Chapter 5

CONTROL DESIGN FOR SET POINT TRACKING

In this chapter, we extend the pole placement, observer-based output feedback design to solve tracking problems. By tracking we mean that the output is commanded to track asymptotically a reference trajectory. We shall first show how to solve the problem using state feedback. Applying the separation principle, we can then solve the problem also using observer-based output feedback design.

While the tracking problem has been solved for much more general classes of reference inputs, we focus on constant step reference inputs. This is the most important class, since the most common tracking problem is that of set point tracking. The results are also easiest to discuss and understand.

5.1 Tracking Constant References Using State Feedback

Consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{5.1}$$

Let y_d denote the desired **constant** value for the output $y(t)$ to track asymptotically. We assume $y_d \neq 0$, and for convenience, we also use $y_d(t) = y_d$ to denote the constant desired reference trajectory. For simplicity, we shall **assume that the number of inputs is equal to the number of outputs**, i.e., $p = m$.

For now, assume that the state x and the reference y_d are avaible. The control objective is to design a control law, which may depend on x and y_d, so that the closed loop control system is stable and that the tracking error $e(t) = y_d(t) - y(t)$ tends to zero as $t \to \infty$.

Since $y_d \neq 0$, the steady state value of $x(t)$ cannot be 0. Assume that a control law has been chosen so that both x and u converge to steady state values as $t \to \infty$. Let

$$\begin{aligned} x^* &= \lim_{t\to\infty} x(t) \\ u^* &= \lim_{t\to\infty} u(t) \end{aligned}$$

For asymptotic tracking, x^* and u^* must satisfy the equation

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x^* \\ u^* \end{bmatrix} = \begin{bmatrix} 0 \\ y_d \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} y_d \tag{5.2}$$

where 0 is a zero matrix of size $n \times p$, and I is the $p \times p$ identity matrix.

By assumption of equal number of inputs and outputs, $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is a square matrix. Since we would like to track any set point changes, y_d is arbitrary. Equation (5.2) can be solved uniquely for $\begin{bmatrix} x^* \\ u^* \end{bmatrix}$ if and only if

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad \text{is nonsingular.} \tag{5.3}$$

Assume that (5.3) is true. We can then express

$$x^* = M_x y_d \tag{5.4}$$
$$u^* = M_u y_d \tag{5.5}$$

where

$$\begin{bmatrix} M_x \\ M_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{5.6}$$

Let $\Delta x = x - x^*$, $\Delta u = u - u^*$, and $\Delta y = y - Cx^* = y - y_d$. We can write a differential equation for Δx

$$\begin{aligned} \dot{\Delta x} &= A\Delta x + B\Delta u \\ \Delta y &= C\Delta x \end{aligned} \tag{5.7}$$

If we can find a feedback law of the form

$$\Delta u = -K\Delta x \tag{5.8}$$

such that the closed loop system for (5.7) is stable, then $\Delta x \to 0$ as $t \to \infty$, resulting in $\Delta y \to 0$ as well. The closed loop system is given by

$$\dot{\Delta x} = (A - BK)\Delta x \tag{5.9}$$

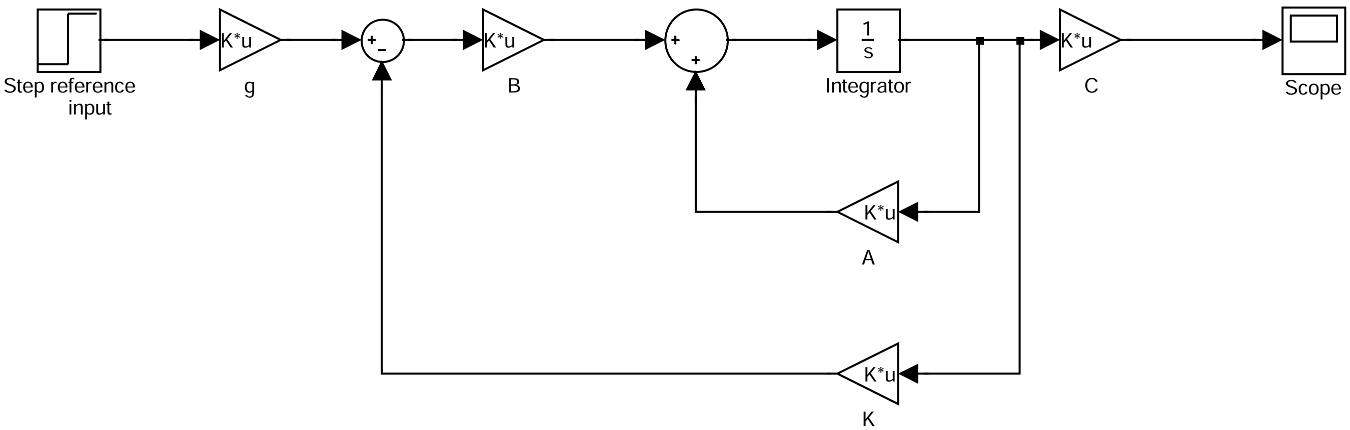
But the stability of (5.9) means $A - BK$ is stable. Hence the control law (5.8). written in the form

$$\begin{aligned} u &= u^* - K(x - x^*) \\ &= (M_u + KM_x)y_d - Kx = gy_d - Kx \end{aligned} \tag{5.10}$$

where $g = M_u + KM_x$, does stabilize (5.1) and solves the required tracking problem. Using (5.6), we can also express

$$g = \begin{bmatrix} K & I \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{5.11}$$

The following simulink block diagram shows the structure of the control design.



Control Configuration for Constant Set Point Tracking

Let us give an interpretation to condition (5.3). Suppose the input to (5.1) is given by $e^{\lambda t}\theta$. For zero initial conditions, the solution $x(t)$ is given by $e^{\lambda t}\xi$. Substituting into (5.1), we get the following equation

$$\lambda\xi = A\xi + B\theta \tag{5.12}$$

Suppose this input results in no output. Then we must also have

$$C\xi = 0 \tag{5.13}$$

In the single-input single-output case (i.e., θ is a scalar), this gives the condition

$$C(\lambda I - A)^{-1}B = 0 \tag{5.14}$$

Such a λ is therefore a zero of the transfer function

$$H(s) = C(sI - A)^{-1}B$$

For the multivariable case, we can combine equations (5.12) and (5.13) into

$$\begin{bmatrix} (A - \lambda I) & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \theta \end{bmatrix} = 0 \tag{5.15}$$

(5.15) can be solved for nonzero $\begin{bmatrix} \xi \\ \theta \end{bmatrix}$ if and only if

$$\begin{bmatrix} (A - \lambda I) & B \\ C & 0 \end{bmatrix} \quad \text{is not full rank} \tag{5.16}$$

By analogy with the single-input single-output case, we call such a λ a transmission zero of the system.

From this discussion, we see that condition (5.3) corresponds to having no transmission zero at the origin. We can now state the conditions for solvability of the tracking problem:

1. (A, B) stabilizable

2. The system (5.1) has no transmission zeros at 0

If, in addition, (A, B) is in fact controllable, then the rate of convergence to 0 of the tracking error can be pre-assigned.

Example 1:
Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 24 & -10 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of the plant is given by

$$det(sI - A) = s^3 + 10s^2 - 24s = s(s^2 + 10s - 24) = s(s+12)(s-2)$$

The transfer function is given by

$$H(s) = \frac{1}{s(s+12)(s-2)}$$

so that there are no transmission zeros at 0. To solve for the steady state values x^* and u^*, we set

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 24 & -10 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^* \\ u^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ y_d \end{bmatrix}$$

Successively from the equation for each row, we see that $x_2^* = 0$, $x_3^* = 0$, $u^* = 0$, and $x_1^* = y_d$. Note that (A, B) is controllable. Suppose we choose the closed loop poles to be at -1, $-1 \pm i$. This corresponds to the desired characteristic polynomial

$$r(s) = (s^2 + 2s + 2)(s + 1) = s^3 + 3s^2 + 4s + 2$$

Since (A, B) is in controllable canonical form, we immediately obtain

$$K = \begin{bmatrix} 2 & 28 & -7 \end{bmatrix}$$

From (5.10), we obtain the desired control law

$$\begin{aligned} u &= -\begin{bmatrix} 2 & 28 & -7 \end{bmatrix} \begin{bmatrix} x_1 - y_d \\ x_2 \\ x_3 \end{bmatrix} \\ &= 2e - 28x_2 + 7x_3 \end{aligned}$$

To determine the transfer function from the reference input $y_d(t)$ to the output $y(t)$, first note that

$$\begin{aligned} A - BK &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 24 & -10 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 28 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \end{aligned} \tag{5.17}$$

Writing

$$u = -K(x - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y_d)$$

we can write the closed loop system as

$$
\begin{aligned}
\dot{x} &= (A - BK)x + BK \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y_d \\
&= (A - BK)x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} y_d
\end{aligned} \tag{5.18}
$$

Noting that (5.18) is again in controllable canonical form, we can immediately write down the transfer function from y_d to y as

$$
\begin{aligned}
y(s) &= C(sI - A + BK)^{-1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} y_d(s) \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}}{s^3 + 3s^2 + 4s + 2} 2y_d(s) \\
&= \frac{2}{s^3 + 3s^2 + 4s + 2} y_d(s) = H_{cl}(s) y_d(s)
\end{aligned} \tag{5.19}
$$

Since $y_d(s) = \frac{y_d}{s}$ and the closed loop system is stable, the steady state value of y can be determined from the final value theorem of Laplace transforms

$$
\begin{aligned}
\lim_{t \to \infty} y(t) &= \lim_{s \to 0} \frac{2}{s^3 + 3s^2 + 4s + 2} y_d \\
&= H_{cl}(0) y_d \\
&= y_d
\end{aligned}
$$

so that asymptotic tracking is indeed achieved. This asymptotic tracking condition can be interpreted as requiring $H_{cl}(s)$, the transfer function from y_d to y to have DC-gain $H_{cl}(0) = 1$.

In this example, the term u^* is not needed since there is a pole at the origin for the open loop plant. From classical control theory, we know that for such (type-1) systems, asymptotic step tracking is guaranteed using unity feedback as long as the closed loop system is stable. The state space formulation gives exactly this structure.

We can also interpret the asymptotic tracking condition directly using the state equations without computing the transfer function from y_d to y. For the example, $g = M_u + KM_x = 2$. The closed loop equation (5.18) can be expressed as

$$\dot{x} = (A - BK)x + Bgy_d \tag{5.20}$$

Hence $H_{cl}(s)$, the transfer function from y_d to y is given by

$$y(s) = C(sI - A + BK)^{-1} Bgy_d(s) \tag{5.21}$$

The DC-gain is therefore given by

$$H_{cl}(0) = -C(A - BK)^{-1} Bg \tag{5.22}$$

which is readily verified to be 1.

Since our control design is carried out using state equations, (5.22) is usually the preferred method to check the DC-gain of the closed loop system.

It is also worth noting that the tracking problem for example 1 can be **reformulated into a regulation problem**. Let $z_1 = x_1 - y_d = y - y_d$, $z_2 = x_2$, and $z_3 = x_3$. Using $z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$ as the state, and noting that y_d is a constant, we get

$$\dot{z}_1 = z_2 \tag{5.23}$$

$$\dot{z}_2 = z_3 \tag{5.24}$$

$$\dot{z}_3 = 24z_2 - 10z_3 + u \tag{5.25}$$

This gives the same form of the state equation for z as x:

$$\dot{z} = Az + Bu$$

Since y_d is known, measuring x is equivalent to measuring z. If we design a control law $u = -Kz$ to cause the state z to be regulated to 0 asymptotically, the tracking error, which is the same as z_1, will also go to 0 asymptotically. We can do this reformulation in example 1 because the only appearance of x_1 in the state equation for x is in the form of $\dot{x}_1$ in the first equation. Equivalently, the first column of A is identically 0.

Example 2:

As another example, consider the linear system (5.1), but with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Since the first column of A is not identically 0, we cannot readily reformulate the tracking problem into a regulation problem. We apply the more general formulation described earlier in this section.

The open loop characteristic polynomial is given by

$$det(sI - A) = s^3 - 2s^2 - s + 2 = (s-1)(s+1)(s-2)$$

To solve for the steady state values of x^* and u^*, put

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^* \\ u^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} y_d$$

First, second, and 4th rows give respectively $x_2^* = 0$, $x_3^* = 0$, $x_1^* = y_d$, while the 3rd row gives $u^* = 2y_d$.

Suppose we would like to place the closed loop poles at -2, $-1 \pm i$, so that the desired characteristic polynomial is

$$r(s) = (s^2 + 2s + 2)(s + 2) = s^3 + 4s^2 + 6s + 4$$

This results in

$$K = \begin{bmatrix} 2 & 7 & 6 \end{bmatrix}$$

The control law is given by

$$\begin{aligned} u &= u^* - K(x - x^*) \\ &= 4y_d - 2x_1 - 7x_2 - 6x_3 \end{aligned} \tag{5.26}$$

The transfer function from y_d to y is easily evaluated to be

$$y(s) = \frac{4}{s^3 + 4s^2 + 6s + 4} y_d(s) \tag{5.27}$$

Once again its DC gain is 1 so that asymptotic tracking is achieved.

Equivalently, we check that

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -4 \end{bmatrix} \quad Bg = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

and

$$-C(A - BK)^{-1}Bg = 1$$

Example 3:

We give an example to show when the set point tracking problem is unsolvable. Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is clearly singular, and the equation

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

has no solution. The transfer function from u to y is given by

$$G(s) = C(sI - A)^{-1}B = \frac{s}{s^2 + 1}$$

which is 0 at $s = 0$. Equivalently, we can verify that the DC-gain $-CA^{-1}B = 0$ directly from the state equations. This means that there is a transmission zero at 0, and the set point tracking problem is not solvable.

5.2 Observer-Based Output Feedback Control

The extension to observer-based output feedback is straightforward using the separation principle. Assuming observability and applying the separation principle, we replace the control law (5.10) with

$$u = (M_u + KM_x)y_d - K\hat{x} \tag{5.28}$$

We now show that the output feedback law (5.28) also solves the tracking problem.

Consider first the case of using the full order observer to estimate x. The estimation error $e = x - \hat{x}$ satisfiies

$$\dot{e} = (A - LC)e$$

The output feedback control law (5.28) can be expressed as

$$u = (M_u + KM_x)y_d - K(x - e) = -Kx + Ke + (M_u + KM_x)y_d \tag{5.29}$$

Again, write $g = M_u + KM_x$. We combine the closed loop equation for x together with the equation for e to give the augmented system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} Bg \\ 0 \end{bmatrix} y_d & (5.30) \\ &= A_{aug}\begin{bmatrix} x \\ e \end{bmatrix} + B_{aug}y_d & (5.31) \\ y &= \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} = C_{aug}\begin{bmatrix} x \\ e \end{bmatrix} & (5.32) \end{aligned}$$

Note that

$$\begin{aligned} (sI - A_{aug})^{-1} &= \begin{bmatrix} sI - A + BK & -BK \\ 0 & sI - A + LC \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (sI - A + BK)^{-1} & (sI - A + BK)^{-1}BK(sI - A + LC)^{-1} \\ 0 & (sI - A + LC)^{-1} \end{bmatrix} & (5.33) \end{aligned}$$

Using (5.33), the transfer function from y_d to y is given by

$$y(s) = C(sI - A + BK)^{-1}Bgy_d(s) \tag{5.34}$$

which, on comparison with (5.21), is the same as that obtained using state feedback. Thus the asymptotic tracking properties are the same whether state feedback or observer-based output feedback is used. This is a vivid demonstration of the power of the separation principle and the state space approach.

To illustrate the design procedure as described by (5.28), we re-visit example 2, using the full order observer. We first check observability of (C, A).

$$\mathcal{O}_{CA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose the observer poles are located at $-4, -4, -4$. With standard pole placement for $A^T - C^TL^T$, we find that

$$L = \begin{bmatrix} 14 \\ 77 \\ 230 \end{bmatrix}$$

The observer is therefore given by the equation

$$\begin{aligned} \hat{x} &= A\hat{x} + Bu + L(y - C\hat{x}) & (5.35) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}\hat{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 14 \\ 77 \\ 230 \end{bmatrix}\left(y - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\hat{x}\right) & (5.36) \end{aligned}$$

Substituting (5.28) into (5.36), we obtain

$$\begin{aligned} \hat{x} &= (A - BK - LC)\hat{x} + Ly + Bgy_d & (5.37) \\ &= \begin{bmatrix} -14 & 1 & 0 \\ -77 & 0 & 1 \\ -234 & -6 & -4 \end{bmatrix}\hat{x} + \begin{bmatrix} 14 \\ 77 \\ 230 \end{bmatrix}y + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}y_d & (5.38) \end{aligned}$$

The two equations (5.38) and (5.28) define the controller for this system. The general result given in (5.34) shows that this controller will achieve asymptotic tracking.

Yet another way to represent the closed loop system is to use $\begin{bmatrix} x & \hat{x} \end{bmatrix}^T$ for the state of the combined system. Putting (5.28) into the state equation and combining with (5.37), we get

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} Bg \\ Bg \end{bmatrix} y_d \tag{5.39}$$

For the example, (5.39) becomes

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 2 & -2 & -7 & -6 \\ 14 & 0 & 0 & -14 & 1 & 0 \\ 77 & 0 & 0 & -77 & 0 & 1 \\ 230 & 0 & 0 & -234 & -6 & -4 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} y_d \tag{5.40}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{5.41}$$

The DC-gain from y_d to y can be readily verified to be 1, as expected.

Observer-based controller design can also be carried out using minimal observers. Since no new concepts are introduced, we leave the details and an illustrative example to the Appendix to Chapter 5.

5.3 Integral Control

Observer-based control design solves the tracking problem provided the system matrices are known exactly. If the system parameters are not precisely known, asymptotic tracking may not be achieved. In this section, we discuss the use of integral control to achieve asymptotic tracking. We shall see that integral control is more robust in that as long as the closed loop system is stable, asymptotic tracking will be attained. The price to be paid is the increase in the dimension of the system.

Again consider the linear system (5.1). We augment the system with a differential equation

$$\dot{\xi} = y - y_d \tag{5.42}$$

We may choose the initial condition ξ_0 arbitrarily, usually taken to be 0. Since y and y_d are both measured, we can determine $\xi(t)$. It is in fact given by, for $\xi_0 = 0$,

$$\xi(t) = \int_0^t [y(\tau) - y_d(\tau)]d\tau$$

We shall be using ξ as part of the feedback, hence the name integral control.

We can write the augmented system in the form

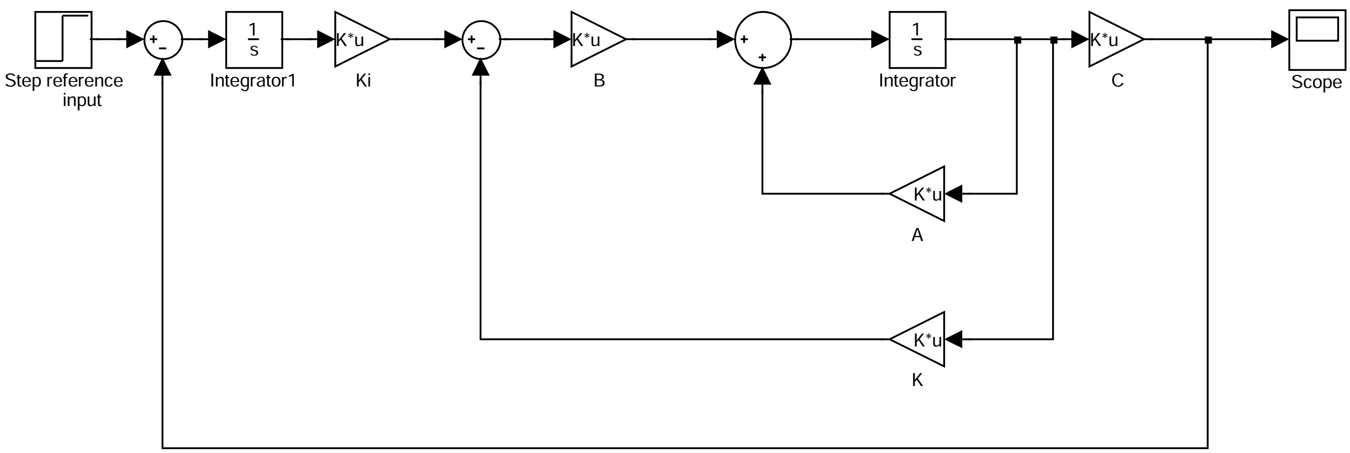
$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} y_d \tag{5.43}$$

Control laws are taken to be of the form

$$u = -Kx - K_I\xi \tag{5.44}$$

Observe that if the augmented system (5.43) can be stabilized with the control law (5.44) for suitable choice of K and K_I, then we will get $\lim_{t\to\infty} \dot{\xi}(t) = 0$. This results in $\lim_{t\to\infty} y(t) = y_d$ so that asymptotic tracking is achieved. Furthermore, this will happen even if the system matrices A and B are perturbed. As long as the closed loop system remains stable under the perturbation, $\lim_{t\to\infty}\dot{\xi}(t) = 0$ still guarantees asymptotic tracking.

The structure of the integral control design is given by the following simulink diagram.



Integral Control Configuration for Tracking

We now examine conditions under which the augmented system is controllable given that (A, B) is controllable. We shall apply the PBH test in the following formulation: (A, B) is controllable if and only if for all complex λ, there is no nonzero vector v such that $v^T \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$.

First note that the augmented system matrix

$$A_a = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$$

has 2 sets of eigenvalues, those of A, and p poles at the origin, where p is the dimension of y. Suppose v_i is a left eigenvector of A corresponding to the eigenvalue λ_i, i.e.,

$$v_i^T A = \lambda_i v_i^T$$

Then we have immediately that

$$\begin{bmatrix} v_i^T & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} v_i^T A & 0 \end{bmatrix} = \lambda_i \begin{bmatrix} v_i^T & 0 \end{bmatrix}$$

so that $\begin{bmatrix} v_i^T & 0 \end{bmatrix}$ is a left eigenvector of A_a. The augmented system input matrix is given by

$$B_a = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

Application of the PBH test will require

$$\begin{bmatrix} v_i^T & 0 \end{bmatrix} B_a = v_i^T B \neq 0 \tag{5.45}$$

Assumption of controllability of (A, B) guarantees that condition (5.45) is satisfied.

For the eigenvalues at 0, we require that there be no nonzero vector v such that $v^T \begin{bmatrix} A_a & B_a \end{bmatrix} = 0$. Noting the special structure of A_a, this is equivalent to having no nonzero vector v such at

$$v^T \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = 0$$

Summarizing the above discussion, we see that the augmented system (A_a, B_a) is controllable if and only if (A, B) is controllable, and that there are no transmission zeros at the origin. Similarly, the augmented system (A_a, B_a) is stabilizable if and only if (A, B) is stabilizable, and that there are no transmission zeros at the origin.

These conditions are the same conditions as those in observer-based feedback control. We can therefore say that by introducing additional dynamics, we can guarantee asymptotic tracking despite parameter perturbations.

As an example, we re-design the controller for the system in Example 2 using integral control. The augmented system (5.43) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} y_d \tag{5.46}$$

Suppose we use state feedback to place the poles of the augmented system at $-2, -2, -1 \pm i$. This results in

$$K = \begin{bmatrix} 14 & 15 & 8 \end{bmatrix} \quad K_I = 8$$

The closed loop system after applying the feedback law (5.44) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -16 & -14 & -6 & -8 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} y_d \tag{5.47}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \tag{5.48}$$

Again, the DC-gain from y_d to y can be verified to be 1.

To illustrate the robustness of the integral controller under parameter variations, suppose the A matrix has been perturbed to

$$A_p = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

Using the same control law, we find that the closed loop system matrix is perturbed to

$$A_{pc} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -15 & -13 & -5 & -8 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which is still stable. One can check that the DC-gain for the perturbed closed loop system remains at 1.

If state feedback is not available, then under the assumption of observability, we can construct a state estimate using an observer. Application of the separation principle shows that asymptotic tracking will again be achieved, now using output feedback only.

Appendix to Chapter 5: Observer-Based Control Design Using Minimal Order Observers

We can also implement (5.28) using a minimal order observer instead of a full order observer. Assume, as in Chapter 4, that $C = \begin{bmatrix} I & 0 \end{bmatrix}$ and that (C, A) is observable. Partitioning A and K as described in Chapter 4 into

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

the estimation error e_2 is given by

$$\hat{e}_2 = (A_{22} - LA_{12})e_2$$

with L chosen to place the poles of the minimal order observer. In terms of e_2, the observer state estimate $\hat{x}$ is given by

$$\hat{x} = x - \begin{bmatrix} 0 \\ e_2 \end{bmatrix}$$

so that (5.28) becomes

$$u = gy_d - Kx + K_2 e_2 \tag{5.49}$$

The augmented system for $\begin{bmatrix} x \\ e_2 \end{bmatrix}$ satisfies

$$\frac{d}{dt}\begin{bmatrix} x \\ e_2 \end{bmatrix} = \begin{bmatrix} A - BK & BK_2 \\ 0 & A_{22} - LA_{12} \end{bmatrix}\begin{bmatrix} x \\ e_2 \end{bmatrix} + \begin{bmatrix} Bg \\ 0 \end{bmatrix} y_d \tag{5.50}$$

$$= A_{raug}\begin{bmatrix} x \\ e_2 \end{bmatrix} + B_{raug} y_d \tag{5.51}$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ e_2 \end{bmatrix} = C_{raug}\begin{bmatrix} x \\ e_2 \end{bmatrix} \tag{5.52}$$

The same analysis shows that the transfer function from y_d to y in the minimal order observer implementation is also

$$y(s) = C(sI - A + BK)^{-1} Bgy_d(s) \tag{5.53}$$

which is the same as that given in (5.34) so that asymptotic tracking is achieved.

As an example, again we go back to example 2, but now using a minimal order observer to estimate x_2 and x_3 and employing the feedback law (5.28). The decomposed system equations are given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y \\ \dot{x}_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Hence the reduced-order observer is given by

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\left(\dot{y} - \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}\right) \tag{5.54}$$

The system matrix for the observer is given by

$$F = \begin{bmatrix} -l_1 & 1 \\ 1-l_2 & 2 \end{bmatrix}$$

where l_1 and l_2 are to be chosen to place the poles of the observer. Its characteristic polynomial is given by

$$det(sI - F) = s^2 + (l_1 - 2)s + (l_2 - 1 - 2l_1)$$

Let us choose the observer poles to be at $-4, -4$. The desired observer characteristic polynomial is given by

$$r_o(s) = s^2 + 8s + 16$$

On matching coefficients, we see that $l_1 = 10$ and $l_2 = 37$. Thus the reduced-order observer is given by

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -36 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y} \tag{5.55}$$

The augmented system for $\begin{bmatrix} x & e_2 \end{bmatrix}^T$ in this case is given by

$$\frac{d}{dt}\begin{bmatrix} x \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -4 & -6 & -4 & 7 & 6 \\ 0 & 0 & 0 & -10 & 1 \\ 0 & 0 & 0 & -36 & 2 \end{bmatrix} \begin{bmatrix} x \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} y_d \tag{5.56}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ e_2 \end{bmatrix} \tag{5.57}$$

Again the DC-gain from y_d to y is easily verified to be 1.

We can also derive the controller transfer function for the minimal order observer-based control design. We start with the minimal order observer equation (5.55). The control law is given by

$$\begin{aligned} u &= u^* + Kx^* - K\hat{x} \\ &= 4y_d - 2\hat{x}_1 - 7\hat{x}_2 - 6\hat{x}_3 \\ &= 4y_d - 2y - 7\hat{x}_2 - 6\hat{x}_3 \end{aligned} \tag{5.58}$$

Substituting the control law into (5.55), we find

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} &= \begin{bmatrix} -10 & 1 \\ -36 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (4y_d - 2y - \begin{bmatrix} 7 \\ 6 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}) + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y} \\ &= \begin{bmatrix} -10 & 1 \\ -43 & -4 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \begin{bmatrix} 0 \\ -4 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y} \end{aligned} \tag{5.59}$$

Using (5.59), we can determine the transfer functions from y and y_d to $\begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}$ as

$$\begin{bmatrix} \hat{x}_2(s) \\ \hat{x}_3(s) \end{bmatrix} = \begin{bmatrix} s+10 & -1 \\ 43 & s+4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \left(\begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 10s \\ 37s \end{bmatrix} \right) y \right)$$

Finally, substituting into (5.58), we obtain

$$\begin{aligned} u &= -\begin{bmatrix} 7 & 6 \end{bmatrix} \begin{bmatrix} s+10 & -1 \\ 43 & s+4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \left(\begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 10s \\ 37s \end{bmatrix} \right) y \right) + 4y_d - 2y \\ &= \frac{s^2+8s+16}{s^2+14s+83} 4y_d - \frac{2s^2+4s-102}{s^2+14s+83} y - \frac{292s^2+179s}{s^2+14s+83} y \\ &= -\frac{294s^2+183s-102}{s^2+14s+83} y + \frac{s^2+8s+16}{s^2+14s+83} 4y_d \end{aligned} \tag{5.60}$$

If we express (5.60) in the form

$$u(s) = -F(s)y(s) + C(s)y_d(s)$$

the closed loop transfer function from y_d to y is given by

$$y(s) = \frac{G(s)}{1+G(s)F(s)} C(s) y_d(s)$$

Substituting, we finally get

$$\begin{aligned} y(s) &= \frac{4(s^2+8s+16)}{s^5+12s^4+54s^3+116s^2+128s+64} y_d(s) \\ &= \frac{4(s^2+8s+16)}{(s+4)^2(s+2)(s^2+2s+2)} y_d(s) \end{aligned} \tag{5.61}$$

In the final transfer function (5.61), the observer poles are in fact cancelled, leaving

$$y(s) = \frac{4}{(s+2)(s^2+2s+2)} y_d(s) \tag{5.62}$$

The transfer function is the same as that obtained using state feedback in (5.27). This derivation is a much more laborious calculation than that shown as a general result in (5.53).