

Chapter 6

LINEAR QUADRATIC OPTIMAL CONTROL

In this chapter, we study a different control design methodology, one which is based on optimization. Control design objectives are formulated in terms of a cost criterion. The optimal control law is the one which minimizes the cost criterion. One of the most remarkable results in linear control theory and design is that if the cost criterion is quadratic, and the optimization is over an infinite horizon, the resulting optimal control law has many nice properties, including that of closed loop stability. Furthermore, these results are intimately connected to system theoretic properties of stabilizability and detectability.

6.1 Quadratic Forms

Before we state the optimal control problem, we review briefly the concept of quadratic forms. Let S be an $n \times n$ symmetric matrix, i.e., $S = S^T$. For $x \in R^n$, the scalar-valued function $x^T S x$ is called a quadratic form associated with S . It is known from linear algebra that if S is symmetric, all its eigenvalues are real, and it is diagonalizable by an orthogonal matrix M , meaning that $M^{-1} = M^T$. If every quadratic form associated with S , $x^T S x \geq 0$, we say S is positive semidefinite and write $S \geq 0$. If for any $x \neq 0$, $x^T S x > 0$, we say S is positive definite and write $S > 0$.

Example: The matrix

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is positive semidefinite, while the matrix

$$S_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

is positive definite.

It can be shown that $S \geq 0$ if and only if all its eigenvalues are nonnegative (and real), and $S > 0$ if and only if all its eigenvalues are positive. Note that the matrix $N^T N \geq 0$ for any matrix N .

6.2 The Optimal Control Problem

Consider a linear system in state space form

$$\dot{x} = Ax + Bu \quad x(0) = x_0 \tag{6.1}$$

We assume that the state x is available. We define the class of admissible control laws Φ to be of the form

$$u = \phi(x) \quad (6.2)$$

such that the following conditions are satisfied:

- (i) ϕ is a continuous function
- (ii) The closed loop system has a unique solution
- (iii) The closed loop system results in $\lim_{t \rightarrow \infty} x(t) = 0$.

The control objective is to find, in the class of admissible control laws, the one which minimizes the following cost criterion

$$J(x_0, \phi) = \int_0^{\infty} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \quad (6.3)$$

where Q is a positive semidefinite matrix, R is a positive definite matrix, and we have indicated explicitly the dependence of the cost criterion on the initial condition and the choice of control law. Denote the optimal cost, when it exists, as $J^*(x_0)$. Then

$$J^*(x_0) = \min_{\phi \in \Phi} \int_0^{\infty} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \quad (6.4)$$

The control law $u = \phi^*(x)$ is optimal if

$$J^*(x_0) = J(x_0, \phi^*) \quad (6.5)$$

We shall refer to the control problem as the linear quadratic optimal control problem, and the control law which solves this optimization problem as the optimal control law. Designing control laws using this optimization approach is referred to as LQR (linear quadratic regulator) design.

We can interpret the cost criterion as follows:

Since Q is positive semidefinite, $x^T(t)Qx(t) \geq 0$ and represents the penalty incurred at time t for state trajectories which deviate from 0. Similarly, since R is positive definite, $u^T(t)Ru(t) > 0$ unless $u(t) = 0$. It represents the control effort at time t in trying to regulate $x(t)$ to 0. The entire cost criterion reflects the cumulative penalty incurred over the infinite horizon. The admissible control requirement (iii) ensures that state regulation occurs as $t \rightarrow \infty$. The choice of the weighting matrices Q and R reflects the tradeoff between the requirements of regulating the state to 0 and the expenditure of control energy. For example, a diagonal matrix

$$Q = \begin{bmatrix} q_1 & 0 & 0 & \cdots & 0 \\ 0 & q_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & q_n \end{bmatrix}$$

gives the quadratic form

$$x^T(t)Qx(t) = \sum_{i=1}^n q_i x_i^2(t)$$

The relative sizes of q_i indicate the relative importance that the designer attaches to $x_i(t)$ being away from 0.

To ensure the control problem is well-posed, we make the following standing assumption throughout this chapter:

Assumption 1: (A, B) is stabilizable.

By stabilizability, there exists a feedback gain K such that the closed loop system

$$\dot{x} = (A - BK)x \quad (6.6)$$

is stable. The feedback law $u = -Kx$ is clearly admissible. The solution of (6.6) is given by

$$x(t) = e^{(A-BK)t}x_0 \quad (6.7)$$

Furthermore, the cost criterion is then given by

$$J(x_0, -Kx) = x_0^T \int_0^\infty e^{(A-BK)^T t} (Q + K^T R K) e^{(A-BK)t} dt x_0$$

which is finite. Hence the optimization problem is meaningful.

To determine the minimizing element u in (6.3), we make use of the properties of solutions of the following quadratic matrix algebraic equation:

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \quad (6.8)$$

Equation (6.8) is called the algebraic Riccati equation, and is one of the most famous equations in linear control theory. The following fundamental result, whose proof may be found in *W.M. Wonham, Linear Multivariable Control: A Geometric Approach*, allows us to write down the solution to the optimal control problem.

Theorem 6.1 *Assume (A, B) is stabilizable, and (Q, A) is detectable. Then there exists a unique solution S , in the class of positive semidefinite matrices, to the algebraic Riccati equation (6.8). Furthermore, the closed-loop system matrix $A - BR^{-1}B^T S$ is stable.*

Remark: If $Q = C^T C$ for some $p \times n$ matrix C , with $p < n$, it can be shown that (Q, A) is detectable if and only if (C, A) is detectable. Such a factorization of M arises naturally if the cost criterion J is given by

$$J^*(x_0) = \min_{\phi \in \Phi} \int_0^\infty [y^T(t)y(t) + u^T(t)Ru(t)] dt \quad (6.9)$$

with $y = Cx$. In this case, the conditions of the theorem become the aesthetic pleasing ones of (A, B) stabilizable, and (C, A) detectable.

Armed with this theorem, we see immediately that if (A, B) is stabilizable, and (Q, A) is detectable, the control law

$$u = -R^{-1}B^T Sx \quad (6.10)$$

results in the closed loop system

$$\dot{x} = (A - BR^{-1}B^T S)x \quad (6.11)$$

which is stable. Hence (6.10) is an admissible control law, resulting in $x(t) \xrightarrow{t \rightarrow \infty} 0$. We now verify that the control law (6.10) is in fact optimal, using the method of completing squares. It is based on the following computation with quadratic forms:

Let $R > 0$. We can *complete the square* in the following quadratic form

$$u^T R u + 2\alpha^T u + \beta = (u + R^{-1}\alpha)^T R (u + R^{-1}\alpha) + \beta - \alpha^T R^{-1}\alpha$$

so that

$$\min_u (u^T R u + 2\alpha^T u + \beta) = \beta - \alpha^T R^{-1}\alpha$$

with the minimizing u given by

$$u = -R^{-1}\alpha$$

Using (6.8), we express Q as

$$Q = SBR^{-1}B^T S - A^T S - SA$$

Substituting into (6.3), we can write, for any admissible u which results in $x \xrightarrow[t \rightarrow \infty]{} 0$,

$$\begin{aligned} J(x_0, u) &= \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \\ &= \int_0^\infty [x^T(t)SBR^{-1}B^T Sx(t) + u^T(t)Ru(t) - x^T(t)(A^T S + SA)x(t)]dt \\ &= \int_0^\infty [(u(t) + R^{-1}B^T Sx(t))^T R(u(t) + R^{-1}B^T Sx(t))]dt \\ &\quad - \int_0^\infty [u^T(t)B^T Sx(t) + x^T(t)SBu(t) + x^T(t)A^T Sx(t) + x^T(t)SAx(t)]dt \\ &= \int_0^\infty \{[(u(t) + R^{-1}B^T Sx(t))^T R(u(t) + R^{-1}B^T Sx(t))]dt - (\dot{x}^T(t)Sx(t) + x^T S\dot{x}(t))\}dt \\ &= \int_0^\infty [(u(t) + R^{-1}B^T Sx(t))^T R(u(t) + R^{-1}B^T Sx(t))]dt - \int_0^\infty \frac{d}{dt}(x^T(t)Sx(t))dt \\ &= x_0^T Sx_0 + \int_0^\infty [(u(t) + R^{-1}B^T Sx(t))^T R(u(t) + R^{-1}B^T Sx(t))]dt \end{aligned}$$

Since $x_0^T Sx_0$ is a constant unaffected by choice of u , and since $u = -R^{-1}B^T Sx$ is admissible and $R > 0$, it is clear that the optimal control law is indeed given by

$$u(t) = -R^{-1}B^T Sx(t)$$

with the optimal cost given by

$$J^*(x_0) = x_0^T Sx_0 \tag{6.12}$$

We summarize the above discussion as

Theorem 6.2. *Let (A, B) be stabilizable and (Q, A) detectable. The optimal control law which minimizes the quadratic cost criterion (6.3) is given by*

$$u(t) = -R^{-1}B^T Sx(t) \tag{6.13}$$

where S is the unique positive semidefinite solution to (6.8). The optimal cost is given by (6.12).

Theorem (6.1) is a deep result. The algebraic Riccati equation is a quadratic matrix equation. As such, it may have no positive semidefinite solutions, or even real solutions. It may also have an infinite number of solutions. However, under verifiable system theoretic properties of stabilizability and detectability, a unique positive semidefinite solution is guaranteed and it gives also the optimal control law. It gives therefore a complete solution to the optimal control problem.

Note that the conditions of stabilizability and detectability are sufficient conditions for the existence and uniqueness of solution of (6.8). Stabilizability is clearly necessary for the solution of the optimal control problem, as has already been mentioned. Without stabilizability, the class of admissible control laws would be empty. It is of interest to examine what can happen if detectability fails. We illustrate with 2 examples.

Example 1:

Consider the system

$$\dot{x} = u$$

with cost criterion

$$J(x_0, u) = \int_0^{\infty} u^2(t) dt$$

Here $Q = 0$, A is unstable, and so (Q, A) is not detectable. The solution to the algebraic Riccati equation is $S = 0$. However the resulting control law $u = 0$ is not admissible, even though it gives $J = 0$, since $x(t) \not\rightarrow 0$ as $t \rightarrow \infty$. Now consider the admissible control law

$$\phi_{\epsilon}(x) = -\epsilon x$$

for $\epsilon > 0$. This gives the closed loop system

$$\dot{x} = -\epsilon x$$

yielding the solution

$$x(t) = e^{-\epsilon t} x_0$$

The corresponding cost is

$$J(x_0, \phi_{\epsilon}(x)) = \int_0^{\infty} \epsilon^2 e^{-2\epsilon t} x_0^2 dt = \frac{\epsilon x_0^2}{2}$$

$J(x_0, \phi_{\epsilon}(x))$ can be made arbitrarily small by decreasing ϵ . Therefore the minimum cost J^* does not exist (i.e., its greatest lower bound, 0, cannot be attained with any admissible control). Hence the optimal control does not exist, although there is a unique positive semidefinite solution to the algebraic Riccati equation.

Example 2:

Consider the system

$$\dot{x} = x + u$$

with cost criterion

$$J(x_0, u) = \int_0^{\infty} u^2(t) dt$$

Here $Q = 0$, and again (Q, A) is not detectable. The algebraic Riccati equation is given by

$$2S - S^2 = 0$$

so that there are 2 positive semidefinite solutions, 0 and 2. Once again, the solution $S = 0$ results in the control law $u = 0$, which is not admissible. On the other hand, $S = 2$ results in

$$u = -2x$$

which is stabilizing and hence admissible. Therefore the optimal control is given by $u = -2x$, even though the solution to the algebraic Riccati equation is not unique.

6.3 Example Applications of Linear Quadratic Optimal Control

We give 2 examples which illustrate the application of linear quadratic optimal control.

Optimal control of the double integrator:

Consider the system

$$\ddot{y} = u$$

with cost criterion

$$J = \int_0^{\infty} [y^2(t) + ru^2(t)]dt \quad r > 0$$

A state space representation of this system is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [1 \quad 0] x \end{aligned}$$

The cost criterion can now be rewritten as

$$J = \int_0^{\infty} [x^T(t)Qx(t) + ru^2(t)]dt$$

where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C^T C = Q$. It is easy to verify that (C, A) is detectable, and that (A, B) is stabilizable. We proceed to solve the algebraic Riccati equation.

Let $S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$, where we have explicitly used the fact that S is symmetric. The elements of S satisfy the following equations:

$$\begin{aligned} -\frac{1}{r}s_2^2 + 1 &= 0 \\ s_1 - \frac{1}{r}s_2s_3 &= 0 \\ 2s_2 - \frac{1}{r}s_3^2 &= 0 \end{aligned}$$

The first gives the solutions $s_2 = \pm\sqrt{r}$. The third equation gives

$$s_3 = \pm(2rs_2)^{\frac{1}{2}}$$

This implies

$$s_2 = \sqrt{r}$$

Furthermore, for S to be positive semidefinite, all its diagonal entries must be nonnegative. Hence

$$s_3 = \sqrt{2r^{\frac{3}{4}}}$$

Finally,

$$s_1 = \frac{1}{r}s_2s_3 = \sqrt{2r^{\frac{1}{4}}}$$

so that

$$S = \begin{bmatrix} \sqrt{2r^{\frac{1}{4}}} & \sqrt{r} \\ \sqrt{r} & \sqrt{2r^{\frac{3}{4}}} \end{bmatrix}$$

S is in fact positive definite since $S_{11} > 0$ and $\det S > 0$ (these are the necessary and sufficient conditions for a 2x2 matrix to be > 0).

The optimal closed loop system is given by

$$\begin{aligned} \dot{x} &= (A - BR^{-1}B^T S)x \\ &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} \sqrt{r} & \sqrt{2r^{\frac{3}{4}}} \end{bmatrix} \right) x \\ &= \begin{bmatrix} 0 & 1 \\ -r^{-\frac{1}{2}} & -\sqrt{2r^{-\frac{1}{4}}} \end{bmatrix} x \end{aligned}$$

The poles of the closed loop system are given by the roots of the polynomial $s^2 + \sqrt{2}r^{-\frac{1}{4}}s + r^{-\frac{1}{2}}$. This is in the form of the standard second order system characteristic polynomial $s^2 + 2\zeta\omega_0s + \omega_0^2$, with $\omega_0 = r^{-\frac{1}{4}}$, and $\zeta = \frac{1}{\sqrt{2}}$. The damping ratio of $\frac{1}{\sqrt{2}}$ of the optimal closed loop system is often referred to as the best compromise between small overshoot and good speed of response, and it is independent of r . Now for a fixed damping ratio, the larger the natural frequency ω_0 , the faster the speed of response (recall that the peak time is inversely proportional to ω_0). Thus, we see that if r decreases, the speed of response becomes faster. Since a small r implies small control penalty and hence allowing large control inputs, this behaviour gives a good interpretation of the role of the quadratic weights in the cost criterion.

Optimal control of a servomotor

Consider the servomotor system given by the transfer function

$$y(s) = \frac{1}{s(s+1)}u(s)$$

A state space representation of this system is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

Let the cost criterion again be

$$J = \int_0^\infty [y^2(t) + ru^2(t)]dt = \int_0^\infty [x^T(t)Qx(t) + ru^2(t)]dt$$

where $Q = C^T C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $r > 0$. Again, it is easy to verify that (C, A) is detectable, and that (A, B) is stabilizable. We proceed to solve the algebraic Riccati equation.

Again, let $S = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix}$. The elements of S satisfy the following equations:

$$\begin{aligned} -\frac{1}{r}s_2^2 + 1 &= 0 \\ s_1 - s_2 - \frac{1}{r}s_2s_3 &= 0 \\ 2(s_2 - s_3) - \frac{1}{r}s_3^2 &= 0 \end{aligned}$$

Solving these equations, we get

$$\begin{aligned} s_2 &= \sqrt{r} \\ s_3 &= r\sqrt{1 + 2r^{-\frac{1}{2}}} - r \\ s_1 &= \sqrt{r + 2r^{\frac{1}{2}}} \end{aligned}$$

so that

$$S = \begin{bmatrix} \sqrt{r + 2r^{\frac{1}{2}}} & \sqrt{r} \\ \sqrt{r} & r\sqrt{1 + 2r^{-\frac{1}{2}}} - r \end{bmatrix}$$

The optimal closed loop system matrix is given by

$$\begin{aligned} A - BR^{-1}B^T S &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{r}s_2 & -1 - \frac{1}{r}s_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -r^{-\frac{1}{2}} & -\sqrt{1 + 2r^{-\frac{1}{2}}} \end{bmatrix} \end{aligned}$$

The characteristic polynomial of the closed loop system is given by $s^2 + \sqrt{1 + 2r^{-\frac{1}{2}}}s + r^{-\frac{1}{2}}$, with poles located at $\frac{-\sqrt{1 + 2r^{-\frac{1}{2}}} \pm \sqrt{1 - 2r^{-\frac{1}{2}}}}{2}$.

Since the algebraic Riccati equation is a nonlinear algebraic equation, solving it analytically in dimensions greater than 3 is intractable in general. However, numerical algorithms have been developed to give effective solution, at least for systems of a moderate size. The Matlab command `lqr` determines the feedback gain, the solution to the algebraic Riccati equation, and the closed loop eigenvalues.

In most control designs based on linear quadratic optimal control, the weighting matrices Q and R are design parameters. We can always choose Q to guarantee detectability of (Q, A) . For example, we can choose $Q > 0$. We will then have (Q, A) detectable (apply PBH test). In such design problems, the sufficient condition for the solution of the optimal control problem is just stabilizability of (A, B) , the same as the condition for solving state regulation problems using pole assignment. We can interpret the use of LQR design as a better way of generating a stabilizing control law compared to pole placement. It results in a unique control law, has reasonably good interpretation for the choice of the design parameters Q and R , avoid the need to specify desired pole locations (often chosen without good design reasons: for example, is $-2, -2$ a better choice than $-2 \pm i?$), has good transient response, and has better numerical properties than pole placement design. For these reasons, LQR design is often preferred over pole placement in control design practice.