SIGNALS AND SYSTEMS II
Assignment 8 Solutions

1. Compute h_k^d first,

$$
\begin{aligned}
h_k^d &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{i\omega})e^{i\omega k}d\omega \\
&= \frac{1-(-1)^k}{k\pi}.
\end{aligned}
$$

   (a) $\delta = 2\%$

$$
\begin{aligned}
A &= -20\log_{10}\delta = 33.98 \\
\alpha &= 0.5842(A-21)^{0.4} + 0.07886(A-21) = 2.652
\end{aligned}
$$

   Using Matlab, we obtain

k	ω_k	$\omega_k h_k^d$
0	1	0
1	0.9592	0.6106
2	0.8434	0
3	0.6714	0.1425
4	0.4704	0
5	0.2702	0.0344

   (b) $\delta = 0.056\%$

$$
\begin{aligned}
A &= -20\log_{10}\delta = 65.04 \\
\alpha &= 0.1102(A-8.7) = 6.208
\end{aligned}
$$

k	ω_k	$\omega_k h_k^d$
0	1	0
1	0.8916	0.5676
2	0.6235	0
3	0.3251	0.0690
4	0.1098	0
5	0.0123	0.0016

   (c) Hamming Window.

k	ω_k	$\omega_k h_k^d$
0	1	0
1	0.9121	0.5807

2	0.6821	0
3	0.3978	0.0844
4	0.1679	0
5	0.08	0.0102

Note that the Hamming weights are between those of Kaiser designs 1 and 2, with the value of $\alpha = 5.4414$ in a Kaiser window design corresponding very closely to that of the Hamming window design. Hence we can say, qualitatively, that Kaiser design 1 has the largest ripple but smallest transition width; Kaiser design 2 has the smallest ripple but the widest transition, while the Hamming design is in between.

2. (a)

$$\begin{aligned} (\frac{\omega_p}{\omega_s})^{2N_c} = \frac{\epsilon^2}{A^2-1} \quad &\Longrightarrow \quad \frac{\omega_p^{2N_c}}{\omega_s^{2N_c}} = \frac{\epsilon^2}{A^2-1} \\ &\Longrightarrow \quad \frac{\omega_p^{2N_c}}{\epsilon^2} = \frac{\omega_s^{2N_c}}{A^2-1} \\ &\Longrightarrow \quad \omega_1(N_c) = \frac{\omega_p}{(\epsilon^2)^{\frac{1}{2N_c}}} = \frac{\omega_s}{(A^2-1)^{\frac{1}{2N_c}}} = \omega_2(N_c) \end{aligned}$$

Therefore, $\frac{\omega_1(N_c)}{\omega_2(N_c)} = 1$.

(b) Notice $N \geq N_c$ and $\frac{\omega_p}{\omega_s} < 1$. So

$$(\frac{\omega_p}{\omega_s})^{2N} \leq (\frac{\omega_p}{\omega_s})^{2N_c} = \frac{\epsilon^2}{A^2-1}$$

Then,

$$\underbrace{\frac{\omega_p}{(\epsilon^2)^{\frac{1}{2N}}}}_{\omega_1(N)} \leq \underbrace{\frac{\omega_s}{(A^2-1)^{\frac{1}{2N}}}}_{\omega_2(N)}$$

i.e., $\omega_1(N) \leq \omega_2(N)$.

(c) From $\omega_c \leq \omega_2(N) = \frac{\omega_s}{(A^2-1)^{\frac{1}{2N}}}$, we have

$$\begin{aligned} \omega_c^{2N}(A^2-1) \leq \omega_s^{2N} \quad &\Rightarrow \\ A^2 \leq (\frac{\omega_s}{\omega_c})^{2N} + 1. \end{aligned}$$

3. Refer to the Example in Chapter 7. Originally, it was intended that you would use the second choice ($\omega_1(N)$ in question 2) for computing ω_c, but it was unfortunately not specified in the problem!

4. (a)

$$\begin{aligned} h_a(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_a(\omega')e^{i\omega' t}d\omega' \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_a(\omega')e^{i\omega' t}d\omega' \end{aligned}$$

So

$$h_k = h_a(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_a(\omega')e^{i\omega' k}d\omega'$$

Thus, the digital filter frequency response is

$$\begin{aligned} H(\omega) &= \sum_{k=-\infty}^{\infty} h_k e^{-i\omega k} \\ \text{with } h_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{i\omega})e^{i\omega k}d\omega \end{aligned}$$

On comparison, we see that

$$H(e^{i\omega}) = H_a(\omega) \qquad |\omega| \le \pi$$

(b) Using the fact that

$$\mathcal{L}^{-1}\left(\frac{\omega_p}{(s+\omega_p)^2+\omega_p^2}\right) = e^{-\omega_p t}\sin(\omega_p t)u(t),$$

we have

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{2\omega_p^2}{(s+\omega_p)^2+\omega_p^2}\right) &= 2\omega_p e^{-\omega_p t}\sin(\omega_p t)u(t) \\ &= (-i\omega_p e^{(i-1)\omega_p t} + i\omega_p e^{-(i+1)\omega_p t})u(t), \end{aligned}$$

by $\sin(\omega_p t) = (e^{i\omega_p t} - e^{-i\omega_p t})/(2i)$. Then

$$h_k = (-i\omega_p e^{(i-1)\omega_p k} + i\omega_p e^{-(i+1)\omega_p k})u(k).$$

And

$$\begin{aligned} H(\omega) &= \sum_{k=0}^{\infty}(-i\omega_p e^{(i-1)\omega_p k} + i\omega_p e^{-(i+1)\omega_p k})e^{-i\omega k} \\ &= \frac{-i\omega_p}{1-e^{(i-1)\omega_p - i\omega}} + \frac{i\omega_p}{1-e^{-(i+1)\omega_p - i\omega}} \\ &= \frac{2\omega_p \sin\omega_p e^{-\omega_p}e^{-i\omega}}{1-2\cos\omega_p e^{-\omega_p}e^{-i\omega} + e^{-2\omega_p}e^{-2i\omega}} \end{aligned}$$