Chapter 3

POLE ASSIGNMENT FOR LINEAR SYSTEMS

3.1 Introduction

In the previous chapters, we have studied analysis of linear systems and structural properties such as stability, and controllability. Beginning with this chapter, we study the design of control laws. We confine ourselves to time-invariant systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \end{aligned} \tag{3.1}$$

and we assume that the state $x(t)$ is available. A natural control law for (3.1) is to use state feedback

$$u = -Kx(t) \tag{3.2}$$

The closed-loop system under (3.2) is then

$$\dot{x} = (A - BK)x(t) \tag{3.3}$$

The closed-loop dynamics is completely determined by $(A - BK)$, and the stability of the closed-loop system as well as the rate of regulation of x to zero is determined by the eigenvalues of $(A - BK)$, which we shall call the poles of the closed-loop system. In particular, the system (3.3) is (asymptotically) stable if and only if all eigenvalues of (A-BK) lie in $Re\ s < 0$. So, as far as our ability to regulate (3.1) using (3.2) is concerned, we need to know how much control we would have over the eigenvalues of $(A - BK)$ for a given (A, B) pair. The problem of finding K to achieve a prescribed set of eigenvalues for $(A - BK)$ is called the pole assignment problem.

To facilitate the subsequent discussion, let us first formulate the problem in a more precise form. The eigenvalues of $A - BK$ are just the roots of the characteristic polynomial of $A - BK$, which we denote by $p_K(s) = \det(sI - A + BK)$. $p_K(s)$ is a monic polynomial of degree n with real coefficients. Specifying the poles of the closed-loop system to be $\lambda_1, ..., \lambda_n$ where a complex λ_i is included if and only if its complex conjugate λ_i^* is also included, is equivalent to specifying the nth degree monic polynomial with real coefficients $r(s) = (s - \lambda_1)(s - \lambda_2)...(s - \lambda_n)$. The pole assignment problem can then be formulated as follows:

Pole Assignment Problem

Given an nth degree monic polynomial with real coefficients $r(s)$, find a matrix K such that $p_K(s) = \det(sI - A + BK) = r(s)$.

If for every given $r(s)$, there exists a K such that $p_K(s) = r(s)$, we say that the closed-loop poles can be arbitrarily assigned. In this chapter, we shall provide necessary and sufficient conditions for the solvability of the pole assignment problem, and give a constructive procedure for finding K.

3.2 Pole Assignment for Single-Input Systems

We first solve the pole assignment problem for single-input systems of the form

$$\dot{x} = Ax + bu$$

The control law is of the form

$$u = -k^T x \tag{3.4}$$

for some **column** vector k, with closed-loop system given by

$$\dot{x} = (A - bk^T)x \tag{3.5}$$

The solution of this problem rests on two observations:

(i) The eigenvalues of $A - bk^T$ are invariant under similarity transformation, i.e., $\det(sI - A) = \det(sI - V^{-1}AV)$. Let $z(t) = V^{-1}x(t)$, V nonsingular. Then

$$\dot{z} = V^{-1}AVz + V^{-1}bu \tag{3.6}$$

Suppose we now solve the pole assignment problem for (3.6), i.e., for a given monic polynomial $r(s)$ whose roots appear in complex conjugate pairs, there exists a vector k_c such that

$$\det(sI - V^{-1}AV + V^{-1}bk_c^T) = r(s) \tag{3.7}$$

Then since

$$\begin{aligned} \det(sI - V^{-1}AV + V^{-1}bk_c^T) &= \det V^{-1}(sI - A + bk_c^T V^{-1})V && (3.8) \\ &= \det(sI - A + bk_c^T V^{-1}) && (3.9) \end{aligned}$$

we find that the pole assignment problem for (3.4) is solved by taking

$$k^T = k_c^T V^{-1} \tag{3.10}$$

(ii) We know how to solve the pole assignment problem if (A, b) were in controllable canonical form,

$$A = A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & & -\alpha_1 \end{bmatrix}, \quad b = b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

This is because for any vector k_c

$$k_c = \begin{bmatrix} k_{c,n} \\ k_{c,n-1} \\ \vdots \\ k_{c,1} \end{bmatrix}$$

we find by direct computation that

$$A_c - b_c k_c^T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & & \ldots & 1 \\ -(\alpha_n + k_{c,n}) & & \ldots & & -(\alpha_1 + k_{c,1}) \end{bmatrix}$$

Thus

$$\det(sI - A_c + b_c k_c^T) = s^n + (\alpha_1 + k_{c,1})s^{n-1} + \ldots + (\alpha_{n-1} + k_{c,n-1})s + (\alpha_n + k_{c,n})$$

which can be made into any monic nth degree $r(s)$ with real coefficients, $r(s) = s^n + r_1 s^{n-1} + \ldots + r_n$ by simply choosing

$$k_c = \begin{bmatrix} r_n - \alpha_n \\ r_{n-1} - \alpha_{n-1} \\ \vdots \\ r_1 - \alpha_1 \end{bmatrix} \tag{3.11}$$

We can now put (i) and (ii) together. We know from Chapter 2 that if (A, b) is controllable, there exists a nonsingular transformation V such that $(V^{-1}AV, V^{-1}b) = (A_c, b_c)$, the controllable canonical form.

We therefore arrive at the following result.

Proposition 3.1: Assume (A, b) is controllable. Let the desired closed-loop characteristic polynomial be given by $r(s)$, a monic polynomial of nth degree with real coefficients. Then the following vector k solves the pole assignment problem of achieving $r(s)$ as the closed-loop system characteristic polynomial:

$$k^T = [(r_n - \alpha_n) \quad (r_{n-1} - \alpha_{n-1}) \cdots (r_1 - \alpha_1)]V^{-1} \tag{3.12}$$

where

$$\begin{aligned} V &= [A^{n-1}b \quad A^{n-2}b \ldots b] \begin{bmatrix} 1 & 0 & & \ldots & 0 \\ \alpha_1 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \\ & & & & 0 \\ \alpha_{n-1} & & \ldots & \alpha_1 & 1 \end{bmatrix} \\ &= [b \ \cdots \ A^{n-1}b] \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & 1 \\ \alpha_{n-2} & \cdots & \alpha_1 & 1 & 0 \\ \vdots & \iddots & 1 & 0 & 0 \\ \alpha_1 & \iddots & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \end{aligned} \tag{3.13}$$

and α_i's are the coefficients of the characteristic polynomial of A, (i.e. $\det(sI - A) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$).

Example

Let (A, b) be given by

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

It is easy to verify that A is unstable. Suppose the control design is based on pole assignment using state feedback with the desired closed-loop poles at -1, $-1 \pm i$.

(i) Check controllability of (A, b).

$$C_{Ab} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Since C_{Ab} is nonsingular, the problem has a solution.

(ii) Find the characteristic polynomial of A.

$$\det(sI - A) = \det \begin{bmatrix} s & 1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & s \end{bmatrix} = s^3 + s$$

(iii) Determine the transformation V

$$V = C_{Ab} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(iv) Determine the desired closed-loop characteristic polynomial from closed-loop poles

$$r(s) = (s+1)[(s+1)^2 + 1] = (s+1)(s^2 + 2s + 2) = s^3 + 3s^2 + 4s + 2$$

(v) Determine the gain k_c to be used if (A, b) were in controllable canonical form.

$$k_c^T = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$$

(vi) Find the required feedback gain k.

$$k^T = k_c^T V^{-1} = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \end{bmatrix}$$

(vii) Verify the closed-loop system has the desired poles.

$$A_{cl} = A - bk^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$$

which indeed has eigenvalues at -1, $-1 \pm i$.

There is an alternative formula, called Ackermann's formula, which can also be used to determine the desired (unique) feedback gain k. A sketch of the proof of Ackermann's formula can be found in K. Ogata, *Modem Control Engineering*.

Ackermann's Formula:

$$k^T = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} C_{Ab}^{-1} r(A)$$

where $r(A) = A^n + r_1 A^{n-1} + \cdots + r_n I$. Applying Ackermann's formula to the above example gives:

$$C_{Ab}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} r(A) &= A^3 + 3A^2 + 4A + 2I \\ &= \begin{bmatrix} -1 & -3 & -3 \\ 3 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

and

$$k^T = \begin{bmatrix} 1 & 3 & 3 \end{bmatrix}$$

the same result as before.

The pole assignment problem for a single-input controllable system is relatively straightforward to solve. The pole assignment problem for multivariable systems, to be presented in the next section, is considerably harder. Interestingly enough though, its solution relies on the solution of the single-input case.

3.3 Pole Assignment for Multivariable Systems

We shall now extend the pole assignment result for single-input systems developed in the previous section to the multivariable case. The generalization rests on the following result, which uses state feedback to convert the multivariable pole assignment problem back to the single-input case.

Theorem 3.1: Let (A, B) be controllable, and let $b_1, ..., b_m$ be the columns of the B matrix. For each i such that $b_i \neq 0$ there exists a $m \times n$ matrix K_i such that $(A - BK_i, b_i)$ is controllable.

Proof: Without loss of generality, let $i = 1$. By controllability, the matrix

$$U = [b_1 \quad Ab_1 ... A^{n-1}b_1 \quad b_2 ... A^{n-1}b_2 ... b_m ... A^{n-1}b_m]$$

has rank n. We now look for the first n linearly independent columns in the matrix U, giving rise to a matrix Q of the form:

$$Q = [b_1 \quad Ab_1 ... A^{\nu_1 - 1}b_1 \quad b_2 \quad Ab_2 ... A^{\nu_2 - 1}b_2 ... b_m \quad Ab_m ... A^{\nu_m - 1}b_m]$$

in which some of the b_i's may be missing (the corresponding $\nu_i = 0$). By controllability

$$\sum_{i=1}^{m} \nu_i = n$$

We now associate the following matrix S with the above Q matrix:

$$\begin{array}{ccccccccccccc} Q & = & [b_1 & ... & A^{\nu_1-1}b_1 & b_2 ... & A^{\nu_2-1}b_2 & \dots & A^{\nu_{m-1}-1}b_{m-1} & b_m & ...A^{\nu_m-1}b_m &] \\ & & & \updownarrow & \updownarrow \ \nu_1 - \text{th column} & \updownarrow & \updownarrow \ (\nu_1+\nu_2)\text{th column} & \updownarrow & \updownarrow \ (\nu_1+\nu_2+\nu_{m-1})\text{th column} & \updownarrow & & \\ S & = & [0 & ...0 & e_2 & 0...0 & e_3 \ ... & 0 & e_m & 0 & ...0 &] \end{array}$$

where e_i is a $m \times 1$ vector with the only nonzero element being 1 in the ith position.

Let $K_1 = -SQ^{-1}$ or $K_1 Q = -S$. By the nonsingularity of Q, such a K_1 is well-defined. We claim $(A - BK_1, b_1)$ is controllable. For,

$$\begin{array}{cccc} K_1 b_1 = 0 & K_1 b_2 = 0 & & K_1 b_m = 0 \\ K_1 A b_1 = 0 & & & \\ \vdots & \vdots & \dots & \vdots \\ K_1 A^{\nu_1 - 1} b_1 = -e_2 & K_1 A^{\nu_2 - 1} b_2 = -e_3 & & K_1 A^{\nu_m - 1} b_m = 0 \end{array}$$

Thus

$$\begin{aligned}
b_1 &= b_1 \\
(A-BK_1)b_1 &= Ab_1 \\
(A-BK_1)^{\nu_1}b_1 &= (A-BK_1)A^{\nu_1-1}b_1 && = A^{\nu_1}b_1 + Be_2 = A^{\nu_1}b_1 + b_2 \\
& && = b_2 + \text{lin. comb. of prev. columns} \\
(A-BK_1)^{\nu_1+1}b_1 &= (A-BK_1)(A^{\nu_1}b_1 + b_2) && = Ab_2 + \text{lin. comb. of prev. columns} \\
(A-BK_1)^{\nu_1+\ldots+\nu_m-1}b_1 &= A^{\nu_m-1}b_m + \text{lin. comb. of} && \text{prev. columns}
\end{aligned}$$

Hence all the above vectors are linearly independent by definition of the ν_i's, so that $[b_1 \ \ (A-BK_1)b_1 \ldots (A-BK_1)^{n-1}b_1]$ is nonsingular. We conclude that

$$\dot{x} = (A-BK_1)x + b_1 v \tag{3.14}$$

is controllable.

The pole assignment problem for the linear multivariable case has now been reduced to the single-input case. The complete procedure is the following:

Let

$$\dot{x} = Ax + Bu \tag{3.15}$$

be controllable with $b_1 \neq 0$. Construct K_1 such that (3.14) is controllable. Find k_1 such that with $v = -k_1^T x$, (3.14) has the pre-assigned poles. The closed-loop system is then

$$\begin{aligned}
\dot{x} &= (A - BK_1 - b_1 k_1^T)x \\
&= [A - B(K_1 + e_1 k_1^T)]x
\end{aligned} \tag{3.16}$$

(3.16) may thus be obtained from (3.15) by letting

$$u = -Kx = -(K_1 + e_1 k_1^T)x \tag{3.17}$$

Visualizing the Construction of Q and S:

Although the construction of the Q and S matrices look complex, in fact it is not difficult to visualize. We illustrate with an example.

Consider the linear system with

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The states x_1 and x_2 are not coupled to those of x_3 and x_4. Inputs u_1 and u_2 can only affect x_1 and x_2, while u_3 can only affect x_3 and x_4. So the system is not controllable with respect to only one column of B. However, it is straightforward to check that (A, B) is controllable.

1. Contruction of the Q matrix:

Begin with b_1. We get

$$\begin{bmatrix} b_1 & Ab_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

These 2 columns are linearly independent.

$$A^2 b_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

is linearly dependent on b_1 and Ab_1. It will therefore not be included in the Q matrix. Once $A^2 b_1$ is thrown out, all $A^k b_1,\ k > 2$ are also thrown out.

The next column to consider is b_2. But note b_2 is linearly dependent on b_1 and Ab_1, the columns that have already been chosen. **(You always check if a candidate column is linearly independent of the columns that have already been included in Q.)** Hence b_2 is **not** included in Q, nor will $A^k b_2,\ k \geq 1$.

The last column to consider is b_3. It is linearly independent of b_1 and Ab_1, so it is accepted as the 3rd column of Q. Finally Ab_3 is linearly independent of b_1, Ab_1 and b_3, and forms the last column of Q.

$$Q = \begin{bmatrix} b_1 & Ab_1 & b_3 & Ab_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

2. Construction of the S matrix:

The S matrix consists as its columns various unit vectors of R^m, m being the number of inputs. In this example, $m = 3$. Where are these unit vectors placed?

The easiest way to see this is to line up the Q and the S matrix, with Q on top. They both have n columns, but S has m rows. The S matrix currently is **empty**, with its elements to be determined.

Step (i):

You look from left to right in the Q matrix, beginning with its first column b_1. At some point, we get to a new column of B being introduced in Q. In the example, it is b_3 in the 3rd column. Note the subscript that specifies the column of B, namely 3.

Step (ii):

Slide down to the S matrix. You are sitting in the 3rd column position.

Step (iii):

Move to the left by 1 column. You are now in the 2nd column position of S.

Step (iv):

A unit vector is placed in the matrix S for this column. The unit vector has the same subscript the one you have in Q just before you slide down, i.e., e_3, the 3rd unit vector in R^3 should be placed in the 2nd column of S.

Step (v) (Climb back to Q and repeat):

Now climb back to the column of Q from which you slid down, i.e., back to the 3rd column of Q. Continue your journey to the next column of Q until you hit another new column of B. If you hit another one, repeat the sliding down to S, move to the left by 1 column, and placing a unit vector. Do this until you have gone through all the columns of Q.

In the example, after b_1, there is only 1 new column of B introduced in Q. Hence there is only 1 unit vector introduced in S.

Step (vi):

Fill the rest of the entries of S with 0.

The S matrix for the examples is then

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Here is how it looks pictorially for the example:
Start with

$$Q = \begin{bmatrix} b_1 & Ab_1 & b_3 & Ab_3 \end{bmatrix}$$

$$S = [\qquad\qquad\qquad\qquad]$$

Go across the columns of Q until you arrive at a new column of B being introduced:

$$Q = \begin{bmatrix} b_1 & Ab_1 & b_3 & Ab_3 \end{bmatrix}$$

(slide down to the S matrix)

$$S = [\qquad\qquad\quad | \quad\;\;]$$

(move 1 column to the left)

$$S = [\qquad | \qquad\qquad\quad]$$

(place the unit vector e_3)

$$S = [\qquad e_3 \qquad\qquad]$$

(write it out)

$$S = \begin{bmatrix} & 0 & & \\ & 0 & & \\ & 1 & & \end{bmatrix}$$

Go back to 3rd column of Q (you slid down from there) and continue to the next column of Q until you reach the end. No new columns of B are found in the rest of Q.

Fill the rest of S with 0:

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Detailed Pole Assignment Example 1:

Let (A, B) be given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

From the structure of A and B, it is clear that the system is not controllable from either the first or the second column of B alone. However, the controllability matrix is given by

$$C_{AB} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 4 & 0 \end{bmatrix}$$

in which columns 1, 2, and 4 are independent, so that the system is controllable. Suppose the desired closed loop poles are at -1, -2, and -3, so that the desired closed loop characteristic polynomial is given by

$$r(s) = (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6$$

To find the desired feedback gain K to achieve these closed loop poles, we proceed as follows.

(i) Find K_1 such that $((A - BK_1), b_1)$ is controllable where b_i is the ith column of B, $i = 1, 2$. We form the Q matrix:

$$Q = \begin{bmatrix} b_1 & b_2 & Ab_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The S matrix is given by

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The inverse of Q is readily determined to be

$$Q^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

so that

$$\begin{aligned} K_1 &= -SQ^{-1} = -\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

This results in

$$\begin{aligned} A_1 &= A - BK_1 \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

The controllability matrix for (A_1, b_1) is

$$C_{A_1 b_1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

which has rank 3, as expected.

(ii) Now find k_1 so that $A_1 - b_1 k_1^T$ has the desired closed loop poles. The characteristic polynomial of A_1 is $(s-1)^2(s-2) = s^3 - 4s^2 + 5s - 2$. The desired V matrix transforming (A_1, b_1) to controllable canonical form is

$$\begin{aligned} V &= C_{A_1 b_1} \begin{bmatrix} 5 & -4 & 1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \end{aligned}$$

so that

$$V^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

This gives

$$\begin{aligned} k_1^T &= \begin{bmatrix} 6-(-2) & 11-5 & 6-(-4) \end{bmatrix} V^{-1} \\ &= \begin{bmatrix} 8 & 6 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 26 & 10 \end{bmatrix} \end{aligned}$$

We verify that the closed loop system matrix $A_1 - b_1 k_1^T$ has the required closed loop poles

$$\begin{aligned} A_1 - b_1 k_1^T &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 24 & 26 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -24 & -26 & -8 \end{bmatrix} \end{aligned}$$

which indeed has eigenvalues at $-1, -2, -3$.

(iii) Finally, the overall feedback gain is

$$\begin{aligned} K &= K_1 + e_1 k_1^T \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 24 & 26 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 26 & 10 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

Detailed Pole Assignment Example 2:

We consider a 4-dimensional system with 3 inputs. Let (A, B) be given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

From the block upper triangular structure of A, we immediately see that A has eigenvalues $1, 0, 0, 0$ and hence is unstable. We would like to find a feedback gain K such that the closed loop poles are at $-1, -2, -3, -4$. We proceed systematically.

(a) Find the matrix Q and verify controllability:

Through the construction of the Q matrix, we can verify controllability at the same time. First compute the controllability matrices of $(A, b_i),\ i = 1, 2, 3$. We have

$$\mathcal{C}_{Ab_1} = \begin{bmatrix} 1 & 1 & & \\ 0 & 0 & & \\ 0 & 0 & \cdots & \\ 0 & 0 & & \end{bmatrix}$$

We stopped after finding the first 2 columns of $\mathcal{C}_{Ab_1}$ since the second column is linearly dependent on the first. The only column from $\mathcal{C}_{Ab_1}$ that will appear in Q is the first column, as the remaining columns cannot contribute to the matrix Q. Similarly,

$$\mathcal{C}_{Ab_2} = \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 1 & 0 & \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \end{bmatrix}$$

so that $\mathcal{C}_{Ab_2}$ will contribute 2 columns to Q (the first 2). Since there is only b_3 left, if the system is controllable, Q must be given by

$$Q = \begin{bmatrix} b_1 & b_2 & Ab_2 & b_3 \end{bmatrix}$$

and must be nonsingular. So we can check controllability by verifying if this Q is nonsingular or not. We have

$$Q = \begin{bmatrix} b_1 & b_2 & Ab_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which indeed is nonsingular.

(b) Find the matrix S:

Using the visual construction, we easily find S to be given by

$$S = \begin{bmatrix} e_2 & 0 & e_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c) Solve for K_1 in the equation $K_1 Q = -S$:

We know K_1 has the same dimension as S. So let

$$K_1 = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \end{bmatrix}$$

We then have

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

It is easy to see that $k_{21} = -1$, $k_{32} = -1$. The other entries in K_1 are all zero. Thus

$$K_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(d) Find $A_1 = A - BK_1$ and its characteristic polynomial:

Computation gives

$$A_1 = A - BK_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

It is straightforward to find $\det(sI - A_1) = \alpha(s) = s^4 - s^3$. We have now the pair (A_1, b_1) controllable. The remaining calculations consist of doing the desired pole assignment using the pair (A_1, b_1).

(e) Find the controllability matrix of (A_1, b_1):

$$\mathcal{C}_{A_1 b_1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(f) Find the matrix V which brings (A_1, b_1) to controllable canonical form:

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(g) Find the desired characteristic polynomial and the gain k_c^T:

The desired characteristic polynomial is $r(s) = (s+1)(s+2)(s+3)(s+4)$. Expanding we get

$$r(s) = s^4 + 10s^3 + 35s^2 + 50s + 24$$

Hence

$$k_c^T = \begin{bmatrix} r_4 - \alpha_4 & r_3 - \alpha_3 & r_2 - \alpha_2 & r_1 - \alpha_1 \end{bmatrix} = \begin{bmatrix} 24 & 50 & 35 & 11 \end{bmatrix}$$

(h) Determine the gain k from the equation $k^T V = k_c^T$:

Let $k^T = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$. Then

$$\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 24 & 50 & 35 & 11 \end{bmatrix}$$

This immediately gives $k_1 = 11$, $k_3 = 35$, $k_4 = 24$, and $k_2 + k_4 = 50$, yielding $k_2 = 26$. Hence

$$k^T = \begin{bmatrix} 11 & 26 & 35 & 24 \end{bmatrix}$$

(i) Finally, we assemble $K = K_1 + e_1 k^T$:

$$K = K_1 + e_1 k^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 11 & 26 & 35 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 26 & 35 & 24 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

3.4 Pole Assignability Implies Controllability

We have proved that controllability is sufficient for pole assignability. We now prove the converse so that controllability is also necessary for pole assignability. Let λ_i, $i = 1, ..., n$ be a set of n distinct real numbers, none of which is an eigenvalue of A. By the assumption of pole assignability, there exists a F (corresponding to $-K$ in the previous sections) such that

$$(A + BF)x_i = \lambda_i x_i$$

where x_i are the eigenvectors associated with λ_i. Thus

$$(\lambda_i I - A)^{-1} BF x_i = x_i \qquad i = 1, ..., n$$

But $(\lambda_i I - A)^{-1} = \sum_{j=0}^{n-1} \rho_j(\lambda_i) A^j$ for some suitable scalar-valued functions ρ_j, $j = 0, ..., n-1$. Hence

$$\sum_{j=0}^{n-1} \rho_j(\lambda_i) A^j BF x_i = x_i \qquad i = 1, ..., n$$

The columns of the left hand side matrix are linear combinations of the columns of $[B \;\; AB ... A^{n-1}B]$. By linear independence of x_i, we conclude Rank$[B \;\; AB \cdots A^{n-1}B] = n$, i.e. (A, B) is controllable.

Combining the above results, we can state the following theorem, usually referred to as the **Pole Assignment Theorem**.

Theorem 3.2: The closed-loop poles of (3.1) can be arbitrarily assigned if and only if (A, B) is controllable.

3.5 Stabilizability

The results of the previous sections show that controllability is equivalent to pole assignability. In applications, however, only a certain portion of the system may be controllable. We would like to see how this affects the ability to assign poles of the closed-loop system.

Recall from Chapter 2 that there exists a nonsingular matrix V such that by letting $z = V^{-1}x$, we can transform the system into

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} u \tag{3.18}$$

We see that the z_2 component is decoupled from z_1 and we have no control whatsoever over z_2. Hence the eigenvalues of $\tilde{A}_{22}$ will be unchanged regardless of what feedback law we choose. On the other hand, since $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable, by suitable choice of feedback

$$u = -\tilde{K}z = -\tilde{K}_1 z_1 - \tilde{K}_2 z_2$$

we can make $\tilde{A}_{11} - \tilde{B}_1\tilde{K}_1$ in the closed-loop system

$$\dot{z} = \begin{bmatrix} \tilde{A}_{11} - \tilde{B}_1\tilde{K}_1 & \tilde{A}_{12} - \tilde{B}_1\tilde{K}_2 \\ 0 & \tilde{A}_{22} \end{bmatrix} z$$

to have any desired set of eigenvalues. Thus we see that we can, by feedback, modify at will q poles of the closed-loop system, corresponding to those in $\tilde{A}_{11}$, but $n - q$ poles, corresponding to those in $\tilde{A}_{22}$, will remain fixed. The eigenvalues which can be modified by feedback are called controllable eigenvalues.

These considerations suggest the following definition, which describes a weaker property than pole assignability.

Definition:

The system (3.1) is said to be stabilizable if there exist a matrix K such that for $u = -Kx$, the closed-loop system (3.3) is stable.

It is now evident from the decomposition (3.18) that the system is stabilizable if and only if $\tilde{A}_{22}$ is (asymptotically) stable. We informally describe this result as:

The system (3.1) is stabilizable if and only if all the unstable modes are controllable.

Example:

Let (A, b) be given by

$$A = \begin{bmatrix} -3 & 1 & 4 \\ -3 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The open loop system is unstable, as the eigenvalues of A are $1, 1, -2$. The controllability matrix is

$$C_{Ab} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

which has rank 2. To see if the system is stabilizable, set

$$V = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Letting $z = V^{-1}x$ gives the system

$$\begin{aligned} \dot{z} &= V^{-1}AVz + V^{-1}bu \\ &= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \end{aligned}$$

We identify

$$A_{11} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \quad b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which is controllable. The uncontrolled eigenvalue is -2. Hence the system is stabilizable.

There is a convenient test for stabilizability, analogous to the PBH test for controllability.

PBH Test for Stabilizability

The pair (A, B) is stabilizable if and only if for all λ that is an eigenvalue of A with $\operatorname{Re}\lambda \geq 0$, $\operatorname{Rank}[A - \lambda I \;\; B] = n$.

Since $\operatorname{Rank}(A - \lambda I) = n$ for all λ which is not an eigenvalue of A, the PBH test for stabilizability is equivalent to saying

The pair (A, B) is stabilizable if and only if for all complex λ with $\operatorname{Re}\lambda \geq 0$, $\operatorname{Rank}[A - \lambda I \;\; B] = n$.

Applying the PBH test to the above example, note that the only unstable eigenvalue (repeated in this case) is $\lambda = 1$. Now

$$[A - I \;\; B] = \begin{bmatrix} -4 & 1 & 4 & 0 \\ -3 & 0 & 3 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

which has rank 3. Hence (A, B) is stabilizable.