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ECE410F Control Systems
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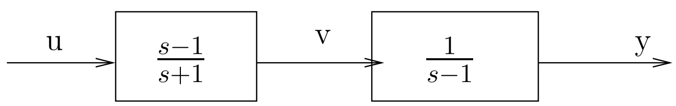
Problem Set #3

1. Consider the linear system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x \end{aligned}$$

Find the transfer function from u to y.

2. Consider the linear system described by the following block diagram



The diagram represents a naive controller design based on pole-zero cancellation to control the unstable plant $\frac{1}{s-1}$. You may recall from ECE311 that designing contollers using unstable pole-zero cancellations is a bad idea. This problem explicitly shows, using state space analysis, that unstable pole-zero cancellations result in an internally unstable system, even though the input-output response appears to be good.

(a) For the left transfer function $\frac{s-1}{s+1}$ with input u and output v, find a first order state space representation of the form

$$\begin{aligned} \dot{x}_1 &= a_1 x_1 + b_1 u \\ v &= c_1 x_1 + d_1 u \end{aligned}$$

Identify the coefficients a_1, b_1, c_1, and d_1.

(b) For the right transfer function $\frac{1}{s-1}$ with input v and output y, find a first order state space representation of the form

$$\begin{aligned} \dot{x}_2 &= a_2 x_2 + b_2 v \\ y &= c_2 x_2 + d_2 v \end{aligned}$$

Identify the coefficients a_2, b_2, c_2, and d_2.

(c) Combine the 2 equations in (a) and (b) to give a state equation for the whole system

$$\dot{x} = Ax + Bu \quad (1)$$
$$y = Cx \quad (2)$$

Identify the matrices A, B, and C.

(d) Using (1) and (2), determine its transfer function from u to y, and verify that it is $\frac{1}{s+1}$.

(e) Show that for almost all initial conditions for x, the total response, which includes the initial condition response, results in the output y as well as the states x_1 and x_2 growing exponentially without bound, even when the input u is bounded. This shows one must be careful in interpreting results based on input-output transfer functions, and that unstable pole-zero cancellations is not a viable control design.

3. The variation of parameters formula (1.39) allows you to write the explicit solution of

$$\dot{x} = Ax + Bu$$

in terms of the initial condition x_0 and the input u. It requires doing a convolution type integral. For specific inputs, other methods may be computationally more effective. Consider the RLC circuit described by (1.49):

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From differential equations, we know the general solution can be represented as

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ is the general solution to the homogeneous equation, and $x_p(t)$ is a particular solution. We already know

$$x_h(t) = e^{At}\alpha$$

where α is an arbitrary n-vector. If we can find $x_p(t)$ directly, this may provide a quicker method for solving the equation.

(a) Suppose $u(t) = 1, \ t \geq 0$. Then a particular solution x_p satisfies $\dot{x}_p = 0$. Use this to determine x_p, which is a constant vector.

(b) Set $x(t) = x_h(t) + x_p$ and set $t = 0$. Substitute the value of x_0 to determine the constant α for $x_h(t)$.

(c) Combine the results of (a) and (b) to get the solution for $x(t)$.

(d) Note that the key to solving for a constant x_p is that A is nonsingular. If A is singular, then a constant x_p generally will not work. Consider the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3}$$

Again, suppose $u(t) = 1, \ t \geq 0$. It is easy to check that the particular solution cannot be a constant vector. Let $x_p(t) = \alpha t + \beta$. Determine the vectors α and β using the state equation. Hence determine the solution to the initial value problem (3).

This method of determining the particular solution is called the **method of undetermined coefficients**.

4. This problem illustrates the fact that $(sI - A)^{-1}$ can be expressed in the form $\sum_{k=0}^{n-1} g_k(s)A^k$, where for each k, $g_k(s)$ is a strictly proper rational function. This is then used to show that e^{At} can be expressed as $e^{At} = \sum_{k=0}^{n-1} \alpha_k(t)A^k$, for some suitable functions $\alpha_k(t)$. We examine the 3×3 case, but the general $n \times n$ case is similar.

(a) Suppose A is a 3×3 matrix, and that $\det(sI - A) = p(s) = s^3 + p_1 s^2 + p_2 s + p_3$. Equations (1.36) and (1.37) of Chapter 1 shows that

$$(sI - A)^{-1} = \frac{B(s)}{p(s)} = \frac{B_1 s^2 + B_2 s + B_3}{p(s)} \tag{4}$$

Use (1.37) to determine B_2 and B_3 explicitly in terns of A and p_1 and p_2 (with $B_1 = I$). Hence write

$$B(s) = q_2(s)A^2 + q_1(s)A + q_0(s)I$$

where $q_i(s), i = 0, 1, 2$ are polynomials. Determine them explicitly. Combining with (4), we get

$$(sI - A)^{-1} = \frac{q_2(s)}{p(s)}A^2 + \frac{q_1(s)}{p(s)}A + \frac{q_0(s)}{p(s)}I = \sum_{k=0}^{2} g_k(s)A^k$$

(b) Using the formula $e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$, show that

$$e^{At} = \sum_{k=0}^{2} \alpha_k(t)A^k$$

for some functions $\alpha_k(t), k = 0, 1, 2$. How is $\alpha_k(t)$ related to $g_k(s)$?

(c) Specialize your results to the case where A is given by

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

This is the matrix whose matrix exponential was determined in Problem 4, Problem Set 2. Use the results of parts (a) and (b) to express e^{At} in the form of $\sum_{k=0}^{2} \alpha_k(t)A^k$.