University of Toronto
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ECE410F Control Systems
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Problem Set #2

1. Consider the homogeneous state equation

$$\dot{x} = Ax$$

Let

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

and $x_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Determine the modal decomposition of $x(t)$.

2. The modal decomposition works for any $n \times n$ matrix A that has n linearly independent eigenvectors. For 2 dimensional linear systems, there is a nice way called the phase portrait to visualize the dynamical behaviour of the system described by the modal decomposition. This problem illustrates how to sketch phase portraits.

   (i) Consider the 2 dimensional linear system

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} x$$

   This gives a decoupled set of differential equations in x_1 and x_2:

$$\begin{aligned} \dot{x}_1 &= 2x_1 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

   so that the general solutions is $x_1(t) = x_{10}e^{2t}$ and $x_2(t) = x_{20}e^{-t}$. Note that $x_1(t) \longrightarrow \infty$ as $t \longrightarrow \infty$ while $x_2(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

   (a) Suppose initially, the solution starts on the x_1 axis, i.e., $x_{20} = 0$. Sketch the trajectory on the $x_1 - x_2$ plane as $t \longrightarrow \infty$. Similarly, if the solutions starts on the x_2 axis, sketch the trajectory on the $x_1 - x_2$ plane as $t \longrightarrow \infty$.

   (b) For solutions not starting on the x_1 or x_2 axes, show that the solution satisfies, for all t, the relationship

$$x_1 x_2^2 = c$$

   where c is a constant. Use this relationship to sketch the solution trajectories not starting on the axes. The combined plot of the solution trajectories from (a) and (b) is called the phase portrait.
   (Hint: Express e^t as a function of x_2.)

   (c) The analysis in parts (a) and (b) is straightforward due to the matrix A being diagonal. If the matrix is not diagonal but diagonalizable, we can still apply the above analysis by a change of variables. Let

$$A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$$

It is straightforward to verify that the matrix

$$P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

diagonalizes A, i.e.

$$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

Let $z = P^{-1}x$. Then

$$\dot{z} = P^{-1}APz = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} z$$

From the results of (a) and (b), we know how to do the phase portrait for z. Use the relationship between x and z to sketch the phase portrait for the system

$$\dot{x} = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix} x$$

You may want to use Matlab to help do the plots.
(Hint: The line $z_1 = 0$ corresponds to a line in the $x_1 - x_2$ plane)

3. The Laplace transform method requires the computation of $(sI - A)^{-1}$. In the course notes, a recursive method is given for its computation. This problem guides you in the derivation of the required formulas.

   First note that

$$(sI - A)^{-1} = \frac{adj(sI - A)}{det(sI - A)}$$

   where adj(M) is the adjoint matrix of the matrix M. Using the definition of the adjoint matrix, we can express

$$adj(sI - A) = B_1 s^{n-1} + B_2 s^{n-2} + \cdots + B_n$$

   Let

$$det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n$$

   Since

$$det(sI - A)I = (sI - A)adj(sI - A),$$

   we have

$$(s^n + a_1 s^{n-1} + \cdots + a_n)I = (sI - A)[B_1 s^{n-1} + B_2 s^{n-2} + \cdots + B_n]$$

   Assume that the coefficients a_i have already been determined. By equating coefficients of the powers of s, derive a set of recursive equations for the matrices B_i, $i = 1, \ldots, n$. Verify that they are the same as those given in Section 1.4 of the course notes.

4. Let A be given by

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

   (i) Find the characteristic polynomial, $p(s)$ of A: $p(s) = \det(sI - A)$. Express $(sI - A)^{-1}$ in the form

$$(sI - A)^{-1} = \frac{B(s)}{p(s)},$$

   using the recursive formulas for computing $B(s)$.

   (ii) Using the results of (i), find e^{At} using inverse Laplace transforms.