Chapter 5

Discrete Time Signal Analysis

5.1 Discrete Time Signals and Fourier Transform

Consider a discrete time sequence

$$x_k \qquad k = 0, \pm 1, \ldots$$

Suppose $\Sigma |x_k| < \infty$. (By Wierstrass's test, we have uniform convergence.)

Define the Fourier transform of x_k by

$$X(e^{i\omega}) = \sum_{k=-\infty}^{\infty} x_k e^{-i\omega k}$$

Now

$$X(e^{i\omega}) \text{ is automatically periodic in } \omega \in [-\pi, \pi]$$

Thus there is no need to consider frequencies extending beyond $[-\pi, \pi]$. This is the phenomenon of aliasing with higher frequencies aliased to that of a lower frequency ($e^{i\frac{7\pi}{2}k} = e^{i\frac{\pi}{2}k}$). We can think of the sequence x_k having a sampling period $T = 1$. In general, for a sampling period T, the Fourier transform is periodic over $[-\frac{\pi}{T}, \frac{\pi}{T}]$. More about sampling later.

Can think of x_k as the Fourier series representation of $X(e^{i\omega})$. Thus can determine x_k by F.S. expansion of $X(e^{i\omega})$ i.e.

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega}) e^{i\omega k} d\omega$$

Virtually all the results on Fourier series discussed in ECE355F apply in this setting, with the only main difference being that the periodic function $X(e^{i\omega})$ is generally complex.

There are, however, 2 results not discussed in the context of Fourier series, which we would like to describe.

Product of 2 sequences

Let

$$m_k = x_k y_k$$

The Fourier transform of m_k can be determined as follows.

$$\begin{aligned} M(e^{i\omega}) &= \sum_{k=-\infty}^{\infty} m_k e^{-i\omega k} \\ &= \sum_{k=-\infty}^{\infty} x_k \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{i\gamma}) e^{i\gamma k} d\gamma e^{-i\omega k} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} x_k e^{-i(\omega-\gamma)k} Y(e^{i\gamma}) d\gamma \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i(\omega-\gamma)})Y(e^{i\gamma})d\gamma \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\gamma})Y(e^{i(\omega-\gamma)})d\gamma
\end{aligned}
$$

Convolution of sequences

$$z_k = \sum_{l=-\infty}^{\infty} x_l y_{k-l}$$

$$
\begin{aligned}
Z(e^{i\omega}) = \sum_k z_k e^{-i\omega k} &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} x_l y_{k-l} e^{-i\omega(k-l)} e^{-i\omega l} \\
&= \sum_l \sum_k x_l y_{k-l} e^{-i\omega(k-l)} e^{-i\omega l} \\
&= Y(e^{i\omega})X(e^{i\omega})
\end{aligned}
$$

Now if $x_k = 1$, $k \geq 0$ and $x_k = 0$, $k < 0$ (unit step), $X(e^{i\omega})$ does not exist, just as the Laplace transform of $x(t) = 1$ does not exist. We therefore introduce the analogue of the Laplace transform for discrete time signals, the z-transform.

5.2 Z-Transforms

Assume

$$|x_k| \leq c r_o^k$$

i.e. x_k is exponentially (geometrically) bounded. Then for all $r > r_0$

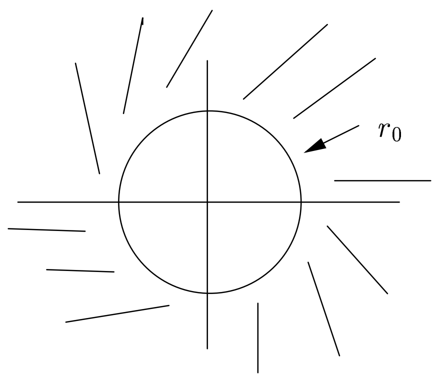
$$\Sigma |x_k| r^{-k}$$

$$\leq c\Sigma \left(\frac{r_0}{r}\right)^k < \infty$$

Define z-transform of x_k as

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

converges in $|z| > r_0$



$X(z)$ is then an analytic function in the region of convergence.

Example:

$$x_k = a^k \quad k \geq 0$$

$$X(z) = \sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1-az^{-1}} \quad |z| > |a|$$

Inversion integral:

$$x_k = \frac{1}{2\pi i} \oint X(z) z^{k-1} dz$$

path inside region of convergence (from Laurent series expansion). Since

$$\begin{aligned} & \frac{1}{2\pi i} \oint \Sigma x_n z^{-n} z^{k-1} dz \\ & = \frac{1}{2\pi i} \Sigma \oint x_n z^{-(n-k)} \frac{dz}{z} \end{aligned}$$

(convergence uniform to permit integration term by term)

$$\begin{aligned} & = x_k \\ & = \Sigma \text{ residues of } X(z) z^{k-1} \text{ inside } C \end{aligned}$$

Example:

$$X(z) = \frac{1}{1-az^{-1}}$$

$$\begin{aligned} x_k &= \frac{1}{2\pi i} \oint \frac{z^{k-1}}{1-az^{-1}} dz \\ &= \frac{1}{2\pi i} \oint \frac{z^k}{z-a} dz = a^k \quad k \geq 0 \end{aligned}$$

Can also do infinite series expansion to get

$$\frac{1}{1-az^{-1}} = \Sigma a^k z^{-k}$$

.

Multiplication of time sequences:

$$w_k = x_k y_k$$

where $X(z)$ has region of convergence $|z| > R_0$ and $Y(z)$ has region of convergence $|z| > R_1$.

$$\begin{aligned} W(z) &= \sum_k x_k y_k z^{-k} = \sum x_k z^{-k} \frac{1}{2\pi i} \oint Y(z) \zeta^{k-1} d\zeta \\ &= \frac{1}{2\pi i} \oint \sum x_k \left(\frac{z}{\zeta}\right)^k Y(\zeta) \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \oint X\left(\frac{z}{\zeta}\right) Y(\zeta) \frac{d\zeta}{\zeta} \end{aligned}$$

where the contour integral is over a circle $|z| > R_1$. It can also be expressed as

$$W(z) = \frac{1}{2\pi i} \oint X(\zeta) Y\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}$$

where the contour integral is over a circle $|z| > R_0$. Since we require

$$\left|\frac{z}{\zeta}\right| > R_0 \qquad |\zeta| > R_1 \Rightarrow |z| > R_0 R_1$$

Convolution of (causal) time sequences:

$$w_k = \sum_{l=0}^{k} x_l y_{k-l}$$

$$\begin{aligned} W(z) &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} x_l y_{k-l} z^{-k} \\ &= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty} x_l y_{k-l} z^{-(k-l)} z^{-l} \\ &= \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} x_l y_j z^{-j} z^{-l} = X(z)Y(z) \end{aligned}$$

Multiplication by a^k:

$$Z\{a^k x_k\} = X\left(\frac{z}{a}\right)$$

since

$$\sum a^k x_k z^{-k} = \sum x_k \left(\frac{z}{a}\right)^{-k}$$

If

$$\begin{aligned} |x_k| &\leq cr^k \\ &\Rightarrow |a^k x_k| \leq c[|a|r]^k \\ &\Rightarrow |z| > |a|r \text{ is region of convergence} \end{aligned}$$

Translation:

$$\begin{aligned} \sum_{k=0}^{\infty} x_{k-m} z^{-k} &= \sum_{k=0}^{m-1} x_{k-m} z^{-k} + z^{-m} X(z) \\ &= x_{-m} + z^{-1} x_{-m+1} + \ldots + x_{-1} z^{-m+1} + z^{-m} X(z) \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{\infty} x_{k+m} z^{-k} &= \sum_{k=0}^{\infty} x_{k+m} z^{-(k+m)} z^m \\ &= \sum_{l=+m}^{\infty} x_l z^{-l} z^{+m} = z^m X(z) - \sum_{l=0}^{m-1} x_l z^{-l} z^{+m} \\ &= z^m X(z) - \{z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}\} \end{aligned}$$

Initial value:

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} f_k z^{-k} = f_0 + \sum_{1}^{\infty} f_k z^{-k} \\ &\Rightarrow f_0 = \lim_{z\to\infty} f(z) \end{aligned}$$

Final value: Assume $f_k \underset{z\to\infty}{\longrightarrow} A < \infty$. Then

$$\lim_{\substack{z \to 1 \\ z > 1 \text{ real}}} (z-1)F(z) = A$$

Proof: Put $f_k = A + b_k$ $b_k \to 0$. Then

$$F(z) = \frac{Az}{z-1} + \sum_0^\infty b_k z^{-k}$$

Second converges since $b_k \to 0$, $|b_k| < B$. Hence $F(z)$ analytic for $|z| > 1$.

To show $\lim_{z\to 1+} (z-1) \sum \frac{b_k}{z^k} = 0$. Given $\epsilon > 0$, choose N so large that $|b_n| < \frac{1}{2}\epsilon$, $n \geq N$. Then

$$\begin{aligned} |(z-1)\sum_N^\infty \frac{b_n}{z^n}| &\leq (z-1)\frac{\epsilon}{2}\sum_N^\infty \frac{1}{z^n} \\ &= \frac{\epsilon}{2}(z-1)\frac{1}{z^N}\frac{z}{z-1} = \frac{\epsilon}{2z^{N-1}} < \frac{\epsilon}{2} \end{aligned}$$

On the other hand $|(z-1)\sum_0^{N-1} \frac{b_n}{z^n}| < \frac{\epsilon}{2}$ for $|z-1| < \delta$ with δ sufficiently small $\Rightarrow$ result.

Solving difference equations:

$$\begin{gathered} y_k + a_1 y_{k-1} + \ldots + a_n y_{k-n} \\ = b_0 u_k + \ldots + b_m u_{k-m} \text{ with } u_k = 0, k < 0. \end{gathered}$$

Putting $a_0 = 1$, we can write the above equation as

$$\sum_{j=0}^n a_j y_{k-j} = \sum_{j=0}^m b_j u_{k-j}$$

Taking z-transform of the left hand side gives

$$\begin{aligned} \sum_0^\infty y_k z^{-k} + \ldots &= Y(z) + a_1 z^{-1} Y(z) + \ldots \\ &= A(z)Y(z) + \sum_{j=0}^n \sum_{k=0}^{j-1} a_j y_{k-j} z^{-k} \end{aligned}$$

$$\therefore Y(z) = \frac{ic(z)}{A(z)} + \frac{B(z)}{A(z)} U(z)$$

where $ic(z)$ is a polynomial depending on the initial condition.

Example:

$$y(k+2) + y(k+1) = u(k+1) - u(k)$$

or

$$y(k) + y(k-1) = u(k-1) - u(k-2)$$

$$u(-2) = u(-1) = 0 \quad u(k) = \left(\frac{1}{2}\right)^k \quad k \geq 0$$

$$y(-1) = 2$$

Taking transforms

$$\begin{gathered} Y(z) + z^{-1}Y(z) + y(-1) = z^{-1}U(z) - z^{-2}U(z) \\ (1 + z^{-1})Y(z) = -2 + z^{-1} - z^{-2}U(z) \\ Y(z) = -\frac{2}{1+z^{-1}} + \frac{z^{-1} - z^{-2}}{1+z^{-1}}U(z) \end{gathered}$$

Since

$$U(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$Y(z) = -\frac{2z}{z+1} + \frac{z-1}{(z+1)(z-\frac{1}{2})}$$

Partial fraction expansion gives

$$Y(z) = \frac{-2z}{z+1} + \frac{4/3}{z+1} - \frac{1/3}{z-1/2}$$

Noting that the z-transform of a^n is $\frac{z}{z-a}$ and that $\frac{a}{z+a} = 1 - \frac{z}{z+a}$, we get

$$y(k) = -2(-1)^k + \frac{4}{3}(\delta(k) - (-1)^k) + 2\left(\frac{1}{3}\right)[\delta(k) - \left(\frac{1}{2}\right)^k]$$