Chapter 2

CONTROLLABILITY

2.1 Reachable Set and Controllability

Suppose we have a linear system described by the state equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

Consider the following problem. For a given vector x in R^n, does there exist a time t, $0 < t < \infty$ and a piecewise continuous control input u, defined on $[0, t]$, such that the solution of (1) $x(t) = x$? We shall refer to this problem as the reachability problem.

If for $x_0 = 0$, there exists an input u which solves the reachability problem, we then say that the state x is reachable from 0 at time t. We refer to such an input u as the control which transfers the state from the origin to x at time t. Denote the set of states reachable from 0 by $\mathcal{R}_0$. If $\mathcal{R}_0 = R^n$, then every state is reachable from 0 by suitable control.

One motivation for studying reachability is the interception or rendezvous problem. Suppose you know the trajectory of some vehicle and you want to maneuver your own vehicle to intercept it. If $\mathcal{R}_0 = R^n$, the interception problem is solvable.

To prepare for the discussion later in the chapter, let us give a slightly more abstract but compact description of the reachability problem. Using the variation of parameters formula, x is reachable from 0 at time t if there exists an input u such that

$$x = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \tag{2.2}$$

Fix the time t. Let $\mathcal{U}$ be the space of all input signals defined on $[0\ t]$. Define the linear operator $\mathcal{L}$ which maps $\mathcal{U}$ to R^n by

$$\mathcal{L}u = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \tag{2.3}$$

Recall that if A is an $m \times n$ matrix (a linear transformation) mapping R^n into R^m, the range of A, denoted $\mathcal{R}(A)$, is the set $\{y \in R^m \mid y = Ax, \text{ for some } x \in R^n\}$. Analogously, $\mathcal{R}(\mathcal{L})$, the range of $\mathcal{L}$, is defined to be the set $\{x \in R^n \mid x = \mathcal{L}u, \text{ for some } u \in \mathcal{U}\}$. Using this formulation, x is reachable from 0 if and only if $x \in \mathcal{R}(\mathcal{L})$.

There is nothing particularly special about the initial state $x_0 = 0$. In fact, if $\mathcal{R}_0 = R^n$, then every state x is reachable from any initial state x_0. To see this, write the variation of parameters formula for the solution of (2.1) in the form

$$x(t) - e^{At} x_0 = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \tag{2.4}$$

We see that x reachable from x_0 is equivalent to $x - e^{At}x_0$ is reachable from 0.

We now define controllability for the system described by (2.1).

Definition: The system (2.1) is said to be **controllable** at time t if for every vector $x \in R^n$ and every initial condition x_0, there exists an input u which transfers the state from x_0 to x at time t.

In view of the preceding discussion, studying the controllability property is equivalent to studying whether $\mathcal{R}_0 = R^n$.

Example: Consider the following network

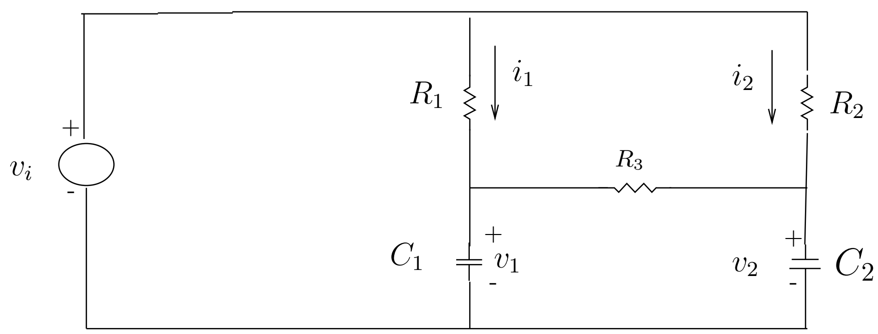

Figure 2.1: An RC bridge circuit

It is straightforward to verify, using Kirchhoff's current law, that v_1, v_2 satisfy the following differential equations:

$$\begin{aligned} C_1\dot{v}_1 &= -\frac{v_1 - v_2}{R_3} + \frac{v_i - v_1}{R_1} \\ C_2\dot{v}_2 &= \frac{v_1 - v_2}{R_3} + \frac{v_i - v_2}{R_2} \end{aligned}$$

In state space form, denoting $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$, we have

$$\dot{v} = \begin{bmatrix} -(\frac{1}{C_1R_1} + \frac{1}{C_1R_3}) & \frac{1}{C_1R_3} \\ \frac{1}{C_2R_3} & -(\frac{1}{C_2R_2} + \frac{1}{C_2R_3}) \end{bmatrix} v + \begin{bmatrix} \frac{1}{C_1R_1} \\ \frac{1}{C_2R_2} \end{bmatrix} v_i$$

Put $\alpha_1 = \frac{1}{C_1R_1}$, $\alpha_2 = \frac{1}{C_2R_2}$, $\beta_1 = \frac{1}{C_1R_3}$, $\beta_2 = \frac{1}{C_2R_3}$. We can write the differential equation in the form

$$\dot{v} = \begin{bmatrix} -(\alpha_1 + \beta_1) & \beta_1 \\ \beta_2 & -(\alpha_2 + \beta_2) \end{bmatrix} v + \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} v_i \tag{2.5}$$

Notice that if $\alpha_1 = \alpha_2$, i.e., $C_1R_1 = C_2R_2$, the difference $v_1 - v_2$ satisfies

$$\frac{d}{dt}(v_1 - v_2) = -(\alpha_1 + \beta_1 + \beta_2)(v_1 - v_2)$$

which is a homogeneous equation with no input. In this case, we cannot manipulate v_1 and v_2 arbitrarily, so that the reachable set is not the whole R^2.

To study the reachability problem, we introduce 2 important matrices arising from the system (2.1).

(i) For $t > 0$, define the **controllability Gramian** at time t

$$W_t = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau \tag{2.6}$$

$$= \int_0^t e^{A(t-s)} BB^T e^{A^T(t-s)} ds \tag{2.7}$$

Finding W_t requires computation of e^{At} and integration.

Example:

$$\begin{aligned}
A &= \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \\
(sI-A)^{-1} &= \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}}{(s+1)(s+2)} \\
&= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix} \\
e^{At} &= \mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \\
B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
W_t &= \int_0^t \begin{bmatrix} e^{-\tau} & e^{-\tau} - e^{-2\tau} \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \;\; 1] \begin{bmatrix} e^{-\tau} & 0 \\ e^{-\tau} - e^{-2\tau} & e^{-2\tau} \end{bmatrix} d\tau \\
&= \int_0^t \begin{bmatrix} e^{-\tau} & e^{-\tau} - e^{-2\tau} \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ e^{-\tau} - e^{-2\tau} & e^{-2\tau} \end{bmatrix} d\tau \\
&= \int_0^t \begin{bmatrix} e^{-2\tau} - 2e^{-3\tau} + e^{-4\tau} & e^{-3\tau} - e^{-4\tau} \\ e^{-3\tau} - e^{-4\tau} & e^{-4\tau} \end{bmatrix} d\tau \\
&= \begin{bmatrix} \frac{1}{2}(1-e^{-2t}) - \frac{2}{3}(1-e^{-3t}) + \frac{1}{4}(1-e^{-4t}) & \frac{1}{3}(1-e^{-3t}) - \frac{1}{4}(1-e^{-4t}) \\ \frac{1}{3}(1-e^{-3t}) - \frac{1}{4}(1-e^{-4t}) & \frac{1}{4}(1-e^{-4t}) \end{bmatrix}
\end{aligned}$$

Note that W_t is a symmetric $n \times n$ matrix.

(ii) Let C_{AB} denote the $n \times nm$ matrix

$$C_{AB} = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] \tag{2.8}$$

C_{AB} is called the **controllability matrix**. For $m = 1$, C_{AB} is a square matrix. For $m > 1$, C_{AB} is a rectangular, wide matrix.

Recall that the rank of a $p \times q$ matrix A is the number of independent columns of A, which is the same as the number of independent rows of A. Note that $rank(C_{AB}) \leq n$. If $rank(C_{AB}) = n$, we say C_{AB} has full rank.

Let us first illustrate the computation of C_{AB} in some examples.

Example 1: For the circuit described by Figure 2.1, we have

$$C_{AB} = \begin{bmatrix} \alpha_1 & -(\alpha_1^2 + \alpha_1\beta_1) + \alpha_2\beta_1 \\ \alpha_2 & -(\alpha_2^2 + \alpha_2\beta_2) + \alpha_1\beta_2 \end{bmatrix}$$

Note that

$$\det C_{AB} = (\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1)(\alpha_1 - \alpha_2)$$

so that C_{AB} is singular (equivalently C_{AB} does not have full rank) if and only if $\alpha_1 = \alpha_2$.

Example 2: Consider the following system of 2 carts connected by a spring

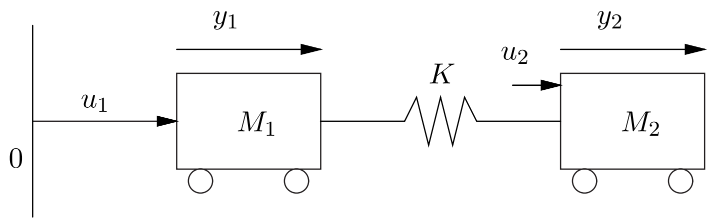

Figure 2.2: A coupled 2-cart system

Newton's law gives

$$\begin{aligned} M_1\ddot{y}_1 &= -K(y_1 - y_2) + u_1 \\ M_2\ddot{y}_2 &= -K(y_2 - y_1) + u_2 \end{aligned}$$

Let us recast the equations in state space form. Define $x = \begin{bmatrix} y_1 & \dot{y}_1 & y_2 & \dot{y}_2 \end{bmatrix}^T$. Then

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K}{M_1} & 0 & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & \frac{-K}{M_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{M_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{M_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The controllability matrix for this system is given by

$$C_{AB} = \begin{array}{c} \begin{bmatrix} 0 & 0 & \frac{1}{M_1} & 0 & 0 & 0 & \frac{-K}{M_1^2} & \frac{K}{M_1M_2} \\ \frac{1}{M_1} & 0 & 0 & 0 & \frac{-K}{M_1^2} & \frac{K}{M_1M_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{M_2} & 0 & 0 & \frac{K}{M_1M_2} & \frac{-K}{M_2^2} \\ 0 & \frac{1}{M_2} & 0 & 0 & \frac{K}{M_1M_2} & \frac{-K}{M_2^2} & 0 & 0 \end{bmatrix} \\ B \qquad\quad AB \qquad\quad A^2B \qquad\qquad A^3B \end{array}$$

Note that C_{AB} has 4 linearly independent columns so that it is full rank.

Before we discuss the main result of Chapter 2, let us review some geometric ideas from linear algebra. Suppose $\mathcal{V}$ is a subspace of R^n. The orthogonal complement of $\mathcal{V}$, denoted by $\mathcal{V}^\perp$, is the set of all vectors w such that $w^Tv = 0$ for all $v \in \mathcal{V}$.

Example: Suppose $\mathcal{V}$ is the $x-y$ plane, a subspace of R^3. Then $\mathcal{V}^\perp$ is the z-axis.

Some important properties of orthogonal complements are:

1. $\mathcal{V}^\perp$ is a subspace.
2. The only vector that lies in both $\mathcal{V}$ and $\mathcal{V}^\perp$ is the zero vector.
3. Every vector $x \in R^n$ can be uniquely decomposed as $x = u + w$ with $u \in \mathcal{V}$ and $w \in \mathcal{V}^\perp$. We refer to the decomposition as R^n is the **direct sum** of $\mathcal{V}$ and $\mathcal{V}^\perp$, written as $R^n = \mathcal{V} \oplus \mathcal{V}^\perp$.
4. $(\mathcal{V}^\perp)^\perp = \mathcal{V}$
5. Let $\mathcal{V}$ and $\mathcal{W}$ be subspaces of R^n. $\mathcal{V} = \mathcal{W}$ if and only if $\mathcal{V}^\perp = \mathcal{W}^\perp$, and $\mathcal{V} \subset \mathcal{W}$ if and only if $\mathcal{W}^\perp \subset \mathcal{V}^\perp$.

The following is the main result of this chapter. It shows how C_{AB} and W_t feature in the solution of the reachability problem.

Theorem 2.1: $\mathcal{R}(\mathcal{L}) = \mathcal{R}(W_t) = \mathcal{R}(C_{AB})$.

Proof: The theorem will be proved using the following 2 steps.

(1) Prove $\mathcal{R}(\mathcal{L}) = \mathcal{R}(W_t)$.

(2) Prove $\mathcal{R}(C_{AB}) = \mathcal{R}(W_t)$.

Step 1: Prove $\mathcal{R}(\mathcal{L}) = \mathcal{R}(W_t)$.

Using property (5) of orthogonal complements, proving $\mathcal{R}(\mathcal{L}) = \mathcal{R}(W_t)$ is equivalent to proving $\mathcal{R}(\mathcal{L})^\perp = \mathcal{R}(W_t)^\perp$.

(a) $\mathcal{R}(\mathcal{L})^\perp \subset \mathcal{R}(W_t)^\perp$: Let $v \in \mathcal{R}(\mathcal{L})^\perp$. Then for all $u \in \mathcal{U}$,

$$v^T \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = 0$$

This is true for all u if and only if

$$v^T e^{A\tau} B = 0, \text{ all } \tau,\ 0 \le \tau \le t.$$

But then

$$v^T W_t = v^T \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau = 0$$

so that $v \in \mathcal{R}(W_t)^\perp$.

(b) $\mathcal{R}(W_t)^\perp \subset \mathcal{R}(\mathcal{L})^\perp$: Let $v \in \mathcal{R}(W_t)^\perp$. Then, in particular,

$$v^T W_t v = 0.$$

But

$$\begin{aligned} v^T W_t v &= v^T \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau\, v \\ &= \int_0^t \| v^T e^{A\tau} B \|^2 \, d\tau \end{aligned}$$

Thus $v^T W_t v = 0$ holds if and only if $v^T e^{A\tau} B = 0$, $0 \le \tau \le t$. This implies $v \in \mathcal{R}(\mathcal{L})^\perp$

This completes Step 1 of the proof.

Note that in the proof, we have also shown that, for $t > 0$, $v^T e^{A\tau} B = 0$, $0 \leq \tau \leq t$ if and only if $v^T W_t = 0$.

Step 2: Prove $\mathcal{R}(C_{AB}) = \mathcal{R}(W_t)$
First note that we can express e^{At} in the form

$$e^{At} = \varphi_0(t)I + \varphi_1(t)A + ... + \varphi_{n-1}(t)A^{n-1} \tag{2.9}$$

for certain functions $\{\varphi_i(t)\}$. An outline of the proof of this result is given in the problem sets. Intutively, we can see that this is a consequence of the **Cayley-Hamilton Theorem**, which states:

If an $n \times n$ matrix A has characteristic polynomial $p(s)$ given by

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n$$

then

$$p(A) = A^n + p_1 A^{n-1} + \cdots + p_{n-1}A + p_n I = 0$$

Cayley-Hamilton Theorem implies $A^k, k \geq n$ is expressible as a linear combination of $A^j, 0 \leq j \leq n-1$. This means that the infinite series that defines e^{At} should be expressible in the form of (2.9).

We can now carry out the proof of Step 2. We first show

$$v^T C_{AB} = 0$$

if and only if for $t > 0$

$$v^T e^{A\tau} B = 0 \text{ for all } \tau, \; 0 \leq \tau \leq t$$

Suppose $v^T C_{AB} = 0$. Then

$$\begin{aligned} v^T e^{A\tau} B &= v^T[\varphi_0(\tau)I + ... + \varphi_{n-1}(\tau)A^{n-1}]B \\ &= v^T[B \;\; AB \; ... \; A^{n-1}B] \begin{bmatrix} \varphi_0(\tau)I \\ \varphi_1(\tau)I \\ \vdots \\ \varphi_{n-1}(\tau)I \end{bmatrix} \\ &= 0 \end{aligned}$$

Conversely, suppose $v^T e^{A\tau} B = 0$ for all $0 \leq \tau \leq t$. Setting $\tau = 0$ gives

$$v^T B = 0$$

For $k = 1, \ldots, n-1$, take the kth derivative of $v^T e^{A\tau} B$ with respect to τ and evaluate the result at $\tau = 0$. This gives successively

$$\begin{aligned} v^T AB &= 0 \\ v^T A^2 B &= 0 \\ &\vdots \\ v^T A^{n-1} B &= 0 \end{aligned}$$

so that

$$v^T C_{AB} = 0$$

Now in the proof of Step 1, we have shown that

$$v^T e^{A\tau} B = 0, \quad 0 \le \tau \le t$$

holds if and only if

$$v^T W_t = 0$$

holds. This allows us to conclude that

$$v^T C_{AB} = 0$$

if and only if

$$v^T W_t = 0$$

Therefore $v \in \mathcal{R}(C_{AB})^\perp$ if and only if $v \in \mathcal{R}(W_t)^\perp$. Hence $\mathcal{R}(C_{AB})^\perp = \mathcal{R}(W_t)^\perp$, which is equivalent to $\mathcal{R}(C_{AB}) = \mathcal{R}(W_t)$. This completes the proof of Step 2 and the proof of the Theorem.

Observe that since $\mathcal{R}(C_{AB})$ is independent ot t, Theorem 2.1 implies that the reachable set and reachability properties are in fact independent of t. We can therefore talk about controllability **without reference to t**. Furthermore, note that for an $n \times n$ matrix M, $\mathcal{R}(M) = R^n$ if and only if M is nonsingular, and for an $n \times nm$ matrix N, $\mathcal{R}(N) = R^n$ if and only $rank(N) = n$. Combining these observations and the Theorem, we obtain the following

Corollary: The following statements are equivalent:

(a) $\mathcal{R}_0 = R^n$

(b) W_t is nonsingular for any $t > 0$

(c) C_{AB} has rank n (full rank)

Since the linear system (2.1) is controllable if and only if $\mathcal{R}_0 = R^n$, the equivalence of statements (a) and (c) of the Corollary can be restated as the following important theorem.

Theorem 2.2: The linear system (2.1) is controllable if and only if Rank $[B \ AB \ \cdots \ A^{n-1}B] = n.$

Theorem 2.2 allows us to check the controllability property using given data A and B. It is a very important result.

Based on the Theorem 2.2, we are justified in saying:

The pair $(\mathbf{A}, \mathbf{B})$ **is controllable** if $Rank[B \ AB \ \cdots \ A^{n-1}B] = n.$

Suppose x_1 is reachable at time t from x_0. It is not difficult to write down explicitly he control that achieves the transfer from x_0 to x_1 at time t. In fact, a control input that achieves the transfer can be verified to be given by

$$u(\tau) = B^T e^{A^T(t-\tau)} \xi \tag{2.10}$$

where ξ is the solution of $W_t \xi = x_1 - e^{At} x_0$. If (A, B) is controllable, W_t^{-1} exists, and (2.10) can be written explicitly as

$$u(\tau) = B^T e^{A^T(t-\tau)} W_t^{-1} (x_1 - e^{At} x_0) \tag{2.11}$$

Note also that if (A, B) is controllable, the state transfer can be accomplished for any $t > 0$, no matter how small, since W_t is nonsingular for any $t > 0$. However, if t is small, W_t^{-1} will be large, and the control input to achieve the transfer will be large too.

2.2 Some Properties of Controllability

In this chapter, we discuss 2 properties of controllability: invariance under a change of basis and invariance under state feedback. We also introduce the controllable canonical form for single input systems, which will be very useful for the pole assignment problem discussed in the next chapter.

(1). **Invariance under change of basis:**

Recall that if x is a state vector, so is $V^{-1}x$ for any nonsingular matrix V. In fact, if we let $z = V^{-1}x$,

$$\begin{aligned} \dot{z} &= V^{-1}x \\ &= V^{-1}AVz + V^{-1}Bu \end{aligned}$$

so that with z as the state vector, the system matrices change from (A, B) to $(V^{-1}AV, V^{-1}B)$. We refer to this as a change of basis because if we let the columns of V form a new basis for R^n, z is then the representation of x in this new basis (See the Appendix to Chapter 2 for more details on change of basis). We call the transformation V featured in the change of basis a **similarity transformation**.

Theorem 2.3: (A, B) is controllable if and only if $(V^{-1}AV, V^{-1}B)$ is controllable for every nonsingular V.

Proof:

$$\begin{aligned} C_{V^{-1}AV,V^{-1}B} &= [V^{-1}B \;\; V^{-1}AVV^{-1}B \cdots] \\ &= V^{-1}[B \;\; AB \cdots] \\ &= V^{-1}C_{AB} \end{aligned}$$

Since

$$Rank(V^{-1}C_{AB}) = Rank(C_{AB})$$

the result is proved.

(2). **Invariance under state feedback:**

A control law of the form

$$u(t) = -Kx(t) + v(t) \tag{2.12}$$

with $v(t)$ a new input, is referred to as state feedback. The closed-loop system equation is given by

$$\dot{x} = (A - BK)x(t) + Bv(t)$$

It is an important property that controllability is also unaffected by state feedback.

Theorem 2.4: (A, B) is controllable if and only if $(A - BK, B)$ is controllable for all K.

A proof of Theorem 2.4 is discussed in the problem sets. Intuitively, the result seems reasonable since the state feedback law (2.12) can be reversed by setting $v = Kx + u$. Therefore, the ability to control the system using the input v should be the same as that of using input u.

2.3 Decomposition into Controllable and Uncontrollable Parts

If the pair (A, B) is not controllable, there is a particular basis in which the controllable and uncontrollable parts are displayed transparently. We illustrate the choice of basis and the computation involved using the circuit example in Section 2.1.

Example: We know from Section 2.1, that the circuit described by Figure 1 is not controllable if $R_1C_1 = R_2C_2$. Using the notation from Section 2.1, and setting $\alpha = \frac{1}{R_1C_1} = \frac{1}{R_2C_2}$, we can rewrite (2.5) as

$$\dot{v} = \begin{bmatrix} -(\alpha+\beta_1) & \beta_1 \\ \beta_2 & -(\alpha+\beta_2) \end{bmatrix} v + \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} v_i \tag{2.13}$$

The controllability matrix is given by

$$C_{AB} = \begin{bmatrix} \alpha & -\alpha^2 \\ \alpha & -\alpha^2 \end{bmatrix}$$

so that $s_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is a basis for $\mathcal{R}(C_{AB})$. Pick a second vector s_2 so that s_1 and s_2 form a basis for R^2, for example, $s_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Set

$$V = \begin{bmatrix} s_1 & s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

We then have

$$\begin{aligned} V^{-1}AV &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -(\alpha+\beta_1) & \beta_1 \\ \beta_2 & -(\alpha+\beta_2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\alpha & \beta_1 \\ -\alpha & -(\alpha+\beta_2) \end{bmatrix} \\ &= \begin{bmatrix} -\alpha & \beta_1 \\ 0 & -(\alpha+\beta_1+\beta_2) \end{bmatrix} \end{aligned}$$

and

$$V^{-1}B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

Note that the change of basis produces the following system equation

$$\dot{z} = \tilde{A}z + \tilde{B}u$$

with

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \tag{2.14}$$

$$\tilde{B} = V^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \tag{2.15}$$

The part corresponding to $(\tilde{A}_{11}, \tilde{B}_1)$, with state component z_1, is controllable, while the part corresponding to the $\tilde{A}_{22}$, with state component z_2, is uncontrollable, since the z_2 equation is decoupled from z_1, and z_2 is unaffected by u. In general, whenever C_{AB} is not full rank, we can find a basis so that in the new basis, A and B take the form given in (2.14) and (2.15), respectively, with $(\tilde{A}_{11}, \tilde{B}_1)$ controllable. The procedure is:

1. Find a basis for $\mathcal{R}(C_{AB})$. Denote the vectors in this basis by $\{v_1, v_2, \cdots, v_q\}$, where $q = rank(C_{AB})$.

2. Select an additional $n-q$ linearly independent vectors $v_{q+1} \cdots v_n$ so that $\{v_1, v_2, \cdots .v_n\}$ form a basis for R^n. Define the matrix V by
$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

3. Compute $\tilde{A} = V^{-1}AV$ and $\tilde{B} = V^{-1}B$. $\tilde{A}$ will take the form (2.14) and $\tilde{B}$ will take the form (2.15).

For more details, the appendix to Chapter 2 contains a formal proof that the above procedure yields $\tilde{A}$ and $\tilde{B}$ as described.

2.4 The PBH Test for Controllability

While we normally check controllability by determining the rank of C_{AB}, there is an alternative useful test for controllability, referred to as the PBH test. We state the result as a theorem.

Theorem 2.5 (PBH Test):
(A, B) is controllable if and only if $Rank[A - \lambda I \quad B] = n$ for all eigenvalues λ of A.

Proof: The statement is equivalent to:

(A, B) is not controllable if and only if rank$[A - \lambda I \;\; B] < n$ for <u>some</u> eigenvalue λ of A.

(i) We first prove $rank[A - \lambda I \;\; B] < n$ implies (A, B) is not controllable. Suppose rank$[A - \lambda I \;\; B] < n$ for some eigenvalue λ, possibly complex. Thus there exists a vector x, possibly complex such that
$$x^*[A - \lambda I \;\; B] = 0$$
where x^* is complex conjugate transpose of x. This results in
$$x^*A = \lambda x^*$$
and
$$x^*B = 0$$
Then
$$x^*AB = \lambda x^*B = 0$$
and
$$x^*A^kB = \lambda^k x^*B = 0$$
Thus $x^*[B \;\; AB ... A^{n-1}B] = 0$ so that $[\mathrm{Re}\, x^*][B \;\; AB ... A^{n-1}B] = 0$ and (A, B) is not controllable.

(ii) We now show (A, B) not controllable implies $rank[A - \lambda I \quad B] < n$ for some eigenvalue λ of A. Assume (A, B) is not controllable. By the results of Section 2.3, we can find a nonsingular V so that $(V^{-1}AV, V^{-1}B) = (\tilde{A}, \tilde{B})$, where $\tilde{A}$ is of the form (2.14) and $\tilde{B}$ is of the form (2.15), with $\tilde{A}_{11}$ $q \times q$ and $\tilde{A}_{22}$ $n - q \times n - q$. Now note that for any λ an eigenvalue of $\tilde{A}_{22}$,
$$Rank(\tilde{A}_{22} - \lambda I) < n - q$$
Hence
$$[\tilde{A} - \lambda I \;\; \tilde{B}] = \text{Rank} \begin{bmatrix} \tilde{A}_{11} - \lambda I & \tilde{A}_{12} & \tilde{B}_1 \\ 0 & \tilde{A}_{22} - \lambda I & 0 \end{bmatrix} < n$$

Next observe that

$$\begin{aligned} V^{-1}[A-\lambda I \quad B]\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} &= V^{-1}[AV-\lambda V \quad B] \\ &= [V^{-1}AV-\lambda I \quad V^{-1}B] \\ &= [\tilde{A}-\lambda I \quad \tilde{B}] \end{aligned}$$

Since V^{-1} and $\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}$ are both nonsingular,

$$\mathrm{Rank}[\tilde{A}-\lambda I \quad \tilde{B}] = \mathrm{Rank}[A-\lambda I \quad B] < n$$

for λ an eigenvalue of $\tilde{A}_{22}$. Finally, note that by the structure of $\tilde{A}$, an eigenvalue of $\tilde{A}_{22}$ is an eigenvalue of $\tilde{A}$. Since a change of basis does not change the eigenvalues, λ is also an eigenvalue of A. This concludes the proof that (A, B) not controllable implies there exists an eigenvalue of A, λ, such that $rank[A-\lambda I \quad B] < n$.

It is useful to note that $Rank[A-\lambda I \quad B] = n$ for all eigenvalues λ of A if and only if $Rank[A-\lambda I \quad B] = n$ for all complex numbers λ. This is because for λ not an eigenvalue of A, $Rank(A-\lambda I) = n$. We refer to the eigenvalue which causes $Rank[A-\lambda I \quad B] < n$ as an uncontrollable eigenvalue. We will see in Chapter 3 that such an eigenvalue is in some sense fixed and not movable.

The PBH test is especially useful for checking when a system is not controllable. We give a couple of examples to illustrate its use.

Example 1: Consider the following (A, B) pair

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This corresponds to the bridge circuit example withe the following parameter values: $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 2$, and $\beta_2 = 1$. We know that since $\alpha_1 = \alpha_2$, this system is not controllable. We can also verify that the controllability matrix is

$$\mathcal{C}_{AB} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

which is singular. Let us apply the PBH test. First $\det(sI-A) = s^2+5s+4$ so that the eigenvalues are -4 and -1. For the eignenvalue -1, PBH test gives

$$\begin{bmatrix} A-(-1)I & B \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

which has rank 2. On the other hand, for the eigenvalue -4, PHB test gives

$$\begin{bmatrix} A-(-4)I & B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

which has rank 1. By the PBH test, the system is not controllable and -4 is an uncontrollable eigenvalue.

Example 2: Consider the following (A, B) pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

By direct expansion along the first row of $(sI - A)$, we find

$$\begin{aligned}
\det(sI - A) &= \det \begin{bmatrix} s & -1 & 0 & 0 \\ 1 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ -2 & 0 & 2 & s \end{bmatrix} \\
&= s^2(s^2+2) + \det \begin{bmatrix} 1 & -1 & 0 \\ 0 & s & -1 \\ -2 & 2 & s \end{bmatrix} \\
&= s^2(s^2+2) + s^2 = s^4 + 3s^2
\end{aligned}$$

By inspection, we immediately see that with the eigenvalue 0, $Rank[A \quad B] < 4$. By the PBH test, this (A, B) pair is not controllable.

2.5 Controllable Canonical Form for Single-Input Controllable Systems

For single input systems, there is a special form of system matrices for which controllability always holds. This special form is referred to as the controllable canonical form. Using a lower case b to indicate explicitly that the input matrix is a column vector for a single input system, the controllable canonical form is given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & & -\alpha_1 \end{bmatrix} \tag{2.16}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.17}$$

It is easy to verify that the controllability matrix for this pair (A, b) always has rank n, regardless of the values of the coefficients α_j. Hence the name controllable canonical form. An A matrix taking the form given in (2.16) is referred to as a companion form matrix. It is straightforward to show that the characteristic polynomial of the companion form matrix is given by

$$\det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

The description of the controllable realization in Section 1.7 reflects in effect the properties of the pair (A, b) given by (2.16) and (2.17).

It will be seen in the next chapter that for applications to pole assignment in single-input systems, the controllable canonical form is particularly convenient for control design. To prepare for that discussion, we now show that if (A, b) is controllable, but not in controllable canonical form, we can always find a similarity transformation V so that $(V^{-1}AV, V^{-1}b)$ will be in controllable canonical form.

Consider the matrix

$$\begin{aligned} V &= [A^{n-1}b \;\; A^{n-2}b \;\; \cdots \;\; b] \begin{bmatrix} 1 & 0 & \ldots & & 0 \\ \alpha_1 & & & & \\ \alpha_2 & & & & \vdots \\ \vdots & & & & \\ \alpha_{n-2} & & & & 0 \\ \alpha_{n-1} & \alpha_{n-2} & \ldots & \alpha_1 & 1 \end{bmatrix} \\ &= [b \;\cdots\; A^{n-1}b] \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & 1 \\ \alpha_{n-2} & \cdots & \alpha_1 & 1 & 0 \\ \vdots & \iddots & 1 & 0 & 0 \\ \alpha_1 & \iddots & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \\ &= [v_1 \;\; \cdots \;\; v_n] \end{aligned}$$

By controllability, V^{-1} exists, so that its columns $v_1, ..., v_n$ form a basis of R^n.

Note that

$$\begin{aligned} v_1 &= A^{n-1}b + \alpha_1 A^{n-2}b + ... + \alpha_{n-1}b \\ v_2 &= A^{n-2}b + \alpha_1 A^{n-3}b + \ldots + \alpha_{n-2}b \\ &\vdots \\ v_{n-1} &= Ab + \alpha_1 b \\ v_n &= b \end{aligned}$$

and that

$$\begin{aligned} Av_1 &= A^n b + ... + \alpha_{n-1}Ab + \alpha_n b - \alpha_n b \\ &= -\alpha_n b \qquad \text{by the Cayley-Hamilton Theorem} \\ &= -\alpha_n v_n \\ Av_2 &= v_1 - \alpha_{n-1}v_n \\ &\vdots \\ Av_n &= v_{n-1} - \alpha_1 v_n \end{aligned}$$

Thus the matrix representation of A with respect to the basis $\{v_1 ... v_n\}$ looks like

$$[A]_v = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & & -\alpha_1 \end{bmatrix}$$

Similarly, the vector b looks like

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = [b]_v$$

But $[A]_v$ and $[b]_v$ are then related to the original matrices through

$$\begin{aligned} [A]_v &= V^{-1}AV \\ [b]_v &= V^{-1}b \end{aligned}$$

so that they are related by a similarity transformation. Thus the new system $z(t) = V^{-1}x(t)$ will satisfy an equation of the form

$$\dot{z} = A_c z + b_c u$$

with (A_c, b_c) in controllable canonical form.

Appendix to Chapter 2: Change of Basis and System Decomposition

Let us first quickly review the operation of a change of basis in linear algebra and how it affects the representation of vectors and linear transformations.

Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ and $\mathcal{V} = \{v_1, \ldots, v_n\}$ be two bases in R^n. Any vector $x \in R^n$ can be represented with respect to either the basis $\mathcal{E}$ or the basis $\mathcal{V}$. Thus we can write

$$x = \sum_{i=1}^{n} \xi_i e_i = \sum_{i=1}^{n} \eta_i v_i$$

The coefficient ξ_i is the ith coordinate of x with respect to the $\mathcal{E}$ basis. To emphasize this dependence on the basis, we write

$$[x]_{\mathcal{E}} = [\xi_1\, \xi_2\, \ldots\, \xi_n]^T$$

Now let A be a linear transformation mapping R^n to R^n. Its action on on the basis vector e_i can be represented by

$$Ae_i = \sum_{k=1}^{n} a_{ki} e_k$$

The coefficients $\{a_{ij}\}$ are then the ijth element of the matrix reprentation of A with respect to the basis $\mathcal{E}$. We write this as $[A]_{\mathcal{E}}= \{a_{ij}\}$. Similarly, if

$$Av_i = \sum_{k=1}^{n} \beta_{ki} v_k$$

then $[A]_{\mathcal{V}}= \{\beta_{ij}\}$.

We now describe the operation of a change of basis. Suppose we wish to change the basis for R^n from $\mathcal{E}$ to $\mathcal{V}$. Define the linear transformation P by

$$Pe_i = v_i = \sum_{k=1}^{n} \alpha_{ki} e_k$$

so that $\{\alpha_{ij}\}$ is the matrix of P in the basis $\mathcal{E}$. Then

$$x = \sum_{i=1}^{n} \xi_i e_i = \sum_{i=1}^{n} \eta_i P e_i = \sum_{k=1}^{n} \sum_{i=1}^{n} \eta_i \alpha_{ki} e_k = \sum_{k=1}^{n} \left[\sum_{i=1}^{n} \alpha_{ki} \eta_i \right] e_k$$

Hence we have

$$\xi_k = \sum_{i=1}^{n} \alpha_{ki} \eta_i$$

which can be written as

$$[x]_{\mathcal{E}} = [P]_{\mathcal{E}} [x]_{\mathcal{V}}$$

or

$$[x]_{\mathcal{V}} = [P]_{\mathcal{E}}^{-1} [x]_{\mathcal{E}}$$

For linear transformations, we have

$$Av_i = \sum_{k=1}^{n} \beta_{ki} v_k = \sum_{k=1}^{n} \beta_{ki} P e_k = \sum_{k=1}^{n} \beta_{ki} \sum_{j=1}^{n} \alpha_{jk} e_j$$

but also

$$Av_i = APe_i = A \sum_{k=1}^{n} \alpha_{ki} e_k = \sum_{k=1}^{n} \alpha_{ki} \sum_{j=1}^{n} a_{jk} e_j$$

On comparing the two representations for Av_i, we get

$$\sum_{k=1}^{n} \alpha_{jk} \beta_{ki} = \sum_{k=1}^{n} a_{jk} \alpha_{ki}$$

In matrix notation, this corresponds to

$$[P]_{\mathcal{E}} [A]_{\mathcal{V}} = [A]_{\mathcal{E}} [P]_{\mathcal{E}}$$

so that

$$[A]_{\mathcal{V}} = [P]_{\mathcal{E}}^{-1} [A]_{\mathcal{E}} [P]_{\mathcal{E}}$$

We are now ready to describe the details of system decomposition for an uncontrollable pair (A, B), described in Section 2.3.

Let Rank$[B \;\; AB ... A^{n-1}B] = q < n$, and let $\{v_1 ... v_q\}$ be a basis for $Range[B \;\; AB ... A^{n-1}B] = \mathcal{M}$. We can pick additional basis vectors $\{v_{q+1}, ..., v_n\}$ so that $\{v_1, \cdots, v_q, v_{q+1}, \cdots, v_n\}$ form a basis for R^n. Now observe that by the Cayley-Hamilton Theorem, $A\mathcal{M} \subset \mathcal{M}$. Hence for $1 \leq i \leq q$,

$$Av_i = \sum_{j=1}^{q} \alpha_{ji} v_j$$

for some scalars α_{ji}, $j = 1, ..., q$. With respect to the basis $\mathbf{V} = \{v_1, ..., v_n\}$, A takes the form

$$[A]_{\mathbf{V}} = \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

Similarly the ith column of B,

$$b_i = \sum_{j=1}^{q} \beta_{ji} v_j$$

Thus, we obtain

$$V^{-1}AV = \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

and

$$V^{-1}B = \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

where $\tilde{A}_{11}$ is a $q \times q$ matrix and $\tilde{B}_1$ is a $q \times m$ matrix. We claim that $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable. For,

$$\text{Rank}[\tilde{B} \quad \tilde{A}\tilde{B} \; ... \; \tilde{A}_{n-1}\tilde{B}] = \text{Rank} \begin{bmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \dots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & 0 & \dots & 0 \end{bmatrix} = q$$

But by the Cayley-Hamilton theorem, for $k \geq 0$, $\tilde{A}_{11}^{q+k}\tilde{B}_1$ is linearly dependent on $\tilde{A}_{11}^{j}\tilde{B}_1$, $j = 0, ..., q-1$. Hence

$$\text{Rank}[\tilde{B}_1 \quad \tilde{A}_{11}\tilde{B}_1 \; ... \; \tilde{A}_{11}^{q-1}\tilde{B}_1] = q$$

so that $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.