University of Toronto
Edwared S. Rogers Sr. Department of
Electrical and Computer Engineering
ECE410F Control Systems
Fall 2008

A Note on Finding R(A), the Range of Matrix A

Suppose A is given by

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$$

A is a 4×5 matrix. Its range is a subspace of R^4 and is given by

$$\mathcal{R}(A) = span(\text{linearly independent columns of } A) \tag{1}$$

There are 2 ways to determine the right hand side of (1).

Method I:

(a) Reduce A^T to row-echelon form E_r (You use A^T because the rows of A^T are the columns of A. Reducing the rows of A^T is the same as reducing the columns of A).

(b) The linearly independent rows of E_r, rewritten as column vectors, has span= $\mathcal{R}(A)$.

The successive steps in the row reduction of A^T for the given A is:

$$\begin{aligned} A^T &= \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 2 \\ 1 & 9 & 5 & 8 \\ 3 & 0 & 1 & -3 \\ 2 & 7 & 8 & 5 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & -2 & -1 & -2 \\ 0 & 6 & 3 & 6 \\ 0 & -9 & -5 & -9 \\ 0 & 1 & 4 & 1 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{7}{2} & 0 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Then

$$\mathcal{R}(A) = span(\begin{bmatrix}1\\3\\2\\2\end{bmatrix}, \begin{bmatrix}0\\2\\1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}) \tag{2}$$

Furthermore, $rank(A) = 3$, as the dimension of $\mathcal{R}(A) = 3$.

Method II:

(a) Reduce A to echelon form E. For the given A, the successive steps result in:

$$A = \begin{bmatrix}1 & 2 & 1 & 3 & 2\\3 & 4 & 9 & 0 & 7\\2 & 3 & 5 & 1 & 8\\2 & 2 & 8 & -3 & 5\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 2 & 1 & 3 & 2\\0 & -2 & 6 & -9 & 1\\0 & -1 & 3 & -5 & 4\\0 & -2 & 6 & -9 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 2 & 1 & 3 & 2\\0 & 1 & -3 & 5 & -4\\0 & 0 & 0 & 1 & -7\\0 & 0 & 0 & 0 & 0\end{bmatrix} = E$$

(b) The columns of E where there is a nonzero entry whose left are all 0 are called the pivot columns. The pivot columns of E from the above are 1, 2, and 4. Then columns 1, 2, and 4 from the original matrix A, **not** E, are linearly independent and span $\mathcal{R}(A)$.

Therefore

$$\mathcal{R}(A) = span(\begin{bmatrix}1\\3\\2\\2\end{bmatrix}, \begin{bmatrix}2\\4\\3\\2\end{bmatrix}, \begin{bmatrix}3\\0\\1\\-3\end{bmatrix}) \tag{3}$$

It is straightforward to verify that $\mathcal{R}(A)$ determined by (2) is the same as that determined by (3). On the other hand, if you had taken columns 1, 2, and 4 from E, you will get

$$span(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\5\\1\\0\end{bmatrix})$$

which clearly is not the same as $\mathcal{R}(A)$.