University of Toronto
Department of Electrical and Computer Engineering
ECE410F Control Systems
Fall 2008
Problem Set #4

1. Consider the system

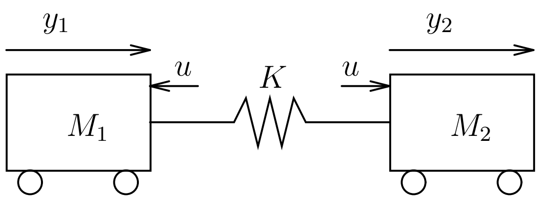
$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

Is this system controllable? If it is not, find $\mathcal{R}_0$, the set reachable from 0, and also a vector x which is not reachable from 0.

2. Is the following (A, B) pair controllable?

$$A = \begin{bmatrix} 0 & -6 & 0 & 4 \\ 1 & 4 & -1 & 0 \\ 1 & 8 & 0 & 0 \\ 1 & 11 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3. In Section 2.1, Example 2 of the course notes, we studied the controllability properties of a coupled 2-cart system with 2 independent inputs. This problem studies the same coupled 2-cart system, but with only 1 input.



The differential equations governing the motion of the 2 carts are given by:

$$\begin{aligned} M_1\ddot{y}_1 &= -u - K(y_1 - y_2) \\ M_2\ddot{y}_2 &= u - K(y_2 - y_1) \\ x &= \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix} \\ \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K}{M_1} & 0 & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & \frac{-K}{M_2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{1}{M_1} \\ 0 \\ \frac{1}{M_2} \end{bmatrix} u \end{aligned}$$

Find the controllability matrix C_{AB} and determine whether or not the system is controllable. Specialize it to the case where $M_1 = K = 1$, $M_2 = \frac{1}{2}$. Compute C_{AB} in this case and find its linearly independent columns (these columns span $\mathcal{R}(C_{AB})$).

4. Theorem 2.2 of Chapter 2 shows that if W_t is nonsingular, then every state x is reachable at time t. Furthermore, an appropriate control input which achieves the transfer is given by

$$u(\tau) = B^T e^{A^T(t-\tau)}\xi \tag{1}$$

where ξ is the unique solution to the equation $W_t\xi = x$. Now if the system is not controllable, W_t will not be invertible, but for x in the reachable set, the equation $W_t\xi = x$ is solvable according to Theorem 2.1. This problem illustrates this situation.

(i) Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with initial state $x_0 = 0$. (A, b) is clearly not controllable. Nevertheless, show that for all states of the form $x = \begin{bmatrix} 0 & \alpha \end{bmatrix}^T$, α arbitrary, there exists a (nonunique) solution to the equation

$$W_t\xi = x$$

Determine all solutions for ξ as a function of α, t, and a free parameter.

(ii) For $t = 1$ and $\alpha = 2$, show that the control input given by (1) transfers the state to $\begin{bmatrix} 0 & 2 \end{bmatrix}^T$ at $t = 1$, regardless of the choice of the free parameter in the solution for ξ (hence it can be chosen to be 0).