SIGNALS AND SYSTEMS II
Assignment 7 Solutions

1. (a) The Fourier Transformation of output signal y_n is

$$\mathcal{F}(y_n) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-N}^{N} c_k u_{n-k}\right) e^{-i\omega n}$$

Change the order of summation, we have

$$\begin{aligned}\mathcal{F}(y_n) &= \underbrace{\sum_{k=-N}^{N} c_k e^{-i\omega k}}_{H(e^{i\omega})} \underbrace{\sum_{n=-\infty}^{\infty} u_{n-k} e^{-i\omega(n-k)}}_{U(e^{i\omega})} \\ &= H(e^{i\omega})\cdot U(e^{i\omega})\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathcal{F}(z_n) &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-N}^{N} c_k u_{n-N-k}\right) e^{-i\omega n} \\ &= \underbrace{\sum_{k=-N}^{N} c_k e^{-i\omega k}}_{H(e^{i\omega})}\cdot \underbrace{\sum_{n=-\infty}^{\infty} u_{n-N-k} e^{-i\omega(n-N-k)}}_{U(e^{i\omega})} \cdot e^{-i\omega N} \\ &= \underbrace{\left(H(e^{i\omega})e^{-i\omega N}\right)}_{H_1(e^{i\omega})}\cdot U(e^{i\omega}) \\ &= H_1(e^{i\omega})\cdot U(e^{i\omega})\end{aligned}$$

Therefore, $|H_1(e^{i\omega})| = |H(e^{i\omega})|$ and $\angle H_1(e^{i\omega}) = \angle H(e^{i\omega}) - \omega N$.

(b) **Method 1:**
First, we have

$$\begin{aligned}\omega_k &= \sum_{j1} h_{j1} u_{k-j1} \quad (*) \\ r_k &= \sum_{j2} h_{j2} u_{k-j2} \\ &= \sum_{j2} h_{j2} \omega_{-k+j2} \quad (**)\end{aligned}$$

From $(*)$ and $(**)$, we have

$$\begin{aligned}y_k &= r_{-k} \\ &= \sum_{j2} h_{j2} \sum_{j1} h_{j1} u_{k+j2-j1}\end{aligned}$$

Let $j = j1 - j2, j1 = j + j2$. Then

$$\begin{aligned} y_k &= \sum_{j2} h_{j2} \sum_j h_{j+j2} u_{k-j} \\ &= \sum_j \underbrace{\left(\sum_{j2} h_{j2} h_{j+j2} \right)}_{h_1(j)} u_{k-j} \end{aligned}$$

Thus, $h_1(k) = \sum_j h_j h_{k+j}$, and

$$\begin{aligned} H_1(e^{i\omega}) &= \sum_k h_1(k) e^{-i\omega k} \\ &= \sum_k \sum_j h_j h_{k+j} e^{-i\omega k} \\ &= \underbrace{\sum_j h_j e^{i\omega j}}_{H^*(e^{i\omega})} \underbrace{\sum_k h_{k+j} e^{-i\omega(k+j)}}_{H(e^{i\omega})} \\ &= H(e^{i\omega}) \cdot H^*(e^{i\omega}) \\ &= |H(e^{i\omega})|^2 \end{aligned}$$

The filter $H_1(e^{i\omega})$ has zero phase.
Method 2:
Similarly, we have

$$\begin{aligned} \omega_k &= \sum_j h_j u_{k-j} \quad (1) \\ r_k &= \sum_j h_j u_{-k+j} \quad (2) \end{aligned}$$

From (1) and (2), we have

$$\begin{aligned} y_k &= \omega_k + r_{-k} \\ &= \sum_j h_j u_{k-j} + \sum_j h_j u_{k+j} \\ &= \sum_j h_j u_{k-j} + \sum_j h_{-j} u_{k-j} \\ &= \sum_j \underbrace{(h_j + h_{-j})}_{h_2(j)} u_{k-j} \end{aligned}$$

Therefore, $h_2(k) = h_k + h_{-k}$, and

$$\begin{aligned}
H_2(e^{i\omega}) &= \sum_k h_2(k)e^{-i\omega k} \\
&= \sum_k (h_k + h_{-k})e^{-i\omega k} \\
&= \sum_k h_k e^{-i\omega k} + \sum_k h_{-k}e^{-i\omega k} \\
&= H(e^{i\omega}) + \sum_k h_k e^{i\omega k} \\
&= H(e^{i\omega}) + H^*(e^{i\omega}) \\
&= 2\Re\{H(e^{i\omega})\}
\end{aligned}$$

The filter $H_2(e^{i\omega})$ has zero phase too.

2. Since $H(f)$ is periodic with period 1, we only need to verify that the relation holds for $-\frac{1}{2} \le f \le \frac{1}{2}$. Furthermore, since the frequency responses are symmetric, we need only check for $0 \le f \le \frac{1}{2}$. From the definition of $H_L(f)$, we have

$$H_L(\frac{1}{2} - f) = \begin{cases} 1 & |\frac{1}{2} - f| \le \triangle \\ 0 & |\frac{1}{2} - f| > \triangle \end{cases}$$

Considering f in the range $0 \le f \le \frac{1}{2}$,

$$\begin{aligned}
|\frac{1}{2} - f| \le \triangle &\iff -\triangle \le \frac{1}{2} - f \le \triangle \\
&\iff \frac{1}{2} - \triangle \le f \le \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
|\frac{1}{2} - f| > \triangle &\iff -\triangle > \frac{1}{2} - f, \ or \ \frac{1}{2} - f > \triangle \\
&\iff 0 \le f < \frac{1}{2} - \triangle
\end{aligned}$$

we have for $0 \le f \le \frac{1}{2}$,

$$H_L(\frac{1}{2} - f) = \begin{cases} 1 & \frac{1}{2} - \triangle \le f \le \frac{1}{2} \\ 0 & 0 \le f < \frac{1}{2} - \triangle \end{cases}$$

Thus, for $0 \le f \le \frac{1}{2}$, $H(f) = H_L(\frac{1}{2} - f)$. By symmetry, $H(f) = H_L(\frac{1}{2} - f)$.
(a) Determine h_k in terms of $h_L(k)$.

$$\begin{aligned}
H_L(\frac{1}{2} - f) &= \sum_k h_L(k)e^{-i2\pi(\frac{1}{2}-f)k} \\
&= \sum_k h_L(k)e^{-i\pi k}e^{i2\pi fk} \\
&= \sum_m h_L(m)(-1)^m e^{i2\pi fm}
\end{aligned}$$

Let $m = -k$. Since $h_L(-k) = h_L(k)$,

$$H_L(\frac{1}{2} - f) = \sum_k \underbrace{h_L(k)(-1)^{-k}}_{h_k} e^{-i2\pi fk}$$

Thus, $h_k = h_L(k)(-1)^{-k} = h_L(k)(-1)^k$.
(b) Verify that $H(f)$ can be implemented as shown.

$$\begin{aligned} y_k &= \left(\sum_j h_L(j) \cdot u_{k-j}(-1)^{k-j} \right) \cdot (-1)^k \\ &= \sum_j h_L(j) u_{k-j}(-1)^{2k-j} \\ &= \sum_j \underbrace{h_L(j)(-1)^{-j}}_{h_j} u_{k-j} \\ &= \sum_j h_j u_{k-j} \end{aligned}$$

as required.

3. (a) (1) N=4.

$$\begin{aligned} c_0 &= 6.0000000e-01 \\ c_1 &= 2.9270510e-01 \\ c_2 &= -8.0901699e-02 \\ c_3 &= -4.2705098e-02 \\ c_4 &= 3.0901699e-02 \end{aligned}$$

(2) N=10. Spec (ii).

$$\begin{aligned} c_0 &= 5.4372580e-01 \\ c_1 &= 2.1437469e-01 \\ c_2 &= 2.7509796e-01 \\ c_3 &= -5.4888802e-01 \\ c_4 &= 3.7823505e-01 \\ c_5 &= 5.0000000e-02 \\ c_6 &= -3.7823505e-01 \\ c_7 &= 4.2528122e-01 \\ c_8 &= -2.7509796e-01 \\ c_9 &= 1.0923211e-01 \\ c_10 &= -2.1862901e-02 \end{aligned}$$

(3) N=10. Spec(iii).

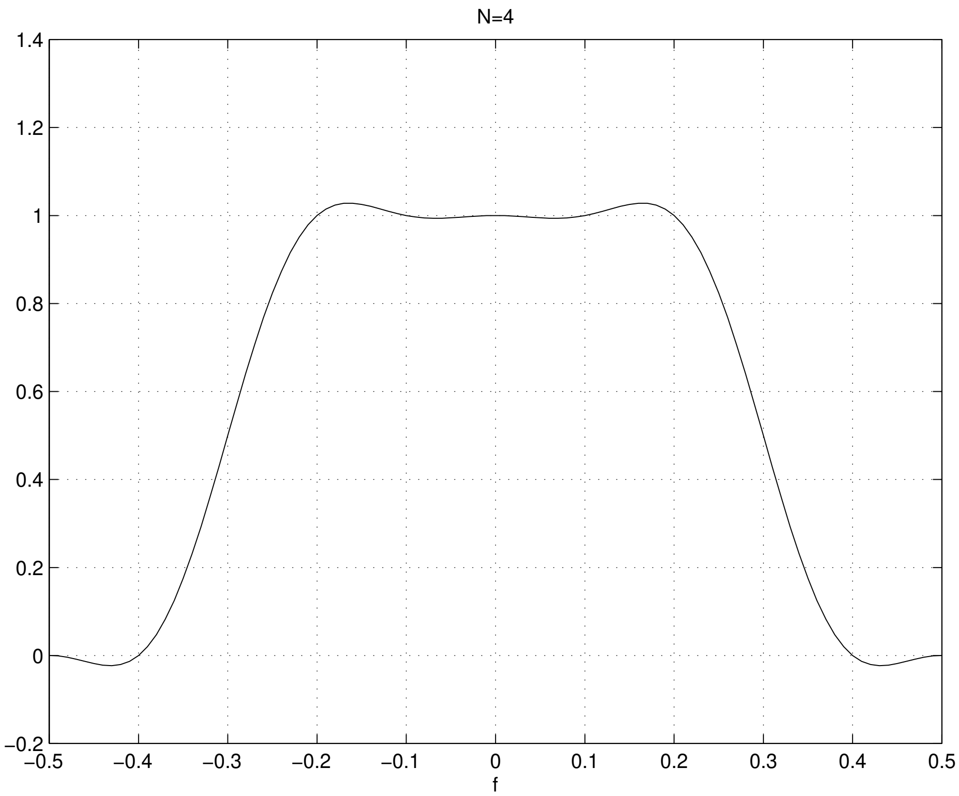
$$
\begin{aligned}
c_0 &= 5.1416492e-01\\
c_1 &= 3.1568758e-01\\
c_2 &= -1.4164916e-02\\
c_3 &= -9.8130525e-02\\
c_4 &= 1.4164916e-02\\
c_5 &= 5.0000000e-02\\
c_6 &= -1.4164916e-02\\
c_7 &= -2.5476272e-02\\
c_8 &= 1.4164916e-02\\
c_9 &= 7.9192220e-03\\
c_10 &= -7.0824579e-03
\end{aligned}
$$

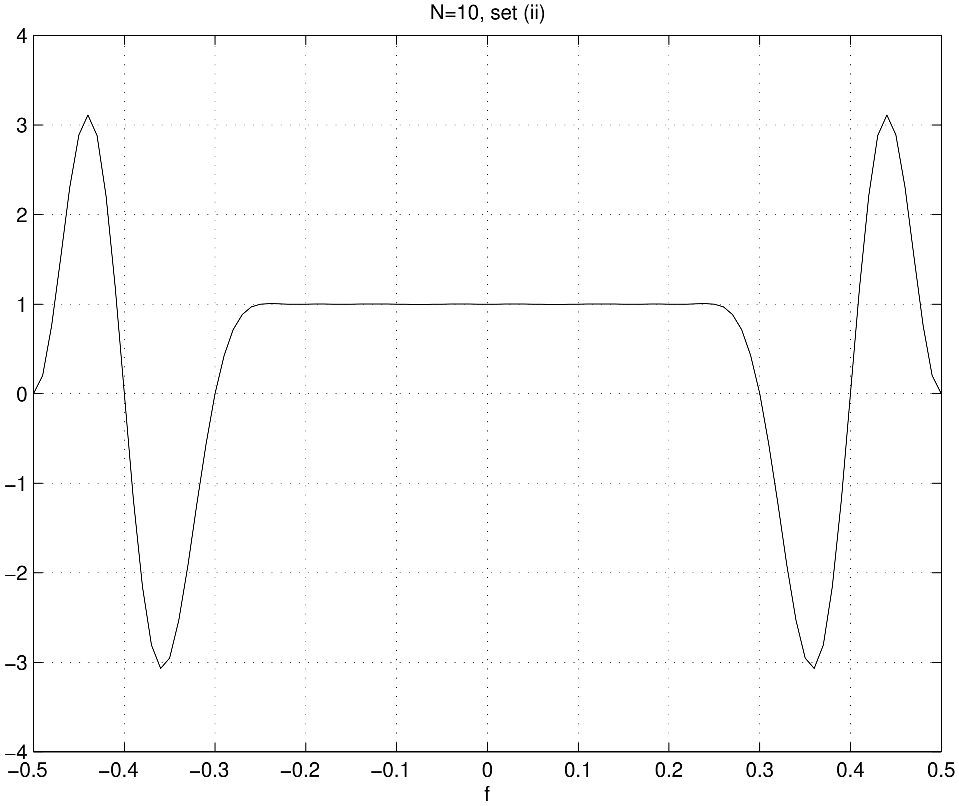
(b) Since the filter $(H(e^{i\omega}))$ has zero phase, we have

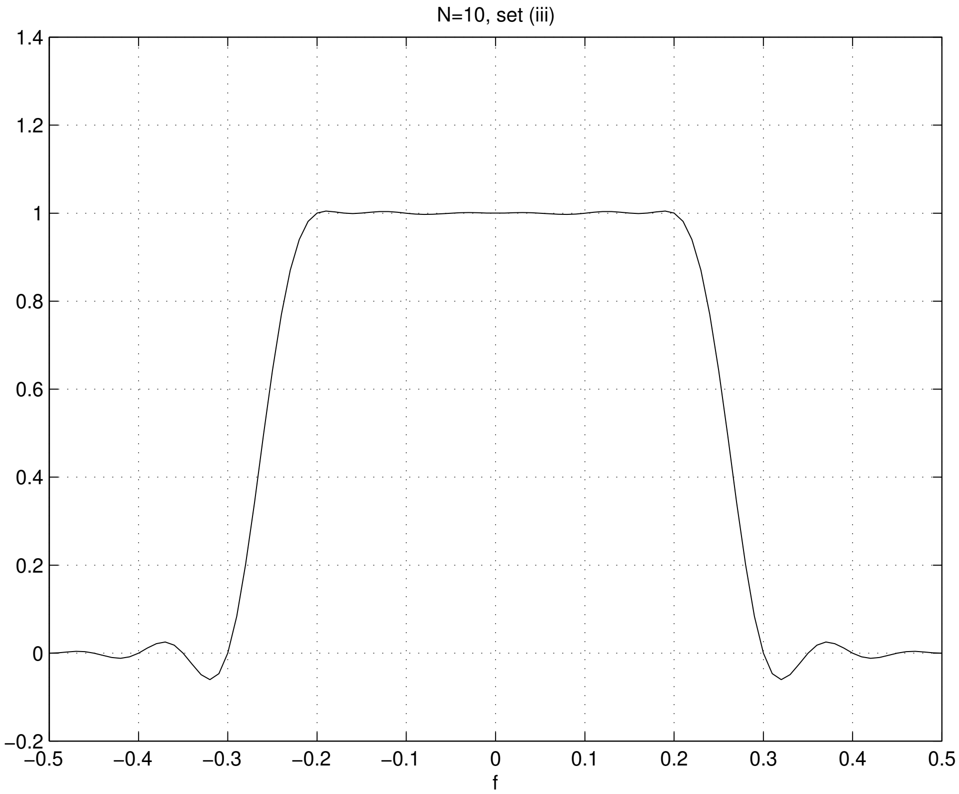
$$y_k = |H(e^{i\omega})| \cos 2\pi f k.$$

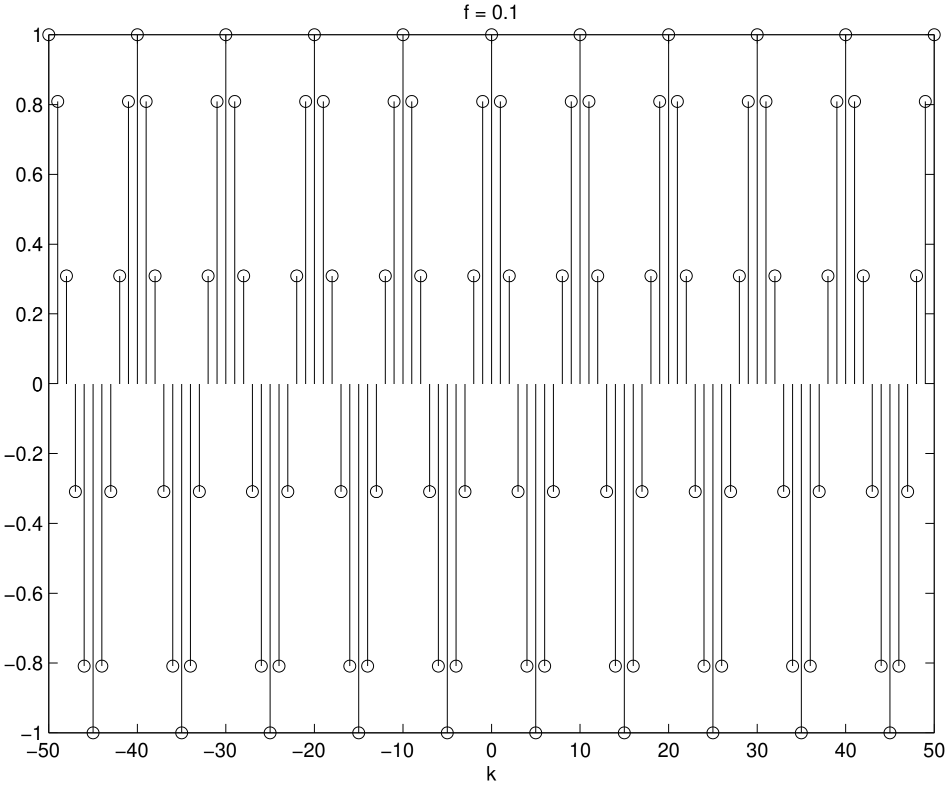
y_k is discrete time signal. So we should use Matlab function **stem** to draw the figure.

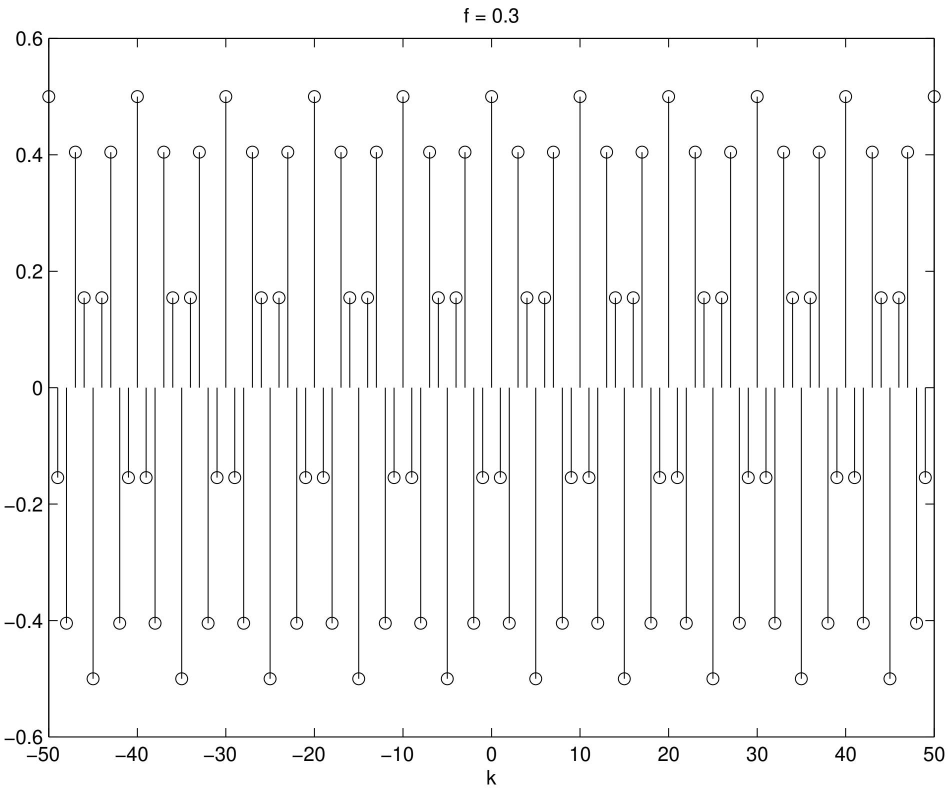
(c)

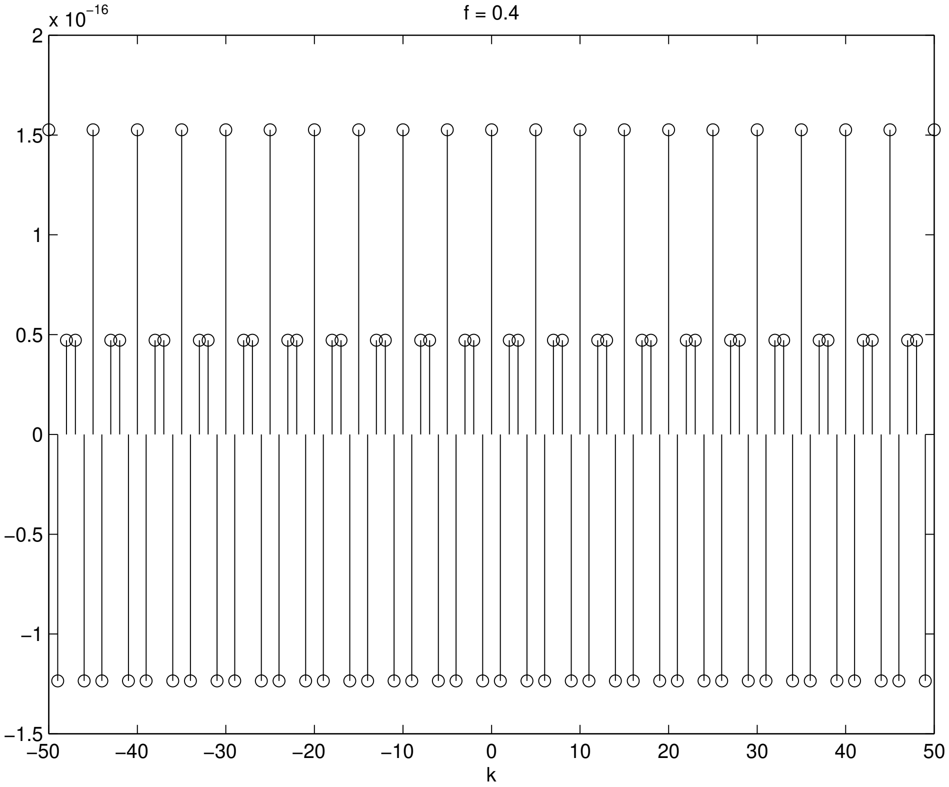


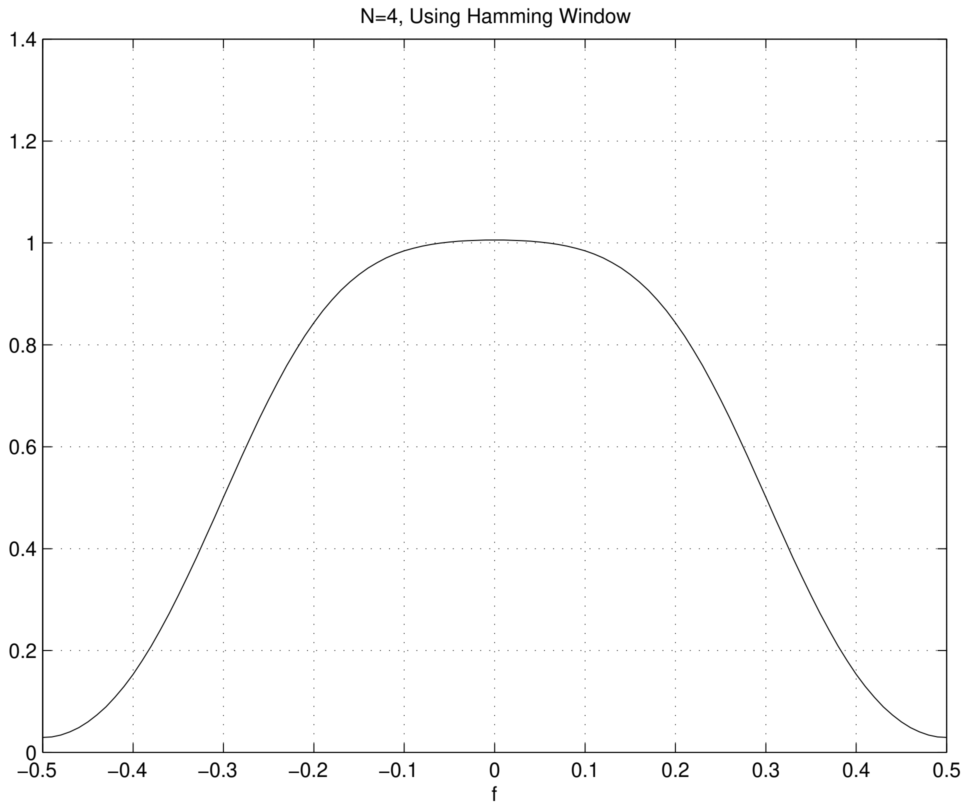

N=10, set (ii)
4
3
2
1
0
−1
−2
−3
−4
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
f


N=10, set (iii)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
f


f = 0.1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−50
−40
−30
−20
−10
0
10
20
30
40
50
k


f = 0.3
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−50
−40
−30
−20
−10
0
10
20
30
40
50
k






```
%% N=4, including the solution for subproblem (b)
```

```
N=4;

s=[0;.1;.2;.4;.5];
hf=[1;1;1;0;0];

T = zeros(N+1, N+1);

for k=1:N+1
    T(k,1)=1;
    for j=2:N+1
T(k,j) = 2*cos(2*pi*s(k)*(j-1));
    end;
end;

c= inv(T)*hf

f=-.5:0.01:.5;
H=zeros(size(f));
l=length(f);

for i=1:l
    H(i)=c(1);
    for j=2:N+1
H(i) = H(i) + 2*c(j)*cos(2*pi*f(i)*(j-1));
    end;
end;

plot(f,H);
xlabel('f');
title('N=4');
grid on;

figure;
f1=0.1;
h1=c(1);
for j=2:N+1
    h1 = h1 + 2*c(j)*cos(2*pi*f1*(j-1));
end;

k=-50:50;
y=abs(h1)*cos(2*pi*f1*k);

stem(k,y);
xlabel('k');
```

```
title('f = 0.1');

figure;
f1=0.3;
h1=c(1);
for j=2:N+1
    h1 = h1 + 2*c(j)*cos(2*pi*f1*(j-1));
end;

k=-50:50;
y=abs(h1)*cos(2*pi*f1*k);

stem(k,y);
xlabel('k');
title('f = 0.3');

figure;
f1=0.4;
h1=c(1);
for j=2:N+1
    h1 = h1 + 2*c(j)*cos(2*pi*f1*(j-1));
end;

k=-50:50;
y=abs(h1)*cos(2*pi*f1*k);

stem(k,y);
xlabel('k');
title('f = 0.4');

figure;
f1=1;
h1=c(1);
for j=2:N+1
    h1 = h1 + 2*c(j)*cos(2*pi*f1*(j-1));
end;

k=-50:50;
y=abs(h1)*cos(2*pi*f1*k);

stem(k,y);
xlabel('k');
title('f = 1');
```

```
%%N=4, using Hamming window
N=4;

w=zeros(N+1,1);
for k=1:N+1
    w(k)=0.54+0.46*cos((k-1)*pi/N)
end;

s=[0;.1;.2;.4;.5];
hf=[1;1;1;0;0];

T = zeros(N+1, N+1);

for k=1:N+1
    T(k,1)=1;
    for j=2:N+1
T(k,j) = 2*cos(2*pi*s(k)*(j-1));
    end;
end;

c= inv(T)*hf

c=c.*w

f=-.5:0.01:.5;
H=zeros(size(f));
l=length(f);

for i=1:l
    H(i)=c(1);
    for j=2:N+1
H(i) = H(i) + 2*c(j)*cos(2*pi*f(i)*(j-1));
    end;
end;

plot(f,H);
xlabel('f');
title('N=4, Using Hamming Window');
grid on;
```