SIGNALS AND SYSTEMS II
Assignment 1 Solutions

1. Solve the equation

$$\begin{aligned} 0 &= rN_1(1-\frac{N_1}{K}) - \alpha N_1 N_2 \\ 0 &= sN_2(1-\frac{N_2}{L}) - \beta N_1 N_2 \end{aligned}$$

We have four equilibrium points

$$\left\{ \begin{array}{rcl} N_1 &=& 0 \\ N_2 &=& 0 \end{array} \right. , \left\{ \begin{array}{rcl} N_1 &=& 0 \\ N_2 &=& L \end{array} \right. , \left\{ \begin{array}{rcl} N_1 &=& K \\ N_2 &=& 0 \end{array} \right. , \left\{ \begin{array}{rcl} N_1 &=& \frac{\alpha KLs - Krs}{\alpha\beta KL - rs} \\ N_2 &=& \frac{\beta KLr - Lrs}{\alpha\beta KL - rs} \end{array} \right. .$$

The Jacobian matrix $\left(\dfrac{\partial f}{\partial x}\right)$ is

$$\left[\begin{array}{cc} r - \dfrac{2rN_1}{K} - \alpha N_2 & -\alpha N_1 \\ -\beta N_2 & s - \dfrac{2sN_2}{L} - \beta N_1 \end{array} \right]$$

Then the linearized equations are

(a) At equilibrium point $(0,0)$,

$$\left(\begin{array}{c} \Delta\dot{N}_1 \\ \Delta\dot{N}_2 \end{array} \right) = \left(\begin{array}{cc} r & 0 \\ 0 & s \end{array} \right) \cdot \left(\begin{array}{c} \Delta N_1 \\ \Delta N_2 \end{array} \right) .$$

(b) At equilibrium point $(0,L)$,

$$\left(\begin{array}{c} \Delta\dot{N}_1 \\ \Delta\dot{N}_2 \end{array} \right) = \left(\begin{array}{ll} -\alpha L + r & 0 \\ -\beta L & -s \end{array} \right) \cdot \left(\begin{array}{c} \Delta N_1 \\ \Delta N_2 \end{array} \right) .$$

(c) At equilibrium point $(K,0)$,

$$\left(\begin{array}{c} \Delta\dot{N}_1 \\ \Delta\dot{N}_2 \end{array} \right) = \left(\begin{array}{ll} -r & -\alpha K \\ 0 & -\beta K + s \end{array} \right) \cdot \left(\begin{array}{c} \Delta N_1 \\ \Delta N_2 \end{array} \right) .$$

(d) At equilibrium point $(\dfrac{\alpha KLs - Krs}{\alpha\beta KL - rs}, \dfrac{\beta KLr - Lrs}{\alpha\beta KL - rs})$, after some algebra:

$$\left[\begin{array}{c} \Delta\dot{N}_1 \\ \Delta\dot{N}_2 \end{array} \right] = -\frac{1}{\alpha\beta KL - rs} \left[\begin{array}{cc} rs(\alpha L - r) & \alpha sK(\alpha L - r) \\ \beta rL(\beta K - s) & rs(\beta K - s) \end{array} \right] \left[\begin{array}{c} \Delta N_1 \\ \Delta N_2 \end{array} \right]$$

2. (a) Solve $\det(\lambda I - A) = 0$, we get two eigenvalues

$$\lambda_1 = \sigma - i\omega, \lambda_2 = \sigma + i\omega.$$

The eigenvectors are

$$v_1 = k_1 \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad v_2 = k_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}, \ k_1, k_2 \in Z.$$

by solving the equations $(\lambda_i I - A)v_i = 0, \quad i = 1, 2.$

Let $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. We have $T^{-1} = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$.

Since

$$e^{T^{-1}ATt} = T^{-1}e^{At}T = e^{\Lambda t} = \begin{pmatrix} e^{(\sigma - i\omega)t} & 0 \\ 0 & e^{(\sigma + i\omega)t} \end{pmatrix},$$

we have

$$e^{At} = Te^{\Lambda t}T^{-1} = e^{\sigma t} \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}.$$

(b) We know that

$$\begin{aligned} \mathcal{L}[e^{At}] &= (sI - A)^{-1} \\ &= \begin{pmatrix} \dfrac{s-\sigma}{(s-\sigma)^2 + \omega^2} & \dfrac{\omega}{(s-\sigma)^2 + \omega^2} \\ \dfrac{-\omega}{(s-\sigma)^2 + \omega^2} & \dfrac{s-\sigma}{(s-\sigma)^2 + \omega^2} \end{pmatrix}. \end{aligned}$$

Then

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}] = e^{\sigma t} \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}.$$

(c)

$$e^{At} = e^{(\sigma I + \omega B)t} = e^{\sigma t} e^{\omega B t},$$

where

$$\begin{aligned} e^{\omega B t} &= I + \frac{\omega B}{1!} t + \frac{\omega^2 B^2}{2!} t^2 + \frac{\omega^3 B^3}{3!} t^3 + \dots \\ &= I + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} t + \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix} \frac{t^3}{3!} + \dots \\ &= \begin{pmatrix} 1 - \frac{\omega^2}{2!}t^2 + \frac{\omega^4}{4!}t^4 + \dots & \omega - \frac{\omega^3}{3!}t^3 + \dots \\ -\omega + \frac{\omega^3}{3!}t^3 + \dots & 1 - \frac{\omega^2}{2!}t^2 + \frac{\omega^4}{4!}t^4 + \dots \end{pmatrix} \\ &= \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}. \end{aligned}$$

3. (a) By the definition of e^{At},

$$e^{At}v = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} v.$$

Consider

$$\begin{cases} Av &= \lambda v \\ A^2 v &= A \cdot (Av) = \lambda(AV) = \lambda^2 v \\ &\vdots \\ A^k v &= \lambda^k v \end{cases},$$

we have

$$e^{At}v = \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} v = e^{\lambda t} v$$

as required.

(b) Since A has distinct eigenvalues, $[v_1, v_2, \ldots, v_n]$ is a basis of n-dimensional linear space. As to any vector x_o, we can write

$$x_o = \sum_{k=1}^{n} c_k v_k \tag{1}$$

Then

$$\begin{aligned} x(t) &= e^{At} \cdot x_o \\ &= \sum_{k=1}^{n} c_k e^{At} v_k \\ &= \sum_{k=1}^{n} c_k e^{\lambda_k t} v_k \end{aligned}$$

c_k is determined by equation 1.

(c) The eigenvalues are

$$\lambda_1 = 4, \quad \lambda_2 = 1.$$

Their corresponding eigenvectors are

$$v_1 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Then solve equation 1, i.e.,

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

we have

$$c_1 = \frac{8}{3}, \quad c_2 = \frac{1}{3}.$$

Thus

$$x(t) = \frac{8}{3} e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{3} e^{t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$