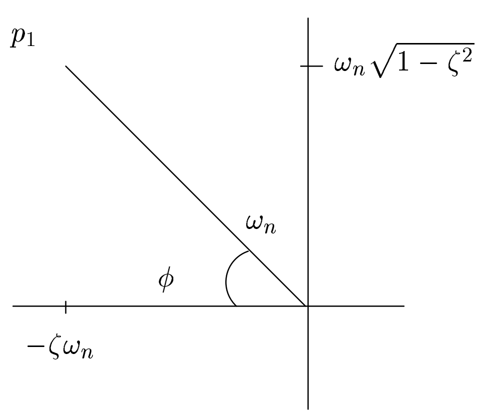
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

the poles are given by

$$\begin{aligned} p_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \quad \zeta \geq 1 \\ &= -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} \quad 0 \leq \zeta < 1 \end{aligned}$$

This can be shown pictorially as

$\zeta < 1$



$\phi = \cos^{-1}\zeta$

The step response is given by:
for $\zeta < 1$,

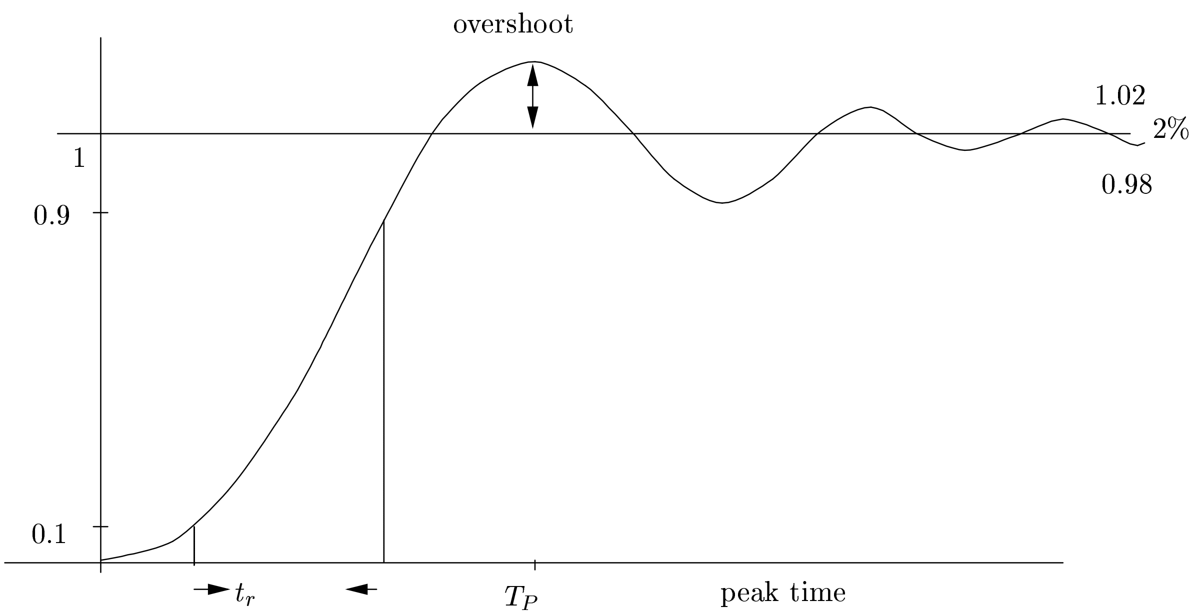
$$\begin{aligned} y(t) &= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t + \phi) \\ &= 1 - \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2} t e^{-\zeta\omega_n t} - \cos \omega_n \sqrt{1-\zeta^2} t e^{-\zeta\omega_n t} \end{aligned}$$

for $\zeta = 1$,

$$y(t) = 1 - (1 + \omega_n t) e^{-\omega_n t}$$

for $\zeta > 1$,

$$\begin{aligned} y(t) = \ & 1 - \tfrac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}} e^{-\omega_n(\zeta-\sqrt{\zeta^2-1})t} \\ & + \tfrac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}} e^{-\omega_n(\zeta+\sqrt{\zeta^2-1})t} \end{aligned}$$



$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Graph of T_p vs. ζ

$$\text{overshoot} = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Graph of overshoot vs. ζ

If we let h_0 to denote the overshoot, we can determine the damping ratio using

$$\zeta^2 = \frac{(log h_0)^2}{\pi^2 + (log h_0)^2}$$

Typically, want small overshoot, small rise time (i.e. fast response) small settling time (not much ringing) t_r estimated using t_R, time to go from 0 to 1.

$$1 = y(t_R) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t + \phi)$$

$$\Rightarrow \sin(\omega_n\sqrt{1-\zeta^2}t_R)+\phi)=0$$

$$\begin{aligned} 0 &= \sin\omega_n\sqrt{1-\zeta^2}t_R\cos\phi+\cos(\omega_n\sqrt{1-\zeta^2}t_R)\sin\phi \\ &= \zeta\sin\omega_n\sqrt{1-\zeta^2}t_R+\cos\omega_n\sqrt{1-\zeta^2}t_R\sqrt{1-\zeta^2} \end{aligned}$$

$$\Rightarrow \tan(\omega_n\sqrt{1-\zeta^2}t_R)=-\frac{\sqrt{1-\zeta^2}}{\zeta}=-\tan(\pi-\omega_n\sqrt{1-\zeta^2}t_R)$$

$$\Rightarrow \omega_n\sqrt{1-\zeta^2}t_R=-\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$t_R=-\frac{1}{\omega_n\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}=\frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}}=\frac{\pi-\cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}}$$

Recall $\tan(180-\alpha)=-\tan\alpha$

$$\therefore \pi-\omega_n\sqrt{1-\zeta^2}t_R=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}=\phi$$

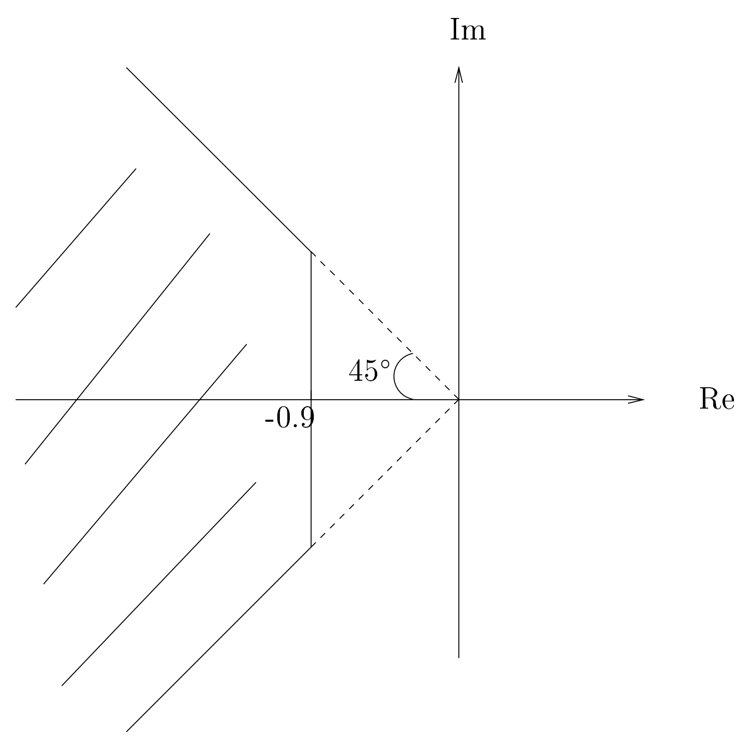
$$\Rightarrow t_R=\frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}}$$

Can estimate settling time t_s as follows: $\zeta<1$

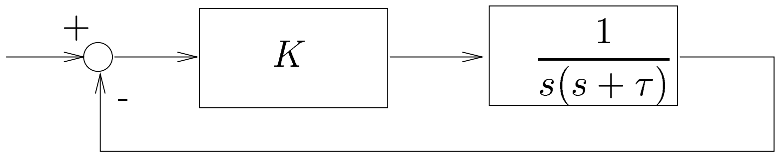
$$\begin{aligned} |e(t)| &= \left|\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t+\phi)\right| \\ &\le \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \end{aligned}$$

If $\zeta<0.8$, $|e(t)|\le 1.7e^{-\zeta\omega_n t_s}$. For $|e(t)|\le 0.02$, set $t_s\ge\frac{4.5}{\zeta\omega_n}=\frac{4.5}{\sigma}$ where σ is real part of pole.

Thus the overshoot requirement specifies the range of values for ζ, which in turn specifies the range of values of the angle ϕ. The settling time requirement specifies the magnitude of the real part of the poles. Combining the 2 requirements gives rise to a region in the complex plane. For example, if the overshoot is required to be less than 5%, then we want which translates into $\phi<45°$. If, in addition, we want the settling time to be less than 5 sec., we would need the poles to lie to left of the line $Re\ s=-0.9$. The desired region in the complex plane for the location of the 2 complex conjugate poles are shown below.



For second order system



$K > 0$

Closed loop transfer function is $\frac{k}{s^2+\tau s+k}$, $\omega_n = \sqrt{K}$, $\zeta = \frac{\tau}{2\sqrt{K}}$

$$\Rightarrow \quad K \uparrow \infty, \omega_n \uparrow \infty$$
$$\zeta \downarrow 0$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\sqrt{K}\sqrt{1-\frac{\tau^2}{4K}}}$$
$$\Rightarrow \quad K \uparrow,\ T_p \downarrow 0$$

Corresponds to arbitrarily fast response.