

**University of Toronto**  
**Department of Electrical and Computer Engineering**  
**ECE410F Control Systems**  
**Fall 2008**  
**Problem Set 6**

1. Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Determine whether  $(C, A)$  is observable or not.

2. Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

- (i) Show that  $(C, A)$  is not observable by determining the rank of the observability matrix  $\mathcal{O}_{CA}$ .
  - (ii) Use the PBH test to show that  $(C, A)$  is not observable.
  - (iii) Using the results of (ii), determine whether or not  $(C, A)$  is detectable.
3. Construct a full order observer for the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} x, \quad \omega > 0 \\ y &= x_1 \end{aligned}$$

with the poles of the observer placed at -1, -2, and -3.

4. Design an output feedback compensator for the system with transfer function  $\frac{1}{(s-1)(s+1)}$  such that all the poles of the closed loop system are located at -2. You are to use pole assignment and full order observer theory. Proceed by following the steps below.
- (i) Realize the transfer function using the controllable canonical form (refer to Chapter 1 of the course notes). This gives rise to the system matrices  $A$ ,  $B$ , and  $C$ .
  - (ii) Find the state feedback gain  $K$  which places the poles of  $A - BK$  at  $-2, -2$ .
  - (iii) Check that  $(C, A)$  is observable. Design the observer gain  $L$  such that the poles of  $A - LC$  are located at  $-2, -2$ . Write down the full order observer equation.

- (iv) Apply the separation principle to obtain the control law  $u = -K\hat{x}$ . Under this control law, the closed loop system with augmented state  $[x \ \hat{x}]^T$  satisfies

$$\frac{d}{dt} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Write down the  $4 \times 4$  system matrix for the closed loop system. Use Matlab to verify that its characteristic polynomial is  $s^4 + 8s^3 + 24s^2 + 32s + 16 = (s + 2)^4$

(Compare this problem with Example 4.2 of Chapter 4.)

5. Let

$$P = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

where  $I$  denotes the identity matrix of dimension  $n \times n$  and  $0$  denotes the zero matrix of dimension  $n \times n$ . Hence  $P$  is a  $2n \times 2n$  matrix.

- (i) Show that  $P^2 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ . This shows that  $P = P^{-1}$ .

(ii) Show that

$$P \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} P = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}$$

Hence conclude that the matrices

$$\begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}$$

have the same eigenvalues.

Note that the matrix

$$\begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}$$

arises in compensator design when we write the closed loop equation for the augmented state  $[x \ \hat{x}]^T$  (see problem 4), while the matrix

$$\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}$$

arises in compensator design when we write the closed loop equation for the augmented state  $[x \ e]^T$  (see eq.(4.29) of the course notes). This problem shows their eigenvalues are the same.