

Chapter 4

OBSERVABILITY, OBSERVERS AND FEEDBACK COMPENSATORS

In Chapter 3, we have studied the design of state feedback laws using pole assignment. Such control laws require the state to be measured. For many systems, we may only get partial information about the state through the measured output. In this chapter, we shall study the property of observability and show that whenever the system is observable, we can estimate the state accurately using an observer. The state estimate can then be used to design feedback compensators.

4.1 Observability

Consider the linear system described by

$$\dot{x}(t) = Ax(t) \tag{4.1}$$

$$x(0) = x_0$$

$$y(t) = Cx(t) \tag{4.2}$$

We assume we only measure the inputs outputs $y(t)$ on an interval $[0, t_1]$. Since the state plays such an important role, we ask whether we can determine the state x from the output measurements on $[0, t_1]$. Since $x(t)$ is fully determined once the initial state x_0 is known (given by $x(t) = e^{At}x_0$), we can equivalently ask whether we can determine x_0 from $y(t), 0 \leq t \leq t_1$. This motivates the following definition.

Definition: The system (4.1), (4.2) is said to be observable at time t_1 if the initial state x_0 can be uniquely determined from $y(t), 0 \leq t \leq t_1$.

The output $y(t)$ is given by

$$y(t) = Ce^{At}x_0 \tag{4.3}$$

In general, C is $p \times n$ with $p < n$ (fewer output components than states). We cannot hope simply to invert C to determine x_0 .

Although (4.1) and (4.2) contains no input u , there is no loss of generality because we can easily incorporate inputs, as follows.

Suppose the system is described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.4}$$

$$x(0) = x_0$$

$$y(t) = Cx(t) + Du(t) \tag{4.5}$$

We assume we can now measure both inputs and outputs, $u(s), y(s)$ $0 \leq s \leq t_1$. By the variations of parameters formula, we have

$$y(t) = Ce^{At}x_0 + C \int_0^t e^{A(t-s)}Bu(s)ds + Du(t) \quad (4.6)$$

which can be rewritten as

$$\tilde{y}(t) = y(t) - C \int_0^t e^{A(t-s)}Bu(s)ds - Du(t) = Ce^{At}x_0 \quad (4.7)$$

Determining x_0 uniquely from $u(s), y(s)$ $0 \leq s \leq t_1$ is equivalent to determining x_0 uniquely from $\tilde{y}(s)$ $0 \leq s \leq t_1$. We are now back to the system (4.1) and (4.2), with the measurement \tilde{y} instead of y .

Fix t_1 . Define the observability Gramian M_{t_1} by

$$M_{t_1} = \int_0^{t_1} e^{A^T\tau}C^TCe^{A\tau}d\tau \quad (4.8)$$

We have the following theorem on observability.

Proposition 4.1: The system (4.1), (4.2) is observable at time t_1 if and only if the observability Gramian M_{t_1} is nonsingular.

Proof: First assume that M_{t_1} is nonsingular. The output is given by

$$y(t) = Ce^{At}x_0 \quad (4.9)$$

Multiplying both sides of (4.9) by $e^{A^Tt}C^T$ and integrating from 0 to t_1 gives

$$M_{t_1}x_0 = \int_0^{t_1} e^{A^T\tau}C^Ty(\tau)d\tau \quad (4.10)$$

Since M_{t_1} is nonsingular, we can invert (4.10) to obtain x_0

$$x_0 = M_{t_1}^{-1} \int_0^{t_1} e^{A^T\tau}C^Ty(\tau)d\tau \quad (4.11)$$

Now suppose M_{t_1} is singular. There exists a nonzero vector v such that $M_{t_1}v = 0$. This in turn implies that $v^TM_{t_1}v = 0$, from which we find

$$Ce^{At}v = 0 \quad 0 \leq t \leq t_1$$

(compare the arguments used in the Chapter 2 in connection with the controllability Gramian). This means that for the vectors v and 0 as initial states of (4.1) both give rise to an output $y = 0$. Hence the system is not observable.

Since observability of the system depends only on the pair (C, A) , and does not depend on B and D even if we have inputs as in (4.4) and (4.5), we shall also say (C, A) is observable.

On comparing the observability Gramian for the pair (C, A) and the controllability Gramian for the pair (A, B) , we see that they are very similar in form. In particular, if we made the correspondence

$$\begin{array}{l} A^T \longleftrightarrow A \\ C^T \longleftrightarrow B \end{array}$$

we would have changed the observability Gramian into a controllability Gramian.

The following result can immediately be obtained using this correspondence.

Theorem 4.1: (Duality Theorem) (C, A) is observable if and only if (A^T, C^T) is controllable.

Let us define the observability matrix \mathcal{O}_{CA} as

$$\mathcal{O}_{CA} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (4.12)$$

From the Duality Theorem, we can immediately deduce the following algebraic criterion for observability, which is the analog of Theorem 2.2.

Theorem 4.2: (Algebraic Test for Observability) (C, A) is observable if and only if

$$\text{Rank}(\mathcal{O}_{CA}^T) = \text{Rank}([C^T \ A^T C^T \ \dots \ (A^T)^{n-1} C^T]) \quad (4.13)$$

$$= n \quad (4.14)$$

Recall the **kernel** or **nullspace** of a $q \times n$ matrix M , denoted by $\text{Ker } M$, is the set $\{x | Mx = 0\}$. We can express the observability test (4.14) equivalently as

$$\text{Ker } \mathcal{O}_{CA} = \mathbf{0} \quad (4.15)$$

where $\mathbf{0}$ is the trivial subspace consisting only of the vector 0.

Similarly, we can write down respectively the analogs of Theorem 2.1 and Theorem 2.5:

Theorem 4.3: For $t > 0$, $\text{Ker } M_t = \text{Ker } \mathcal{O}_{CA}$.

PBH Test for Observability: (C, A) is observable if and only if

$$\text{Rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \text{ for all } \lambda \text{ an eigenvalue of } A$$

4.2 Full Order Observers

In the previous chapter, we have seen how control laws can be designed based on pole placement whenever the state is available. When the state is not available for measurement, a natural approach is to use the input-output measurements to estimate the state. We have already seen in the previous section that for the system

$$\dot{x} = Ax + Bu \quad x(0) = x_0 \quad (4.16)$$

$$y = Cx + Du \quad (4.17)$$

the initial state, and hence the state trajectory, can in principle be determined if (C, A) is observable. However, the procedure involves integration and inversion of the observability Gramian which is often ill-conditioned. Thus an alternative approach is desirable.

The following idea suggests how we can design a state estimator. If we build a duplicate of system described by (4.16) and (4.17), in general the initial conditions would not match and the outputs of the

two systems would be different. The error can then be used as a feedback signal to improve the state estimation. Specifically, we seek an estimator of the form

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{x}(0) &= \hat{x}_0 \\ \hat{y} &= C\hat{x} + Du\end{aligned}\tag{4.18}$$

where \hat{x}_0 can be arbitrarily chosen (often taken to be 0). The error in the state estimate, defined as $e = x - \hat{x}$, is governed by the equation

$$\begin{aligned}\dot{e} &= (A - LC)e \\ e(0) &= x_0 - \hat{x}_0\end{aligned}\tag{4.19}$$

The error equation does not depend on the input u . Note also that if the eigenvalues of $A - LC$ all lie in the left half plane, then regardless of $e(0)$, $e(t) \rightarrow 0$ exponentially and the goal of perfect state estimation is achieved asymptotically. The question of the speed of convergence of \hat{x} to x is precisely the dual of the pole assignment problem.

Theorem 4.4: There exists an $n \times p$ matrix L such that $\det(sI - A + LC) = r(s)$, where $r(s)$ is any n th degree monic polynomial with real coefficients, if and only if (4.1), (4.2) is observable.

Proof: By duality, (C, A) is observable if and only if (A^T, C^T) is controllable. By the pole assignment theorem, this is equivalent to the existence of a matrix L^T such that $\det(sI - A^T + C^T L^T)$ is any pre-assigned monic polynomial. Since $\det(sI - A^T + C^T L^T) = \det(sI - A + LC)$, the result follows.

The eigenvalues of the system matrix of the error equation (4.19), $A - LC$ are usually referred to as the observer poles, since the observer equation (4.18) can be rewritten as

$$\dot{\hat{x}} = (A - LC)\hat{x} + (B - LD)u + Ly\tag{4.20}$$

which also has the system matrix $A - LC$.

Example 4.1

Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\tag{4.21}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x\tag{4.22}$$

This can be interpreted as the state space realization of the transfer function $\frac{1}{s(s+1)}$ in controllable canonical form. To construct an observer to estimate the states with observer poles at $-10, -10$, we just need to determine the feedback gain L to place the poles of $A - LC$ at $-10, -10$. We apply the pole placement algorithm to the pair (A^T, C^T) .

$$C_{A^T C^T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The characteristic polynomial of A^T is $s^2 + s$. The transformation which brings (A^T, C^T) to controllable canonical form is

$$V = C_{A^T C^T} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence

$$V^{-1} = - \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

The desired observer characteristic polynomial is $r_o(s) = (s+10)^2 = s^2 + 20s + 100$. So the desired feedback gain L^T is

$$L^T = [100 \quad 19] \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = [19 \quad 81]$$

Finally, the observer equation is given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ &= \left(\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 19 & 0 \\ 81 & 0 \end{bmatrix} \right) \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 19 \\ 81 \end{bmatrix} y \\ &= \begin{bmatrix} -19 & 1 \\ -81 & -1 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 19 \\ 81 \end{bmatrix} y \end{aligned}$$

The state estimator (4.18) is often called a **full order observer** or identity observer since it estimates the entire state of the system. For simplicity, we shall refer to it as an observer. The convergence of the estimate to the true state is exponential, with convergence rate dependent on the choice of the observer poles, which can be arbitrarily assigned whenever the system is observable.

4.3 Feedback Compensation

Now that we know how to generate state estimates which converge exponentially to the true states, we are in a position to look at feedback compensation using only the outputs of the system. Combining the pole placement design together with the observer, it would seem to be a reasonable methodology to replace the states in the pole placement design with the state estimates generated by the observer. This design strategy is referred to as

The Separation Principle:

Control design can be separated into 2 steps:

Step 1: Design a state feedback control law assuming that the states are available.

Step 2: Design a state estimator to estimate the states of the system. Replace the states in the state feedback control law from Step 1 by the state estimates to give the final output feedback control design.

For linear systems specifically, we can first design the state feedback control law $u = -Kx$, choosing the gain K by pole placement design. Next, we design an observer to estimate the state x by choosing appropriate observer poles. The final feedback control law is given by

$$u = -K\hat{x} \tag{4.23}$$

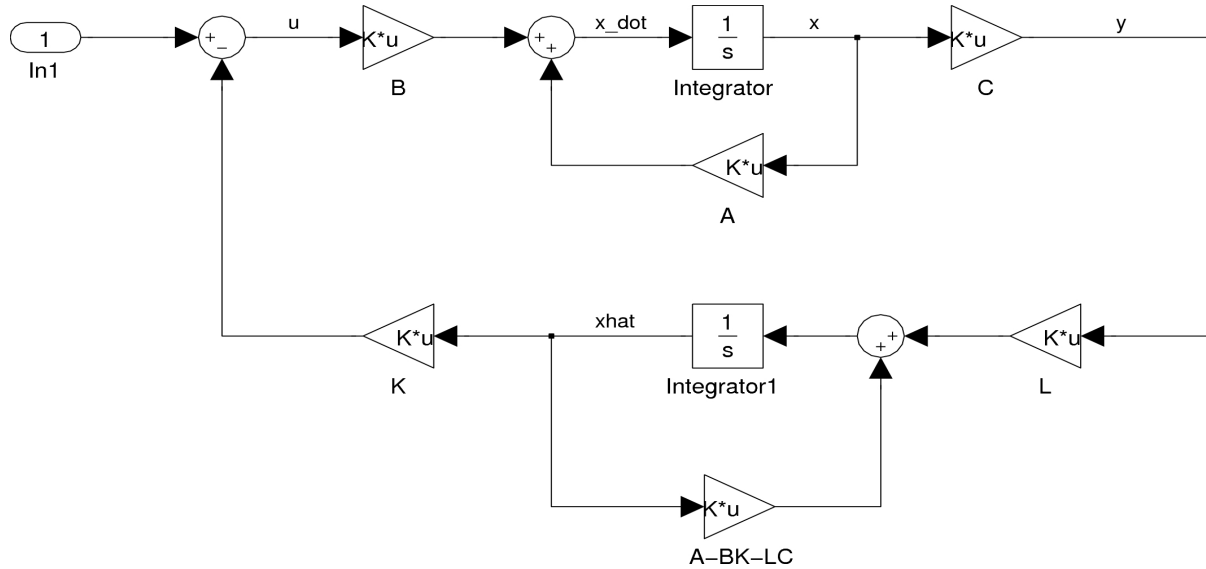
Substituting (4.23) into (4.20), we obtain

$$\dot{\hat{x}} = (A - BK - LC + LDK)\hat{x} + Ly \tag{4.24}$$

We can interpret the control law as a linear system with state equation (4.24) driven by y , with the output u given by the equation (4.23). The feedback law can also be described in transfer function form by

$$U(s) = -K(sI - A + BK + LC - LDK)^{-1}LY(s) \tag{4.25}$$

Taking $D = 0$ for simplicity, the simulink diagram representation of the compensator is given in the following figure.



Observer-based Compensator Design

We now study the properties of the closed loop system using this control design based on the Separation Principle. For this purpose, it is more convenient to use the estimation error equation, which is independent of the input u :

$$\dot{e} = (A - LC)e \quad (4.26)$$

Recall that the estimation error $e = x - \hat{x}$. Hence we can express the control law (4.23) also as

$$u = -K(x - e) = -Kx + Ke \quad (4.27)$$

Substituting (4.27) into the system equation (4.1), we obtain the closed loop equation

$$\dot{x} = (A - BK)x + BKe \quad (4.28)$$

Combining (4.28) together with the estimation error equation (4.26), we have

$$\frac{d}{dt} \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (4.29)$$

Note that the poles of the closed loop system are exactly the union of the eigenvalues of $A - BK$ and $A - LC$, which may be pre-assigned if controllability of (A, B) and observability of (C, A) both hold. Hence the control design based on the Separation Principle, combining pole placement and observer designs, achieves state regulation with an exponential decay rate governed by the choice of closed loop system poles.

Example 4.2

Let us revisit Example 4.1, but instead of just state estimation, we would like to achieve state regulation using output feedback control. The system is in controllable canonical form. Suppose the design specification is that all states must be regulated to 0 with transients decaying at least as fast as e^{-2t} . Suppose we choose the desired eigenvalues of $A - BK$, the control poles, to be at $-2, -2$, and the observer poles at $-10, -10$. This guarantees the design specifications are met. The desired control characteristic polynomial is $r_c(s) = \det(sI - A + BK) = (s + 2)^2 = s^2 + 4s + 4$. The desired pole placement gain K is therefore given by

$$K = [4 \quad 3] \quad (4.30)$$

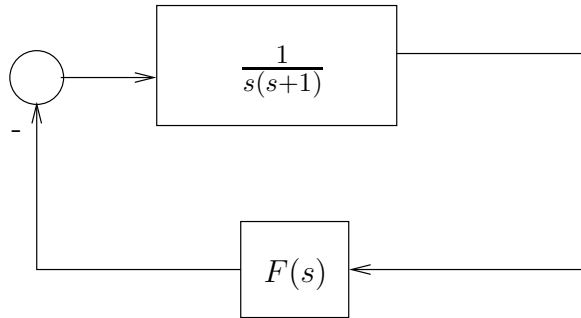
The desired observer gain L has already been determined in Example 4.1. Substituting the value of K and the observer gain L into (4.24), we obtain the following description for the feedback law

$$\begin{aligned}\dot{\hat{x}} &= \left(\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \end{bmatrix} - \begin{bmatrix} 19 & 0 \\ 81 & 0 \end{bmatrix} \right) \hat{x} + \begin{bmatrix} 19 \\ 81 \end{bmatrix} y \\ &= \begin{bmatrix} -19 & 1 \\ -85 & -4 \end{bmatrix} \hat{x} + \begin{bmatrix} 19 \\ 81 \end{bmatrix} y \\ u &= - \begin{bmatrix} 4 & 3 \end{bmatrix} \hat{x}\end{aligned}$$

The transfer function for the feedback law is

$$\begin{aligned}U(s) &= - \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} s+19 & -1 \\ 85 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 19 \\ 81 \end{bmatrix} Y(s) \\ &= - \begin{bmatrix} 4 & 3 \end{bmatrix} \frac{\begin{bmatrix} s+4 & 1 \\ -85 & s+19 \end{bmatrix}}{s^2 + 23s + 161} \begin{bmatrix} 19 \\ 81 \end{bmatrix} Y(s) \\ &= - \begin{bmatrix} 4 & 3 \end{bmatrix} \frac{\begin{bmatrix} 19s + 157 \\ 81s - 76 \end{bmatrix}}{s^2 + 23s + 161} Y(s) \\ &= - \frac{319s + 400}{s^2 + 23s + 161} Y(s)\end{aligned}\tag{4.31}$$

Recall that in terms of transfer functions, the state space description given by (4.21) and (4.22) is the controllable canonical form realization of the transfer function $\frac{1}{s(s+1)}$. Thus, the overall system under feedback control can be described by the following block diagram

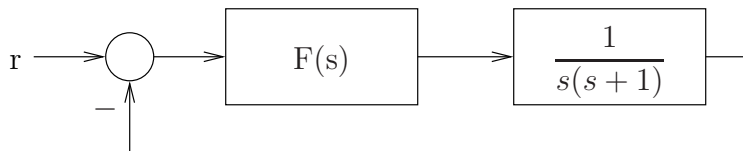


where the controller $F(s) = \frac{319s+400}{s^2+23s+161}$. Note that the negative sign in (4.31) has been incorporated into the block diagram as negative feedback. Finally, the overall transfer function is given by

$$\frac{\frac{1}{s(s+1)}}{1 + \frac{1}{s(s+1)} \frac{319s+400}{s^2+23s+161}} = \frac{s^2 + 23s + 161}{s^4 + 24s^3 + 184s^2 + 480s + 400} = \frac{s^2 + 23s + 161}{(s+2)^2(s+10)^2}$$

with poles at $-2, -2, -10, -10$, as expected.

The compensator can also be arranged in the standard single loop feedback configuration



where for state regulation, $r = 0$.

Note that the order of the feedback compensator is n since the full order observer is of dimension n . Furthermore, the feedback compensator is always strictly proper rational.

4.4 Detectability

There is a property for (C, A) which is analogous to the property of stabilizability for (A, B) . We make the following

Definition: A pair (C, A) is said to be detectable if there exists a matrix L such that $A - LC$ is stable.

Since the eigenvalues of $A - LC$ are the same as those of $(A - LC)^T = A^T - C^T L^T$, we immediately see that (C, A) is detectable if and only if (A^T, C^T) is stabilizable. Thus all properties about detectability can be inferred from those of stabilizability. For example, we have

PBH Test for Detectability: (C, A) is detectable if and only if

$$\text{Rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \text{ for all } \lambda \text{ an eigenvalue of } A \text{ with } \text{Re } \lambda \geq 0$$

Moreover, observability of (C, A) implies detectability of (C, A) (compare with controllability implies stabilizability).

Note that by the estimation error equation (4.19), the state estimation error converges to 0 if and only if $A - LC$ is stable. Thus, an observer to estimate $x(t)$ perfectly asymptotically exists if and only if the weaker condition of detectability of (C, A) holds. However, since some stable eigenvalues of A cannot be changed, we would not be able to control completely the rate at which the estimate \hat{x} converges to x .

4.5 Minimal Order Observers*

In the previous 2 sections, we have developed a complete output feedback control design procedure. However it is not hard to see that there is redundancy in the observer design. Indeed, the output y measures exactly a part of the state so that we really only need to estimate the remaining part. For example, in Example 4.1, the state x_1 is the output y . This leads to the idea of minimal order observers or reduced order observers.

For simplicity, we shall assume $D = 0$. Suppose the pair (C, A) is observable. We shall assume that the matrix C is of full rank p and that a basis has been chosen so that

$$C = [I_p \ 0] \quad I_p \text{ the } p \times p \text{ identity matrix} \quad (4.32)$$

There is no loss of generality in making these assumptions. Partition $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $y = x_1$. Partition A and B correspondingly. Equation (4.1) can then be written as

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \quad (4.33)$$

$$\dot{x}_2 = A_{22}x_2 + A_{21}x_1 + B_2u \quad (4.34)$$

where A_{11} is $p \times p$ and A_{22} is $(n - p) \times (n - p)$. We then have

Lemma 4.1: Consider the system $\dot{x} = Ax + Bu$, $y = Cx$, with C in the form of (4.32). The system is observable if and only if $\dot{x}_2 = A_{22}x_2 + v$, $w = A_{12}x_2$ is observable, i.e., the pair (A_{12}, A_{22}) is observable.

Proof: Since x_1 and u are known exactly, we may write (4.1) as

$$\underbrace{\dot{x}_1 - A_{11}x_1 - B_1u}_w = A_{12}x_2 \quad (4.35)$$

$$\dot{x}_2 = A_{22}x_2 + \underbrace{A_{21}x_1 + B_2u}_v \quad (4.36)$$

The lemma now follows from the definition of observability.

The same result can be proved more formally using the PBH test.

To estimate x_2 , we imitate the development of the full order observer. Let us interpret $w = \dot{x}_1 - A_{11}x_1 - B_1u$ as the observation on the state x_2 . The state equation for x_2 is given by (4.34), where $A_{21}x_1 + B_1u = A_{21}y + B_1u$ is the known input. We can therefore write the observer equation for x_2 as

$$\begin{aligned} \dot{\hat{x}}_2 &= A_{22}\hat{x}_2 + A_{21}x_1 + B_2u + L(w - A_{12}\hat{x}_2) \\ &= A_{22}\hat{x}_2 + A_{21}x_1 + B_2u + L(A_{12}x_2 - A_{12}\hat{x}_2) \end{aligned} \quad (4.37)$$

Subtracting (4.37) from (4.34), the estimation error $e_2 = x_2 - \hat{x}_2$ satisfies

$$\dot{e}_2 = (A_{22} - LA_{12})e_2 \quad (4.38)$$

By observability of (C, A) which implies observability of (A_{12}, A_{22}) , the eigenvalues of $A_{22} - LA_{12}$ may be pre-assigned.

Notice that (4.37) on the surface would have to be implemented in the form

$$\begin{aligned} \dot{\hat{x}}_2 &= A_{22}\hat{x}_2 + A_{21}x_1 + B_2u + L(\dot{x}_1 - A_{11}x_1 - B_1u - A_{12}\hat{x}_2) \\ &= (A_{22} - LA_{12})\hat{x}_2 + (A_{21} - LA_{11})y + (B_2 - LB_1)u + L\dot{x}_1 \end{aligned} \quad (4.39)$$

This would require differentiating $x_1 = y$ which may not be desirable.

The problem is now removed by writing

$$z = \hat{x}_2 - Lx_1 = \hat{x}_2 - Ly$$

Then z satisfies the equation

$$\begin{aligned} \dot{z} &= (A_{22} - LA_{12})z + (A_{22} - LA_{12})Ly + (A_{21} - LA_{11})y + (B_2 - LB_1)u \\ &= (A_{22} - LA_{12})z + (B_2 - LB_1)u + [(A_{22} - LA_{12})L + (A_{21} - LA_{11})]y \end{aligned} \quad (4.40)$$

The complete state estimate is then given by

$$\hat{x} = \begin{bmatrix} y \\ z + Ly \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ L & I_{n-p} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad (4.41)$$

Example 4.3

Consider the system described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x \\ y &= [1 \ 0 \ 0] x \end{aligned}$$

It is readily verified that the system is observable. Since the first component x_1 of x is observed, we need only estimate the last 2 components of x in the minimal order observer. Let x_2 denote the **last 2** components of x . We can decompose the system into the following set of equations:

$$\begin{aligned}\dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 \\ \dot{x}_1 &= \begin{bmatrix} -1 & 0 \end{bmatrix} x_2 \\ y &= x_1\end{aligned}$$

The observer equation for x_2 is given by

$$\dot{\hat{x}}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + L(\dot{x}_1 - \begin{bmatrix} -1 & 0 \end{bmatrix} \hat{x}_2) \quad (4.42)$$

where the gain L is chosen to place the observer poles. Suppose we pick the observer poles to be $-1, -1$. It is straightforward to find that

$$L = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Substituting into (4.42) and using y for x_1 , we get the equation

$$\dot{\hat{x}}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ -1 \end{bmatrix} (\dot{y} - \begin{bmatrix} -1 & 0 \end{bmatrix} \hat{x}_2) \quad (4.43)$$

$$= \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \hat{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ -1 \end{bmatrix} \dot{y} \quad (4.44)$$

To eliminate the differentiation of y , set

$$z = \hat{x}_2 - \begin{bmatrix} -2 \\ -1 \end{bmatrix} y$$

We get

$$\begin{aligned}\dot{z} &= \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} (z - \begin{bmatrix} 2 \\ 1 \end{bmatrix} y) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{y} \\ &= \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \dot{y}\end{aligned}$$

Finally, the complete state estimate is given by

$$\hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

4.6 Feedback Compensation Using Minimal Order Observers*

We have seen in the previous section that the minimal order observer, though more complex to construct, also gives exponentially convergent state estimates. By the Separation Principle, we can also design feedback compensators using state estimates generated by the minimal order observer. The construction

is analogous to the full order observer case. The minimal order observer equation, from (4.40) and (4.41), has the form

$$\begin{aligned}\dot{z} &= Fz + Gu + Hy \\ \hat{x} &= M \begin{bmatrix} y \\ z \end{bmatrix}\end{aligned}$$

with

$$\hat{x}_1 = y = x_1$$

where the eigenvalues of F may be pre-assigned if (C, A) is observable. The estimation error equation is

$$\dot{e}_2 = Fe_2$$

Suppose the state feedback gain K has been determined. If we put $u = -K\hat{x}$, we find, on partitioning $K = [K_1 \ K_2]$

$$\begin{aligned}u &= -K_1\hat{x}_1 - K_2\hat{x}_2 = -K_1x_1 - K_2\hat{x}_2 \\ &= -Kx + K_2e_2\end{aligned}$$

The closed loop system is then

$$\begin{aligned}\dot{x} &= (A - BK)x + BK_2e_2 \\ \dot{e}_2 &= Fe_2\end{aligned}$$

Again, the closed loop system may take any pre-assigned set of poles provided (A, B) is controllable and (C, A) is observable.

Example 4.4

We again re-visit the control design problem described in Example 4.2, but using minimal order observer to generate the state estimate. Since we require a decay rate of at least e^{-2t} , we can achieve the design specifications by choosing the 2 control poles at -2 , and the minimal order observer, which is first order, to have its pole at -10 . The required control gain has already been determined in Example 4.2 to be

$$K = [4 \ 3]$$

We now do

Minimal order observer design: Since x_2 is not available for measurement, we design an observer to estimate x_2 . The dynamics of the observer should give an error system with a pole at -10 to meet the design specifications. Now the state equation (4.21) may be written more explicitly as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u\end{aligned}$$

Thus the minimal order observer is

$$\begin{aligned}\dot{\hat{x}}_2 &= -\hat{x}_2 + u + L(\dot{x}_1 - \hat{x}_2) \\ &= -(1 + L)\hat{x}_2 + u + L\dot{x}_1\end{aligned}$$

This determines $L = 9$. Defining $z = \hat{x}_2 - 9x_1 = \hat{x}_2 - 9y$, we get

$$\dot{z} = -10\hat{x}_2 + u = -10z - 90y + u \quad (4.45)$$

$$\hat{x}_2 = z + 9y \quad (4.46)$$

We can now determine the overall

Compensator design: On putting the state feedback controller and the minimal order observer together, we obtain the following controller

$$u = -4y - 3\hat{x}_2 = -31y - 3z \quad (4.47)$$

with z satisfying

$$\begin{aligned} \dot{z} &= -10z - 90y + (-3z - 31y) \\ &= -13z - 121y \end{aligned} \quad (4.48)$$

The compensator transfer function $F(s)$ can be obtained as follows. From (4.48), we have

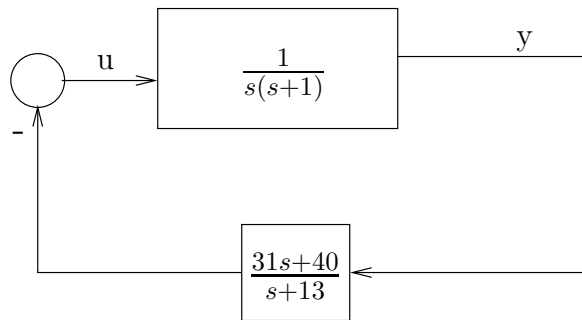
$$\frac{Z(s)}{Y(s)} = -\frac{121}{s+13}$$

Hence

$$\begin{aligned} U(s) &= -F(s)Y(s) = -31Y(s) + \frac{363}{s+13}Y(s) \\ &= -\frac{31s+40}{s+13}Y(s) \end{aligned}$$

Note that with the direct transmission from y to u , the compensator transfer function is proper rational when a minimal order observer is used (as opposed to strictly proper rational for the full order observer).

The closed loop system is given by



with transfer function

$$\frac{\frac{1}{s(s+1)}}{1 + \frac{1}{s(s+1)} \frac{31s+40}{s+13}} = \frac{s+13}{s^3 + 14s^2 + 44s + 40} = \frac{s+13}{(s+2)^2(s+10)}$$