Design of Reach Controllers on Simplices

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***Abstract*— We examine currently available methods to design reach controllers on simplices under the assumption of a preferred triangulation of the state space. Using an example-driven approach, we show that existing methods completely resolve how reach controllers can be designed under this triangulation. The paper begins by examining the simplest cases using affine feedbacks. Then using the reach control indices we look at cases when the design calls for either discontinuous piecewise affine feedback or time-varying affine feedback.**

I. Introduction

This paper studies the *reach control problem* (RCP) on simplices. The problem is to make the closed-loop trajectories of an affine system leave a simplex $\mathcal{S}$ through a prespecified exit facet $\mathcal{F}_0$ in finite time. The problem has been developed in a series of papers, particularly [1], [3]–[9], [11]. A brief historical overview follows. RCP first appeared in [7]. In [8], so-called invariance conditions which guarantee that trajectories only exit $\mathcal{S}$ through $\mathcal{F}_0$ were shown to be necessary for solvability of RCP by continuous state feedback. Also [8] proposed an appealing design procedure for synthesizing affine feedbacks $u = Kx + g$. The modern formulation of RCP, relaxing a restriction of [8] that trajectories exit $\mathcal{S}$ at the first time they reach $\mathcal{F}_0$, appeared in [9], [11]. Both [9] and [11] presented necessary and sufficient conditions for a given affine feedback to solve RCP: (a) the affine feedback satisfies the invariance conditions, and (b) there are no closed-loop equilibria in $\mathcal{S}$.

The shortcoming of the results of [9], [11] was that the no-equilibrium condition (b) cannot be verified a priori. This condition only gives a trial and error style of design: choose an affine feedback satisfying the invariance conditions. If there is no closed-loop equilibrium in $\mathcal{S}$, RCP is solved. Otherwise choose another affine feedback, and repeat. With the aim to make RCP a relevant design tool for reachability problems, [3] initiated a new line of inquiry on feedback synthesis. Specifically, the goal was to replace the no-equilibrium condition (b) by a condition depending directly on the system data (A, B, a) and the simplex data. To make the synthesis problem tractable, [3] introduced a "preferred" triangulation of the state space. This triangulation is focused on the placement of the set $\mathcal{O}$ of possible equilibria of the control system relative to simplices of the triangulation. Several outcomes were achieved in [3]. First, necessary and sufficient conditions for solvability of RCP by affine feedback in terms of the problem data were obtained. These conditions are checkable a priori and easily lead to a control synthesis. Second, affine feedback and continuous state feedback were shown to be equivalent with respect to solvability of RCP.

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From [3] it emerged that there can be cases when RCP is solvable by open-loop controls but not by continuous state feedback. To close this gap, discontinuous feedbacks were investigated in [5], [6]. The method is based on the reach control indices [4], [6]. These indices bookkeep how many vertices of $\mathcal{S}$ directly depend on each input. They expose how affine or continuous state feedbacks may fail - such feedbacks induce closed-loop equilibria in sub-simplices that are inherently starved of sufficient inputs.

In [5], [6] a subdivision procedure was proposed that triangulates the simplex into sub-simplices with sub-reach control problems that are solvable by affine feedback. The final outcome is that if RCP is solvable by open-loop controls, then it is solvable by discontinuous piecewise affine feedback. Because discontinuous feedbacks may lead to chattering due to measurement error, in this paper we ask the following question: *can time-varying feedbacks solve RCP when continuous feedbacks fail?* We show that under the preferred triangulation of the state space it is possible to construct a simple time-varying feedback that solves RCP. The present paper is the conference announcement of [1]. Here we present context, examples, and intuition not available in [1], whereas [1] contains all proofs.

Notation. For a vector $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\text{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation $\text{sp}\{y_1, y_2, \ldots\}$ denotes the span of vectors $y_i \in \mathbb{R}^n$. A matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive. A $\mathscr{Z}$-matrix M is a nonsingular $\mathscr{M}$-matrix if every real eigenvalue of M is positive [2].

II. Reach Control Problem

Consider an n-dimensional simplex $\mathcal{S} := \text{co}\{v_0, \ldots, v_n\}$, the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. Let its vertex set be $V := \{v_0, \ldots, v_n\}$ and its facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$. The facet is indexed by the vertex it does not contain. Let h_j, $j \in \{0, \ldots, n\}$, be the unit normal vector to each facet $\mathcal{F}_j$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet*. Let $I := \{1, \ldots, n\}$ and define $I(x)$ to be the minimal index set among $\{0, \ldots, n\}$ such that $x \in \text{co}\{v_i \mid i \in I(x)\}$. For $x \in \mathcal{S}$ define the closed, convex cone $\mathcal{C}(x) := \left\{ \; y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \; j \in I \setminus I(x) \; \right\}$. Also we write cone($\mathcal{S}$) $:= \mathcal{C}(v_0)$, since $\mathcal{C}(v_0)$ is the tangent cone to $\mathcal{S}$ at v_0. We consider the affine control system

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{S}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and rank$(B) = m$. Let $\mathcal{B} = \text{Im}(B)$, the image of B. Let $\phi_u(t, x_0)$ denote the trajectory of (1) starting at x_0 under some control law u. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{S}$.

Problem 2.1 (Reach Control Problem (RCP)): Consider system (1) defined on $\mathcal{S}$. Find a state feedback $u = f(x)$ such that for each $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\delta > 0$ such that (i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$; (ii) $\phi_u(T, x_0) \in \mathcal{F}_0$; and (iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \delta)$.

In the sequel we use the shorthand notation $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ to denote that conditions (i)-(iii) of Problem 2.1 hold under some control law. To solve RCP we require conditions that disallow trajectories to exit from the facets $\mathcal{F}_i$, $i \in I$. We say the *invariance conditions are solvable* if there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that,

$$Av_i + Bu_i + a \in \mathcal{C}(v_i)\,, \qquad i \in \{0, \ldots, n\}\,. \tag{2}$$

The inequalities (2) called *invariance conditions* are necessary conditions for solvability of RCP by continuous state feedback [8] and by open-loop controls [6]. The following result provides the foundation for solving RCP by affine feedback.

Theorem 2.1 ([9], [11]): Given system (1) on a simplex $\mathcal{S}$ and an affine feedback $u = Kx + g$, with $K \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$, and $u_0 = u(v_0), \cdots, u_n = u(v_n)$, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using u if and only if

(a) The invariance conditions (2) hold,
(b) There is no closed-loop equilibrium in $\mathcal{S}$.

Theorem 2.1 provides for a trial and error style of design: choose an affine feedback satisfying (2). If there is no closed-loop equilibrium the problem is solved. Otherwise, select another affine feedback, and repeat. We wish to find a true synthesis method, particularly by replacing (b) with a checkable condition. To that end, we define $\mathcal{O} := \{\, x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}$ and $\mathcal{O}_\mathcal{S} := \mathcal{S} \cap \mathcal{O}$. One can show that either $\mathcal{O} = \emptyset$ or $\mathcal{O}$ is an affine space with $m \leq \dim(\mathcal{O}) \leq n$. Notice that $Ax + Bu + a$ for $x \in \mathcal{O}$ can vanish for an appropriate choice of u, so $\mathcal{O}$ is the set of possible (open- or closed-loop) equilibria of the system.

To obtain constructive conditions for solvability of RCP we invoke the *preferred triangulation* of [3]. See the algorithm in [10] on the placing triangulation.

Assumption 2.1: Simplex $\mathcal{S}$ and system (1) satisfy the following condition: if $\mathcal{O}_\mathcal{S} \neq \emptyset$, then $\mathcal{O}_\mathcal{S}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \leq \kappa \leq n$.

Under the preferred triangulation, all cases when RCP is solvable by affine feedback (or equivalently continuous state feedback) are known.

Theorem 2.2 ([3]): Suppose Assumption 2.1 holds. We have $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback if and only if the invariance conditions (2) are solvable and (at least) one of the following conditions holds: (i) $\mathcal{O}_\mathcal{S} = \emptyset$. (ii) $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$. (iii) $\mathcal{O}_\mathcal{S} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$ and there exists a linearly independent selection $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$.

Cases (i) and (ii) are the trivial cases when RCP is solvable by affine feedback. Condition (ii) implies that the control input can be selected sufficiently large so that the closed-loop vector field can be made to point in the direction of $\mathcal{F}_0$; hence trajectories will exit. Instead, condition (iii) regards the degree of actuation of the control system. When (iii) fails, we say the system is *underactuated* with respect to $\mathcal{S}$.

III. Reach Control Indices

We turn to more complex designs when RCP fails with continuous state feedback, and we continue to work with the preferred triangulation. Because the case $v_0 \in \mathcal{O}_\mathcal{S}$ is trivially resolved by Remark 8.1 of [3], we only study cases when $\mathcal{O}_\mathcal{S}$ is a κ-dimensional face in $\mathcal{F}_0$. Let $I_{\mathcal{O}_\mathcal{S}} := \{1, \ldots, \kappa+1\}$ be the vertex index set of $\mathcal{O}_\mathcal{S}$.

Assumption 3.1: Simplex $\mathcal{S}$ and system (1) satisfy the following conditions.

(A1) $\mathcal{O}_\mathcal{S} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < n$.
(A2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$.
(A3) $\mathcal{B} \cap \mathcal{C}(v_i) \neq \mathbf{0}$, $i \in I_{\mathcal{O}_\mathcal{S}}$.
(A4) The maximum number of linearly independent vectors in any selection $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ is $\widehat{m} \leq \kappa$.

In light of (A4), define $p := \kappa + 1 - \widehat{m}$.

Theorem 3.1 ([4], [6]): Suppose Assumption 3.1 holds. Then there exist integers $r_1, \ldots, r_p \geq 2$ such that w.l.o.g. (by reordering indices) for each $k = 1, \ldots, p$

$$\mathcal{B} \cap \mathcal{C}(v_i) \subset \text{sp}\{b_{m_k}, \ldots, b_{m_k+r_k-1}\}, i = m_k, \ldots, m_k + r_k - 1\,. \tag{3}$$

where $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$ and $m_k := r_1 + \cdots + r_{k-1} + 1$ for $k = 1, \ldots, p$. Moreover, for each $k = 1, \ldots, p$, $\{b_{m_k}, \ldots, b_{m_k+r_k-2}\}$ are linearly independent and

$$b_{m_k+r_k-1} = c_{m_k} b_{m_k} + \cdots + c_{m_k+r_k-2} b_{m_k+r_k-2} \tag{4}$$

with $c_i < 0$.

We observe that due to (4) the lists (3) have the property that any vector in a list on the right is dependent on all the other vectors in its list. Also, if any vector is removed from a list, the remaining vectors are linearly independent. In particular, the kth list contains $r_k - 1$ linearly independent vectors in $\mathcal{B}$. The integers $\{r_1, \ldots, r_p\}$ are called the *reach control indices* of system (1) with respect to simplex $\mathcal{S}$.

The implication of Theorem 3.1 for control design is as follows. First we observe that if we assign controls such that every vertex in $\mathcal{O}_\mathcal{S}$ is an equilibrium, then using affine feedback and by convexity, every point in $\mathcal{O}_\mathcal{S}$ is a closed-loop equilibrium. This case represents the "maximal set" of equilibria achievable in $\mathcal{S}$ using affine feedback. The question addressed by Theorem 3.1 is how to identify a (not necessarily unique) "minimal set" of equilibria that is not removable by affine (or continuous) feedback. The theorem addresses this question in the following sense. For each $k = 1, \ldots, p$ define $I_k := \{m_k, \ldots, m_k + r_k - 1\}$ and define the sub-simplices $\mathcal{O}_k := \text{co}\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$. Select control inputs at these vertices so that the corresponding closed-loop velocity vectors are $\{b_{m_k}, \ldots, b_{m_k+r_k-1}\}$ for $k = 1, \ldots, p$, where the b_i's are given by the theorem. Then select an affine feedback $u = Kx + g$ which satisfies $u(v_i) = b_i$, $i = 1, \ldots, r$, where $r := r_1 + \cdots + r_p$. Now observe that (4) implies that $0 \in \text{co}\{b_{m_k}, \ldots, b_{m_k+r_k-1}\}$ for each $k = 1, \ldots, p$. Since the closed-loop vector field is convex, each $\mathcal{O}_k$ contains an equilibrium. Indeed, Theorem 7.1 of [3] implies each $\mathcal{O}_k$, $k = 1, \ldots, p$ contains an equilibrium for any choice of control inputs. Each sub-simplex $\mathcal{O}_k$ has a shortfall of exactly one independent control direction (since removal of any one vector makes the set $\{b_{m_k}, \ldots, b_{m_k+r_k-1}\}$ linearly independent) and it is in this sense that the equilibrium set is "minimal" (linearly independent vectors cannot contain 0 in their convex hull). In sum, Theorem 3.1 tells us exactly where equilibria arise using affine feedbacks. We wish to exploit this information to construct other feedbacks which can eliminate or work around those equilibria.

Example 3.1: Let $\mathcal{S} = \text{co}\{v_0, \ldots, v_4\}$ with $v_0 = (-1, -1, 0, -1)$, $v_1 = (0, 0, 1, 0)$, $v_2 = (0, -1, 0, 0)$, $v_3 = (0, -1, 1, 0)$, and $v_4 = (0, -1, 0, 1)$. Consider the affine system

$$\dot{x} = \begin{bmatrix} -4 & 4 & -6 & -3 \\ -5 & -2 & 2 & 3 \\ -4 & 1 & -4 & 3 \\ -8 & 3 & -4 & -3 \end{bmatrix} x + \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 1 & 3 \\ -1 & 1 \end{bmatrix} u + \begin{bmatrix} 7 \\ -3 \\ 3 \\ 5 \end{bmatrix}.$$

We find $\mathcal{O} = \{x \mid x_1 = 0\}$, so $\mathcal{O}_{\mathcal{S}} = \text{co}\{v_1, v_2, v_3, v_4\}$. The normal vectors for $\mathcal{S}$ are $h_0 = (1, 0, 0, 0)$, $h_1 = (0, -1, 0, 0)$, $h_2 = (-2, 0, 1, 1)$, $h_3 = (0, 1, -1, 0)$, and $h_4 = (1, 0, 0, -1)$. The invariance conditions for $\mathcal{S}$ give $v_0 : -1 \le u_{01} \le 2, -\frac{2}{3} \le u_{02} \le 3$, $v_1 : u_{11} \le 1, 0 \le u_{12} \le 0$, $v_2 : u_{21} \ge 1, -1 \le u_{22} \le -1$, $v_3 : -1 \le u_{31} \le -1, u_{32} \le 1$, and $v_4 : -2 \le u_{41} \le -2, u_{42} \ge -1$. To simplify the calculation of the reach control indices, we apply the affine feedback transformation suggested in [4]. We let $u = K_1 x + g_1 + G_1 w$, where w is a new exogenous input. This yields a new affine system $\dot{x} = \hat{A}x + \hat{B}w + \hat{a} = (A + BK_1)x + BG_1 w + (a + Bg_1)$. The transformation is chosen so that $\hat{A}v_i + \hat{a} = 0$ for $i \in I_{\mathcal{O}_{\mathcal{S}}}$; $\hat{A}v_i + \hat{a} \in \mathcal{C}(v_i) \setminus \mathcal{B}$ for $i \notin I_{\mathcal{O}_{\mathcal{S}}}$; and $\hat{B} = BG_1$. The fact that $\mathcal{O}$ is unaffected by this transformation is shown in [4]. For this example we get

$$u = \begin{bmatrix} 4 & 2 & -2 & -3 \\ -1 & -1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} w.$$

The new (linear) system is

$$\dot{x} = \begin{bmatrix} -10 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ -13 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & -4 \\ -1 & 0 \\ -1 & -6 \\ 1 & -2 \end{bmatrix} w.$$

Here we have chosen $b_1 := (1, -1, -1, 1) \in \mathcal{B} \cap \mathcal{C}(v_1)$, $b_3 := (-4, 0, -6, -2) \in \mathcal{B} \cap \mathcal{C}(v_3)$, and $\hat{\mathcal{B}} = \text{sp}\{b_1, b_3\}$. Also $b_2 := -2b_1 = (-2, 2, 2, -2) \in \mathcal{B} \cap \mathcal{C}(v_2)$, and $b_4 := -\frac{1}{2}b_3 = (2, 0, 3, 1) \in \mathcal{B} \cap \mathcal{C}(v_4)$. Now it is clear there are two reach control indices $r_1 = 2$ and $r_2 = 2$, and $\mathcal{O}_1 = \text{co}\{v_1, v_2\}$ and $\mathcal{O}_2 = \text{co}\{v_3, v_4\}$. $\triangleleft$

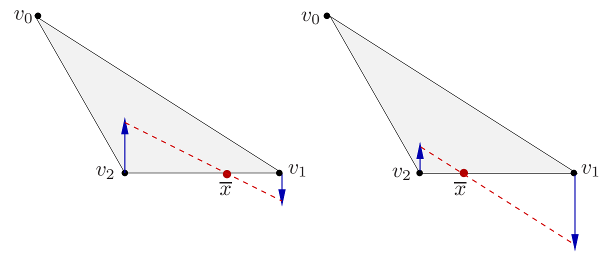

Fig. 1. Sliding the equilibrium along $\mathcal{F}_0$.

IV. Time-Varying Affine Feedback

For the preferred triangulation and in the case when continuous state feedbacks fail, discontinuous piecewise affine feedbacks have been proposed to solve RCP [6]. A natural question is: *can time-varying feedbacks solve RCP?*

To motivate the method, we look at the simplest example when continuous state feedback fails. Consider the simplex $\mathcal{S} = \text{co}\{v_0, v_1, v_2\}$ with facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$, as in the left of Figure 1. Consider the single-input system $\dot{x} = Ax + bu + a$ and suppose $b = (0, 1)$ and $\mathcal{O} = \{v_1, v_2\}$. Label the vertex velocity vectors as $y_0 = Av_0 + bu_0 + a$, and $b_i = Av_i + bu_i + a$, $i = 1, 2$. To satisfy the invariance conditions, at v_1, b_1 has to point down, but at v_2, b_2 has to point up. If we continuously interpolate along $\mathcal{F}_0$ from v_1 to v_2, the continuous vector field, always in $\text{Im}(b)$ along $\mathcal{F}_0$, must pass through zero (by the Intermediate Value Theorem). The reach control indices tell us that the defect is that there are two vertices v_1 and v_2 that share one control input.

The proposed technique to overcome this problem is to "move" the equilibrium that appears along $\mathcal{F}_0$ by using a specially design time-varying affine feedback. First, we select two affine controllers: $u^0(x)$ which places an equilibrium at v_1 and nowhere else, and $u^\infty(x)$ which places an equilibrium at v_2, and nowhere else. Second, define a controller that interpolates between these two: $u(x, \alpha) := (1 - \alpha)u^0(x) + \alpha u^\infty(x)$. By playing with $\alpha \in [0, 1]$, we can "slide" the equilibrium along $\mathcal{F}_0$. See Figure 1. Effectively we have introduced a new degree of freedom that is not available with continuous state feedback. Our method will be to slide the equilibrium from v_1 to v_2, opposite to the natural direction of flow of trajectories in $\mathcal{S} \setminus \mathcal{F}_0$. The technical difficulty arises in carrying out the design for multi-input systems. The development is divided into two parts. First we establish that a flow-like condition holds on $\mathcal{S}$. By a *flow-like condition* we mean a condition of the form $\xi^* \cdot (Ax + Bu(x) + a) \ge 0$, $x \in \mathcal{S}$, where $0 \ne \xi^* \in \mathbb{R}^n$. Geometrically this condition corresponds to a foliation of parallel hyperplanes $\mathcal{H}_c := \{x \in \mathbb{R}^n \mid \xi^* \cdot x = c\}$, $c \in \mathbb{R}$, with normal vector ξ^* such that closed-loop trajectories on $\mathcal{S}$ flow in one sense only with respect to the hyperplanes. Second, we propose a time-varying compensator whose role is to dynamically shift the set of equilibria generated by affine feedback in a direction opposite to the direction indicated by the flow-like condition.

A. Flow-like Condition

In this section the flow-like condition is developed. First, it is a fact of linear system theory that on any convex set $\mathcal{P}$ not intersecting $\mathcal{O}$, trajectories only flow in one sense relative to $\mathcal{B}$; that is, there exists $\xi^2 \in \mathrm{Ker}(B^T)$ such that $\xi^2 \cdot (Ax + Bu + a) > 0$ for all $x \in \mathcal{P}$, $u \in \mathbb{R}^m$. Because of the preferred triangulation that places $\mathcal{O}$ on the boundary of $\mathcal{S}$, this property is guaranteed on $\mathcal{S} \setminus \mathcal{O}_{\mathcal{S}}$. Instead on $\mathcal{O}$, trajectories only flow in the direction of $\mathcal{B}$, so automatically $\xi^2 \cdot (Ax + Bu + a) = 0$, $x \in \mathcal{O}_{\mathcal{S}}$, $u \in \mathbb{R}^m$. Second, we require a flow-like condition on $\mathcal{O}_{\mathcal{S}}$ with the property that a hyperplane associated with it strongly separates at least two vertices in each group $\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$, $k = 1, \ldots, p$. The requirement forces a deeper study of system structure, and $\mathscr{M}$-matrices are the vehicle to extract it. Relevant $\mathscr{M}$-matrices are identified in Lemma 4.1, and they immediately leads to candidate vectors β_k for separating vertices in $\mathcal{O}_{\mathcal{S}}$. These vectors only succeed to separate vertices in $\mathcal{O}_{\mathcal{S}}$ if $\mathcal{B}$ is not parallel to $\mathcal{F}_0$. Finally, a combination of the β_k's and ξ^2 will give ξ^*.

Assumption 4.1: Simplex $\mathcal{S}$ and system (1) satisfy Assumption 3.1 and the following condition.

(A5) $\mathrm{sp}\{b_{m_k}, \ldots, b_{m_k+r_k-1}\} \not\subset \mathcal{H}_0 := \{y \in \mathbb{R}^n \mid h_0 \cdot y = 0\}, \quad k = 1, \ldots, p.$

Recall $r = r_1 + \cdots + r_p$ and define the matrix $\widehat{B} = \left[\, b_1 \cdots \; b_{m_1+r_1-2} \cdots \; b_{m_p} \cdots \; b_{m_p+r_p-2} \right] \in \mathbb{R}^{n \times (r-p)}$ and let $\widehat{\mathcal{B}} = \mathrm{Im}(\widehat{B})$. Note that the columns of $\widehat{B}$ are ordered according to Theorem 3.1 and that vectors $b_{m_k+r_k-1}$, $k = 1, \ldots, p$, do not appear in the columns of $\widehat{B}$. Also, velocity vectors associated with v_i, $i = r+1, \ldots, \kappa+1$ are not yet defined, so they do not appear. By Theorem 3.1, $\mathrm{rank}(\widehat{B}) = r - p$. To set up the relevant $\mathscr{M}$-matrices, define matrices $H_{q,s} := [h_q \cdots h_s]$, $Y_{q,s} := [b_q \cdots b_s]$, and $M_{q,s} := H_{q,s}^T Y_{q,s}$. Since $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $i \in I_{\mathcal{O}_{\mathcal{S}}}$, each $M_{q,s}$ is a $\mathscr{Z}$-matrix.

Lemma 4.1: [1] Suppose Assumption 4.1 holds. Then for each $k = 1, \ldots, p$,

(i) Each principal submatrix of M_{m_k, m_k+r_k-1} is a nonsingular $\mathscr{M}$-matrix.
(ii) Matrix $M_{m_k, m_k+r_k-1} \in \mathbb{R}^{r_k \times r_k}$ is irreducible.
(iii) Matrix M_{m_k, m_k+r_k-1} is a singular $\mathscr{M}$-matrix.

Let $k \in \{1, \ldots, p\}$. Since M_{m_k, m_k+r_k-1} is a singular, irreducible $\mathscr{M}$-matrix, so is $M_{m_k, m_k+r_k-1}^T$. By Theorem 6.4.16(2) of [2], there exists $d_k \in \mathbb{R}^{r_k}$ with $d_k \prec 0$ such that $M_{m_k, m_k+r_k-1}^T d_k = 0$. Define

$$\beta_k := H_{m_k, m_k+r_k-1} d_k\,. \tag{5}$$

It can be shown that $\mathrm{Ker}(\widehat{B}^T) = \mathrm{sp}\{\beta_1, \ldots, \beta_p, h_{r+1}, \ldots, h_n\}$. The next result shows that β_k can be used to strongly separate at least two vertices in $\mathcal{O}_k$.

Lemma 4.2: Suppose Assumption 4.1 holds. For each $k \in \{1, \ldots, p\}$, there exist $i_k, j_k \in I_k$ such that $\beta_k \cdot (v_{i_k} - v_{j_k}) \neq 0$.

In light of Lemma 4.2, we assume w.l.o.g. (by reordering the indices within each group I_k) that for $k = 1, \ldots, p$

$$v_{m_k} \in \arg\max_{i \in I_k} \beta_k \cdot v_i\,, \;\; v_{m_k+r_k-1} \in \arg\min_{i \in I_k} \beta_k \cdot v_i\,. \tag{6}$$

We define the sets $\mathcal{E}^0 := \mathrm{co}\big\{v_{m_1}, v_{m_2}, \ldots, v_{m_p}\big\}$ and $\mathcal{E}^\infty := \mathrm{co}\big\{v_{m_1+r_1-1}, v_{m_2+r_2-1}, \ldots, v_{m_p+r_p-1}\big\}$. The following is the main result on a flow-like condition.

Theorem 4.1: Suppose Assumption 4.1 holds. Let $u(x,t)$ be a time-varying affine feedback such that for $t \geq 0$

$$Av_i + Bu(v_i,t) + a \in \mathcal{C}(v_i)\,, \quad i = 0, r+1, \ldots, n \tag{7a}$$
$$Av_i + Bu(v_i,t) + a \in \widehat{\mathcal{B}}\,, \qquad i = 1, \ldots, r\,. \tag{7b}$$

Then there exists $0 \neq \xi^* \in \mathrm{Ker}(\widehat{B}^T)$ such that for $t \geq 0$

$$\xi^* \cdot (Ax + Bu(x,t) + a) \geq 0\,, \qquad x \in \mathcal{S}\,, \tag{8}$$

and such that $\mathcal{H}^*$ strongly separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$ where $\mathcal{H}^* := \{x \in \mathbb{R}^n \mid \xi^* \cdot (x - v_0) = 1\}$.

Example 4.1: We continue Example 3.1. Using the reach control indices and Lemma 4.1, the relevant $\mathscr{M}$-matrices are $M_{1,2}^T = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, and $M_{3,4}^T = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix}$ whose kernels are spanned by $d_1 = (1, \frac{1}{2})$ and $d_2 = (1, 3)$ respectively. Using (5) we get $\beta_1 = H_{1,2} d_1 = (1, 1, -\frac{1}{2}, -\frac{1}{2})$ and $\beta_2 = H_{3,4} d_2 = (-3, -1, 1, 3)$. This gives a basis $\mathrm{Ker}(B^T) = \mathrm{sp}\{\beta_1, \beta_2\}$. Following (6), we obtain $v_1 = v_{m_1} \in \arg\max_{i \in I_1} \beta_1 \cdot v_i$, $v_2 = v_{m_1+r_1-1} \in \arg\min_{i \in I_1} \beta_1 \cdot v_i$, $v_4 = v_{m_2} \in \arg\max_{i \in I_2} \beta_2 \cdot v_i$, and $v_3 = v_{m_2+r_2-1} \in \arg\min_{i \in I_2} \beta_2 \cdot v_i$. We use these to define the sets $\mathcal{E}^0 = \mathrm{co}\{v_1, v_4\}$ and $\mathcal{E}^\infty = \mathrm{co}\{v_2, v_3\}$. All elements are now in place for designing the time-varying compensator. $\triangleleft$

B. Time-varying Compensator

The time-varying compensator will be constructed so as to exploit the flow-like condition (8) and the separation property of $\mathcal{H}^*$. First, we define two affine feedbacks $u^0(x)$ and $u^\infty(x)$ that place equilibria at $\mathcal{E}^0$ and $\mathcal{E}^\infty$, respectively. Then we define a compensator $u(x, \alpha)$ with additional state $\alpha \in \mathbb{R}$. This compensator simply interpolates between $u^0(x)$ and $u^\infty(x)$ as α varies from 0 to 1. By construction when $\alpha = 0$, all closed-loop equilibria are in $\mathcal{E}^0$. When $\alpha = 1$, they are in $\mathcal{E}^\infty$. Thus, as α varies from 0 to 1, the set of closed-loop equilibria crosses from one side of $\mathcal{H}^*$ to the other in a direction with decreasing ξ^* component. Informally, we can say that trajectories flow downstream according to (8) while equilibria flow upstream, so that no trajectory can be "stuck" at an equilibrium. Ultimately, this enables all trajectories to exit $\mathcal{S}$, as shown in Theorem 4.2.

Suppose the invariance conditions for $\mathcal{S}$ are solvable; thus, there exist inputs $u_0^0, \ldots, u_n^0 \in \mathbb{R}^m$ such that (2) hold. Let $y_i^0 := Av_i + Bu_i^0 + a$, for $i = 0, \ldots, n$. We choose $u_1^0, \ldots, u_{\kappa+1}^0 \in \mathbb{R}^m$ such that

$$y_i^0 = 0\,, i \in \{m_1, m_2, \ldots, m_p\} \tag{9a}$$
$$y_i^0 = b_i\,, i \in I_{\mathcal{O}_{\mathcal{S}}} \setminus \{m_1, m_2, \ldots, m_p\}\,, \tag{9b}$$

where $b_i \in \widehat{\mathcal{B}} \cap \mathcal{C}(v_i)$, $i = 1, \ldots, r$, are provided by Theorem 3.1; and $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $i = r+1, \ldots, \kappa+1$, are selected so that $\widehat{m}$ independent directions in $\mathcal{B}$ are associated with $\mathcal{O}_{\mathcal{S}}$, as per (A4). Finally, construct the associated affine feedback $u^0(x) = K^0 x + g^0$, and let $\phi^0(t, x_0)$ denote trajectories of the closed-loop system. Note that the closed-loop system has equilibria at $v_{m_1}, \ldots, v_{m_p}$.

Next we define a symmetrical controller $u^\infty(x)$ which is identical to $u^0(x)$ except that it places equilibria at $v_{m_k+r_k-1}$, $k = 1, \ldots, p$. Let $y_i^\infty := \hat{A} v_i + B u_i^\infty + a$, for $i = 0, \ldots, n$. First set $u_i^\infty = u_i^0$, $i = 0, r+1, \ldots, n$. Then we choose $u_1^\infty, \ldots, u_{\kappa+1}^\infty \in \mathbb{R}^m$ such that

$$
\begin{aligned}
y_i^\infty &= b_i \,, i \in \{1, \ldots, r\} \setminus \{m_1+r_1-1, \ldots, m_p+r_p-1\} && \text{(10a)}\\
y_i^\infty &= 0 \,, i \in \{m_1+r_1-1, \ldots, m_p+r_p-1\} \,, && \text{(10b)}
\end{aligned}
$$

where again $b_i \in \widehat{\mathcal{B}} \cap \mathcal{C}(v_i)$, $i = 1, \ldots, r$, are provided by Theorem 3.1. Finally, construct the associated affine feedback $u^\infty(x) = K^\infty x + g^\infty$, and let $\phi^\infty(t, x_0)$ denote trajectories of the closed-loop system. Note that this closed-loop system has equilibria at $v_{m_k+r_k-1}$, $k = 1, \ldots, p$.

Now we extend the state x by an additional scalar state α with dynamics $\dot{\alpha} = -c\alpha + c$, $\alpha(0) = 0$, where $c > 0$ is a to-be-determined constant. Define a feedback

$$
u(x, \alpha) := (1-\alpha) u^0(x) + \alpha u^\infty(x) \,. \tag{11}
$$

Clearly the role of $u(x, \alpha)$ is to interpolate from $u^0(x)$ to $u^\infty(x)$ as α varies from 0 to 1.

Theorem 4.2: Suppose Assumption 4.1 holds and suppose the invariance conditions (2) for $\mathcal{S}$ are solvable. There exists $c > 0$ sufficiently small such that $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using feedback $u(x, \alpha)$ in (11).

Example 4.2: We continue Example 4.1. First we design $w^0(x) = K^0 x + g^0$ such that $y_0^0 = \hat{A} v_0 + \hat{a}$, $y_{m_1}^0 = y_1^0 = 0$, $y_{m_1+r_1-1}^0 = y_2^0 = b_2$, $y_{m_2+r_2-1}^0 = y_3^0 = b_3$, $y_{m_2}^0 = y_4^0 = 0$. This yields the control values $w_0^0 = (0,0)$, $w_1^0 = (0,0)$, $w_2^0 = (-2,0)$, $w_3^0 = (0,1)$, and $w_4^0 = (0,0)$. By solving $\begin{bmatrix} K^0 & g^0 \end{bmatrix} = \begin{bmatrix} w_0^0 & \ldots & w_4^0 \end{bmatrix} \begin{bmatrix} v_0 & \ldots & v_4 \\ 1 & \ldots & 1 \end{bmatrix}^{-1}$ we find that $w^0(x) = K^0 x + g^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Second we design $w^\infty(x) = K^\infty x + g^\infty$ such that $y_0^\infty = \hat{A} v_0 + \hat{a}$, $y_{m_1}^\infty = y_1^\infty = b_1$ $y_{m_1+r_1-1}^\infty = y_2^\infty = 0$, $y_{m_2+r_2-1}^\infty = y_3^\infty = 0$, and $y_{m_2}^\infty = y_4^\infty = b_4$, This yields the control values $w_0^\infty = (0,0)$, $w_1^\infty = (1,0)$, $w_2^\infty = (0,0)$, $w_3^\infty = (0,0)$, and $w_4^\infty = (0, -\frac{1}{2})$. By the same procedure as above we find that $w^\infty(x) = K^\infty x + g^\infty = \begin{bmatrix} -4 & 0 & 2 & 2 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{bmatrix} x + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. We now extend the state with $\alpha \in \mathbb{R}$ to get $w(x, \alpha) = (1-\alpha) w^0(x) + \alpha w^\infty$ where $\alpha(t) = 1 - e^{-ct}$. Since we have used a feedback transformation, we must determine our actual input u to be

$$
\begin{aligned}
u(x, \alpha) &= K_1 x + g_1 + G_1 w(x, \alpha)\\
&= (1-\alpha) u^0(x) + \alpha u^\infty(x)\\
&= (1-\alpha) \left(\begin{bmatrix} 4 & 1 & -2 & -3 \\ -1 & 3 & -2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right)\\
&\quad + \alpha \left(\begin{bmatrix} 8 & 2 & -4 & -5 \\ -3 & -1 & 2 & 2 \end{bmatrix} x + \begin{bmatrix} 5 \\ -2 \end{bmatrix} \right)
\end{aligned}
$$

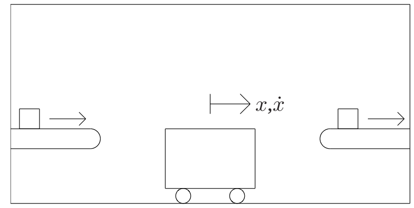

Fig. 2. Schematic of material transfer system.

V. Application

We illustrate the use of reach controllers and time-varying compensation in an application. Consider the cart-conveyor system shown in Figure 2. The objective is to move goods in a production facility between two locations autonomously, in this case two conveyors. The dynamics of the cart are given by

$$
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,. \tag{12}
$$

The control objective is to design a state feedback to achieve the following *complex specifications*:
(S) **Safety**: $|x_1| \le 3$, $|x_2| \le 3$, $|x_1 + x_2| \le 3$;
(L) **Liveness**: $|x_1| + |x_2| \ge 1$;
(TS) **Temporal Sequence**: Every box arriving at conveyor 1 is always eventually delivered to conveyor 2.

The safety requirements disallow the cart from colliding with the conveyors and from exceeding actuator limits. The liveness requirement imposes fast motion far from the conveyors. These specifications generate a non-convex polytope $\mathcal{P}$ as shown in Figure 3. We triangulate $\mathcal{P}$ using only the vertices of $\mathcal{P}$ to obtain ten simplices $\mathcal{S}_i$, $i = 1, \ldots, 10$; see the figure. The arrows in the figure indicate the direction of flow in order to achieve the desired temporal sequence. We design a control law that solves RCP for each simplex with exit facets determined by the desired direction of flow. To handle the switching between the controllers, we employ a discrete supervisory controller. The states of the supervisor coincide with membership of the continuous time state in a particular simplex $\mathcal{S}_i$, $i = 1, \ldots, 10$. The transitions between the states of the DES correspond to *events*, which occur when the continuous state crosses exit facets. The exit facet for each simplex is given by $\mathcal{F}_0^i := \mathcal{S}_i \cap \mathcal{S}_{i+1}$, $i = 1, \ldots, 9$, and for simplex $\mathcal{S}_{10}$ the exit facet is $\mathcal{F}_0^{10} := \mathcal{S}_{10} \cap \mathcal{S}_1$.

By examining (12) we find that $\mathcal{O} = \{x \mid x_2 = 0\}$ and we define $\mathcal{O}_{\mathcal{S}}^i := \mathcal{S}_i \cap \mathcal{O}$, $i = 1, \ldots, 10$. It can be verified

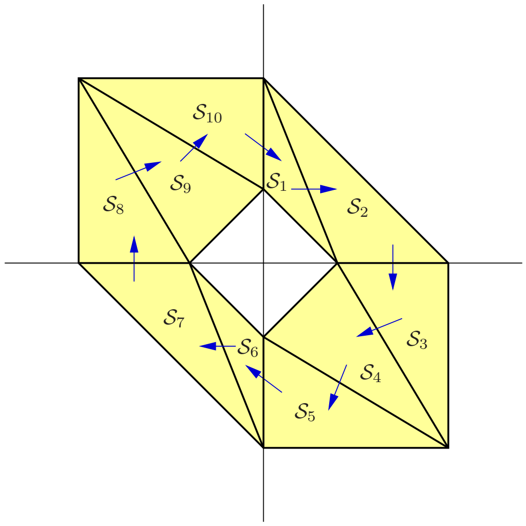

Fig. 3. Polytope $\mathcal{P}$ and triangulation into simplices $\mathcal{S}_i$.

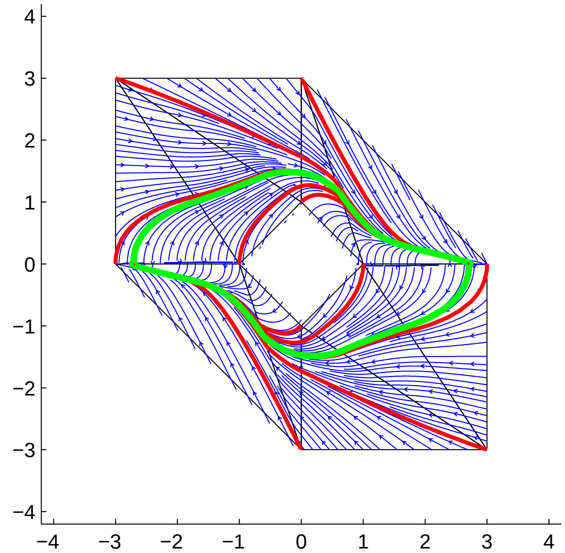

Fig. 4. Simulation of material transfer system with $c = 0.01$ for time-varying affine feedbacks.

that the invariance conditions (2) are solvable for each $\mathcal{S}_i$. By inspection, $\mathcal{O}^5_{\mathcal{S}} = \emptyset$ and $\mathcal{O}^{10}_{\mathcal{S}} = \emptyset$. Therefore, by Theorem 6.1 of [3], $\mathcal{S}_5 \xrightarrow{\mathcal{S}_5} \mathcal{F}^5_0$ $\mathcal{S}_{10} \xrightarrow{\mathcal{S}_{10}} \mathcal{F}^{10}_0$ by affine feedback. Also, $\mathcal{O}^i_{\mathcal{S}} \neq \emptyset$ and $\mathcal{B} \cap \text{cone}(\mathcal{S}_i) \neq \mathbf{0}$ for $i \in \{1, 3, 4, 6, 8, 9\}$. Therefore, by Theorem 6.2 of [3], $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{F}^i_0$ for $i \in \{1, 3, 4, 6, 8, 9\}$.

It remains to design control laws for $\mathcal{S}_2$ and $\mathcal{S}_7$. For these simplices, $\mathcal{O}^i_{\mathcal{S}} \neq \emptyset$, Assumption 4.1 holds, $\mathcal{B} \cap \text{cone}(\mathcal{S}_i) = \mathbf{0}$, and the system is underactuated on these two simplices. By the results of [3], RCP is not solvable by continuous state feedback on $\mathcal{S}_2$ and $\mathcal{S}_7$. However, by Theorem 4.2, it is solvable by time-varying affine feedback. Figure 4 shows the phase portrait of the resulting closed-loop system. For $\mathcal{S}_2$ and $\mathcal{S}_7$ we choose $c = 0.01$ and we plot the vector field for $\alpha = 0$. The green curve corresponds to a steady-state limit cycle behavior after trajectories initiated at the vertices of the polytope have iterated around 15 simplices.

VI. Discussion

We have examined currently available methods to design reach controllers on simplices under the Assumption 2.1 of a preferred triangulation. The simplest designs are based on affine feedbacks, and among those the simplest cases are either when $\mathcal{O}_{\mathcal{S}} = \emptyset$, $v_0 \in \mathcal{O}_{\mathcal{S}}$, or $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$. If these three cases do not apply, then the next property to check is whether the system is *underactuated* with respect to the given simplex. If the system is sufficiently actuated, then the design procedure is to assign a maximal number of linearly independent control directions b_i at the vertices in $\mathcal{O}_{\mathcal{S}}$, and to construct an affine feedback satisfying the invariance conditions.

If $\mathcal{O}_{\mathcal{S}} \neq \emptyset$, $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$, and the system is underactuated, then affine feedbacks always fail under the preferred triangulation; indeed so do continuous state feedbacks [3]. In that case, the natural candidates are discontinuous feedback [6] or time-varying feedback. Both of these solutions depend on computing the reach control indices, and the most complex designs involve multiple indices. This design flow resolves all cases of interest for RCP under the preferred triangulation since the time-varying affine feedback presented here will solve RCP in all cases when the problem is solvable by open-loop controls, but not by affine feedback [6].

References

[1] G. Ashford and M.E. Broucke. Time-varying affine feedback for reach control on simplices. *Automatica*. vol. 49, issue 5, pp. 1365-1369, May 2013.

[2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, New York. 1979.

[3] M.E. Broucke. Reach control on simplices by continuous state feedback. *SIAM Journal on Control and Optimization*. vol. 48, issue 5, pp. 3482-3500, February 2010.

[4] M.E. Broucke. On the reach control indices of affine systems on simplices. *8th IFAC Symposium on Nonlinear Control Systems*. August 2010.

[5] M.E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *American Control Conference*. June 2011.

[6] M.E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *SIAM J. Control and Optimization*. Submitted, December 2012. arXiv:1212.6450 [math.OC]

[7] L.C.G.J.M. Habets and J.H. van Schuppen. Control of piecewise-linear hybrid systems on simplices and rectangles, in: M.D. Di Benedetto and A.L. Sangiovanni-Vincentelli (Eds.) *Hybrid Systems: Computation and Control, Lecture Notes in Computer Science*. Springer Verlag, vol. 2034, pp. 261–274, 2001.

[8] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.

[9] L.C.G.J.M. Habets, P.J. Collins, and J.H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Trans. Automatic Control*. no. 51, pp. 938–948, 2006.

[10] C. W. Lee. Subdivisions and triangulations of polytopes. *Handbook of Discrete and Computational Geometry*. CRC Press Series Discrete Math. Appl., pp. 271–290, 1997.

[11] B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. vol. 42, no. 11, pp. 1913–1918, November 2006.