Stability and Controllability of Planar, Conewise Linear Systems

Ari Arapostathis [a,2] and Mireille E. Broucke [b,*,1]

[a] *Department of Electrical and Computer Engineering, University of Texas, Austin TX 78712, U.S.A.*

[b] *Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada*

Abstract

This paper presents a fairly complete treatment of stability and controllability of piecewise-linear systems defined on a conic partition of $\mathbb{R}^2$. This includes necessary and sufficient conditions for stability and controllability, as well as establishing that controllability implies stabilizability by piecewise-linear state feedback. A key tool in the approach is the study of the Poincaré map.

Key words: switched systems, piecewise linear systems, stability, controllability

1 Introduction

This paper studies stability and controllability of piecewise-linear systems defined on a conic partition of $\mathbb{R}^2$, which we call *conewise linear systems* (CLS). We derive necessary and sufficient conditions for stability and for controllability, as well as establish that controllability implies stabilizability via piecewise-linear state feedback. The analysis relies on the study of the Poincaré map. As long as the standard assumptions are posed concerning the lack of trajectories following unstable eigenvectors or unstable sliding modes, the properties of the Poincaré map are the determining factor in stability. The Poincaré map is

* Corresponding author: `broucke@control.utoronto.ca`.
1 Supported by the Natural Sciences and Engineering Research Council of Canada.
2 Supported in part by the Office of Naval Research through the Electric Ship Research and Development Consortium and in part by the National Science Foundation under Grants ECS-0218207 and ECS-0424169.

again used to study controllability, thus providing a unifying theme. Assuming there are no one-dimensional controlled invariant subspaces or half-lines (those on sliding surfaces), a Poincaré type map of the boundary of the funnel of the controlled trajectories provides necessary and sufficient conditions for controllability.

Pachter and Jacobson [1] also obtain a necessary and sufficient condition for stability of switched linear systems in the plane with conic switching by calculating the gain of a Poincaré map. In this paper we go one step further by obtaining explicit algebraic expressions for what we refer to as the *characteristic values* of the CLS. Roughly speaking, for a CLS there are two mechanisms that lead to stability or instability. One is the effect of the time-average of the eigenvalues of the individual linear components on each partition weighted by the fraction of the time that trajectories spend on each partition. The other is induced by the non-commutativity of the individual linear maps. The expressions obtained in this paper distinguish between the two components and thus shed some new light on the issue of stability.

Xu and Antsaklis [2] obtain necessary and sufficient conditions for asymptotic stabilizability of second-order switched linear systems, and they construct a stabilizing control law. Their results are obtained via a detailed analysis of the Poincaré map and the phase portraits of individual vector fields; they obtain a conic switching law essentially by selecting the linear system along each ray that points most directly to the origin. While the underlying geometric approach based on the study of the Poincaré map is the same, the primary difference between the present work and theirs lies in the problem formulation: they start with a collection of autonomous linear vector fields and address the problem of selecting the switching boundaries so as to obtain an asymptotically stable system; in the present work we consider a controlled piecewise linear system on a given partition and address the problem of existence of a stabilizing control law.

Several results on necessary and sufficient conditions for stability pertain to switched systems with arbitrary time switching. Boscain [3] obtained necessary and sufficient conditions for the stability of a time-switched linear system with two subsystems and arbitrary switching between them. Holcman and Margaliot [4] obtained a necessary and sufficient condition for stability of two homogeneous subsystems with arbitrary switching by constructing an appropriate common Lyapunov function. Margaliot [5] studied the problem of stability of switched systems with arbitrary switching using a variational approach in order to analyze the most unstable trajectory of the switched system. See the references therein for related work on worst-case switching laws and Lie-algebraic methods. The identification of worst case trajectories arises in the present work in our analysis of stabilizability.

The paper is organized as follows. In Section 2 we present a preliminary result on trajectories escaping convex cones in $\mathbb{R}^d$. In Section 3 we give our main result on stability by computing characteristic values. In Section 4 we present necessary and sufficient conditions for controllability, and in Section 5 results are given on stabilizability.

2 Preliminaries

In this section, we present some preliminary definitions and results. In particular, we show that if a closed, convex cone contains no subspaces and no eigenvectors of the system matrix, then all trajectories escape the cone.

Definition 1 *Let $\dot{x} = Ax$ be the dynamics on a convex cone $\mathcal{K}$ of $\mathbb{R}^d$. We define an eigenvector of A to be* visible *if it lies in $\bar{\mathcal{K}}$, the closure of $\mathcal{K}$.*

The following result appeared in [6] and relies on Lefschetz's fixed point theorem.

Lemma 2 (Pachter, [6]) *Let $\mathcal{K}$ be a non-empty closed convex cone in $\mathbb{R}^d$ but not a linear subspace. If $\mathcal{K}$ is invariant under the semigroup $\{e^{At}\}$, i.e., $e^{At}\mathcal{K} \subset \mathcal{K}$ for all $t \geq 0$, then $\mathcal{K}$ contains an eigenvector of A.*

Lemma 2 clearly implies the following result. Its relevance is in enabling us to argue that the characteristic values computed in Section 3 are well-defined.

Theorem 3 *Let $\mathcal{K}$ be a closed convex cone in $\mathbb{R}^d$, and suppose $\mathcal{K}$ does not contain a subspace of $\mathbb{R}^d$. Suppose no eigenvectors of $A \in \mathbb{R}^{d\times d}$ lie in $\mathcal{K}$. Then for any initial condition $x_0 \in \mathcal{K}$, $x_0 \neq 0$, there exists $t_0 \in \mathbb{R}$ such that $e^{At_0}x_0 \notin \mathcal{K}$.*

PROOF. Suppose that for some non-zero initial condition $x_0 \in \mathcal{K}$, $e^{At}x_0 \in \mathcal{K}$, for all $t \geq 0$. Let $\hat{\mathcal{K}}$ denote the maximal invariant set under the semigroup $\{e^{At}\}$ contained in $\mathcal{K}$; that is, $\hat{\mathcal{K}}$ is formed by the union of trajectories that lie in $\mathcal{K}$ for all $t \geq 0$. Clearly $\hat{\mathcal{K}} \neq \varnothing$, and since the dynamics are linear, it is evident that $\hat{\mathcal{K}}$ is also a closed convex cone. Moreover, $\hat{\mathcal{K}}$ is not a subspace since $\mathcal{K}$ does not contain a subspace of $\mathbb{R}^d$. Thus, by Lemma 2, $\hat{\mathcal{K}}$ contains an eigenvector of A, leading to a contradiction. □

3 Stability

In this section we define the characteristic values of a planar CLS and express them as explicit functions of the system parameters. The method amounts to computing the growth of trajectories over one cycle around the origin and using this parameter to obtain the asymptotic behavior of the CLS. Let $\mathcal{A} = \{A_j \in \mathbb{R}^{2\times 2}\,,\ j = 1, \dots, k_0\}$ be a collection of matrices and let $\{v_1, \dots, v_{\ell+1}\}$ be a set of unit vectors in $\mathbb{R}^2$ directed counterclockwise such that $v_{\ell+1} = v_1$. We define $\Theta(\cdot\,,\cdot)$ to be the angle in radians between two vectors in $\mathbb{R}^2$ in the counterclockwise sense, and assume, without loss of generality, that $\Theta\big(v_i, v_{i+1}\big) < \pi$. Let $\{\mathcal{K}_1, \dots, \mathcal{K}_\ell\}$ be a set of open convex cones that form a partition of $\mathbb{R}^2$ such that $\mathcal{K}_i$ is generated by $\{v_i, v_{i+1}\}$. On each $\mathcal{K}_i$ we have the dynamics $\dot{x} = A_i x$ with $A_i \in \mathcal{A}$. We denote the resulting CLS by $\Sigma = \{(\Sigma_i, \mathcal{K}_i)\,,\ i = 1, \dots, \ell\}$ where Σ_i denotes the dynamics on $\mathcal{K}_i$. Let $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and define the index set $\mathcal{I} = \{1, \dots, \ell\}$. For $i \in \mathcal{I}$, we define $\mathcal{V}_i = \{\lambda v_i : \lambda \in (0, \infty)\}$. Let n_i denote the unit vector orthogonal to $\mathcal{V}_i$ satisfying $n_i^{\mathsf{T}} v_{i+1} > 0$ (i.e., $\{n_1, \dots, n_\ell\}$ is a collection of unit normal vectors to $\{\mathcal{V}_1, \dots, \mathcal{V}_\ell\}$ ordered counterclockwise).

The asymptotic behavior of the system Σ is determined by the visible eigenvectors, sliding modes, and by the trajectories which encircle the origin. First we place conditions on the visible eigenvectors and sliding modes to insure stability. Let

$$\alpha_i^+ \triangleq n_i^{\mathsf{T}} A_i v_i\,, \qquad \alpha_i^- \triangleq n_{i+1}^{\mathsf{T}} A_i v_{i+1}\,, \quad i \in \mathcal{I}\,. \tag{3.1}$$

If $\alpha_i^+ \alpha_{i-1}^- \le 0$ and $|\alpha_i^+| + |\alpha_{i-1}^-| \neq 0$, let $r_i \in [0, 1]$ denote the (unique) number satisfying $r_i \alpha_i^+ + (1 - r_i)\alpha_{i-1}^- = 0$. Let

$$\xi_i \triangleq v_i^{\mathsf{T}} (r_i A_i + (1 - r_i) A_{i-1}) v_i\,.$$

Clearly, all trajectories that lie on $\mathcal{V}_i$ are asymptotically stable if and only if $\xi_i < 0$. In the case $\alpha_i^+ = \alpha_{i-1}^- = 0$, v_i is an eigenvector of both A_i and A_{i-1}, and as a result all trajectories that lie on $\mathcal{V}_i$ are asymptotically stable if and only if the corresponding eigenvalues are both negative. We summarize this in the following lemma.

Lemma 4 *In order for Σ to be asymptotically stable it is necessary that*

(i) *All visible eigenvectors are associated with stable eigenspaces.*
(ii) *If $\alpha_i^+ \alpha_{i-1}^- \le 0$ and $|\alpha_i^+| + |\alpha_{i-1}^-| \neq 0$, then $\xi_i < 0$, i.e., all sliding modes are stable.*

Next we compute the time needed for a trajectory to transverse a cone, as well as its growth in the cone. These calculations are used later to determine the asymptotic behavior of the trajectories that encircle the origin. Fix $i \in \mathcal{I}$. Suppose that A_i has no visible eigenvectors relative to $\mathcal{K}_i$. Without loss of generality we may assume that $\alpha_i^+ > 0$. Then necessarily $\alpha_i^- > 0$, for otherwise $\bar{\mathcal{K}}_i$ is invariant under the semigroup $\{e^{A_i t}\}$, and by Lemma 2 must contain an eigenvector of A_i contradicting the hypothesis. Thus the trajectory of Σ_i starting at v_i exits the cone crossing the set $\mathcal{V}_{i+1}$ in finite time by Theorem 3. We consider three cases depending on the Jordan form of A_i.

Case 1: $A_i \in \mathbb{R}^{2\times 2}$ has a pair of complex eigenvalues $\lambda_i \pm j\omega_i$.

Let $P_i \in \mathbb{R}^{2\times 2}$ denote the transformation such that $A_i = P_i\Big(\lambda_i I + \omega_i J\Big)P_i^{-1}$. The time τ_i that it takes the system $\dot{z} = (\lambda_i I + \omega_i J)z$ to traverse the cone $\{P_i^{-1}v_i, P_i^{-1}v_{i+1}\}$ is $\tau_i = \frac{\Theta(P_i^{-1}v_i,\, P^{-1}v_{i+1})}{\omega_i}$. This is the same as the time that it takes the original system $\dot{x} = A_i x$, with $x(0) = v_i$, to traverse $\mathcal{K}_i$. We define:

$$v_i' = P_i^{-1}v_i\,, \qquad v_i'' = P_i^{-1}v_{i+1}\,, \tag{3.2}$$

and

$$\alpha_i = \lambda_i\,, \qquad \beta_i = \log\biggl(\frac{\|v_i'\|}{\|v_i''\|}\biggr)\,. \tag{3.3}$$

A simple computation yields

$$x(\tau_i) = e^{\mu_i\tau_i}v_{i+1}\,, \qquad \mu_i = \alpha_i + \frac{\beta_i}{\tau_i}\,, \tag{3.4}$$

where $\tau_i = \frac{\Theta(v_i', v_{i+1}')}{\omega_i}$.

Case 2: $A_i \in \mathbb{R}^{2\times 2}$ has two distinct real eigenvalues $\lambda_i' > \lambda_i''$.

Let $P_i \in \mathbb{R}^{2\times 2}$ denote the transformation such that $A = P_i \left(\begin{smallmatrix} \lambda_i' & 0 \\ 0 & \lambda_i'' \end{smallmatrix}\right) P_i^{-1}$ and define v_i', v_i'' by (3.2). Then (3.4) holds with

$$\begin{aligned} \tau_i &= \frac{1}{\lambda_i' - \lambda_i''}\log\biggl(\frac{v_{i2}'v_{i1}''}{v_{i1}'v_{i2}''}\biggr) \\ \alpha_i &= \frac{\lambda_i' + \lambda_i''}{2} \\ \beta_i &= \frac{1}{2}\log\biggl(\frac{v_{i1}'v_{i2}'}{v_{i1}''v_{i2}''}\biggr)\,. \end{aligned} \tag{3.5}$$

Note that since $\mathcal{K}_i$ contains no eigenvectors of A_i it has to be the case that v_i' and v_i'' have the same sign (component wise). Also $v_{i1}'v_{i2}'' \neq 0$, otherwise $\mathcal{K}_i$ contains an eigenvector of A_i. Therefore formulas (3.5) are well-defined.

Case 3: $A_i \in \mathbb{R}^{2\times 2}$ has a real eigenvalue λ_i of multiplicity 2 in its minimal polynomial.

Let $P_i \in \mathbb{R}^{2\times 2}$ denote the transformation such that $A_i = P_i \left(\begin{smallmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{smallmatrix}\right) P_i^{-1}$ and define v_i', v_i'' by (3.2). Then (3.4) holds with

$$\begin{aligned} \tau_i &= \frac{v_{i1}''}{v_{i2}''} - \frac{v_{i1}'}{v_{i2}'} = \frac{1}{v_{i2}'v_{i2}''} \det \begin{pmatrix} v_{i1}'' & v_{i1}' \\ v_{i2}'' & v_{i2}' \end{pmatrix} \\ \alpha_i &= \lambda_i \\ \beta_i &= \log\left(\frac{v_{i2}'}{v_{i2}''}\right) . \end{aligned} \tag{3.6}$$

Note that $v_{i2}'v_{i2}'' \neq 0$, otherwise $\mathcal{K}_i$ contains an eigenvector of A_i.

The following may be proved by direct computation so we omit the proof.

Lemma 5 *The expressions for α_i and β_i are independent of the choice of the P_i's.*

We now present the main result of this section. Let

$$\tau = \sum_{i\in\mathcal{I}} \tau_i .$$

Theorem 6 *The planar CLS $\Sigma = \{(\Sigma_i, \mathcal{K}_i)\,,\ i = 1, \dots, \ell\}$ is asymptotically stable if and only if*

(a) Conditions (i) and (ii) of Lemma 4 hold.
(b) If there are no visible eigenvectors or sliding modes, then with τ_i, α_i, and β_i as defined in (3.3), (3.5), (3.6),

$$\mu := \sum_{i=\mathcal{I}} \frac{(\tau_i\alpha_i + \beta_i)}{\tau} < 0 \,. \tag{3.7}$$

PROOF. First show that if there are no visible eigenvectors or sliding modes, then (3.7) is necessary and sufficient. Without loss of generality suppose that $\alpha_1^+ > 0$. Then, as mentioned in the paragraph following Lemma 4 we must have $\alpha_1^- > 0$, and hence also $\alpha_2^+ > 0$. By induction $\alpha_i^+ > 0$, $\alpha_i^- > 0$ for all $i \in \mathcal{I}$. Thus the trajectory x of Σ satisfying $x(0) = v_1$, encircles the origin and crosses $\mathcal{V}_1$ at time τ. Using the results in Cases 1–3 above, we have

$$\begin{aligned} \|x(\tau)\| &= \left\| P_\ell e^{B_\ell \tau_\ell} P_\ell^{-1} \cdots P_1 e^{B_1\tau_1} P_1^{-1} v_1 \right\| \\ &= e^{\mu_1\tau_1} \left\| P_\ell e^{B_\ell \tau_\ell} P_\ell^{-1} \cdots P_2 e^{B_2\tau_2} P_2^{-1} v_2 \right\| \\ &\ \vdots \\ &= e^{\mu\tau} \|v_{\ell+1}\| = e^{\mu\tau} , \end{aligned} \tag{3.8}$$

where $B_i \triangleq P_i^{-1} A_i P_i$, $i \in \mathcal{I}$. Therefore, $\|x(k\tau)\| = e^{k\mu\tau}$, for all $k \in \mathbb{N}$, implying that (3.7) is necessary. It is also sufficient since if $\hat{x} \in \mathbb{R}^2 \setminus \{0\}$, then $\varrho\hat{x} \in \{x(t) : 0 \leq t < \tau\}$, for some $\varrho > 0$. Thus, the trajectory starting from $\hat{x}$ converges asymptotically to 0, provided (3.7) holds. Necessity of (a) is asserted in Lemma 4. It remains to show that if Σ has visible eigenvectors or sliding modes, then (a) is sufficient. It is evident that in this case, a trajectory cannot revisit a cone it exits. Therefore it has to get trapped in some cone $\mathcal{K}_i$ after some time t_0. Then necessarily either A_i has a visible eigenvector relative to $\mathcal{K}_i$, or there is a stable sliding mode in $\bar{\mathcal{K}}_i$. In both cases it is fairly straightforward to show that the trajectory converges asymptotically to the origin. It is also evident that trajectories are bounded uniformly over any bounded set of initial conditions. This completes the proof. □

Remark 7

(1) *When there are no visible eigenvectors or sliding modes the stability of Σ is determined by the complex numbers $\mu \pm j\omega$, where*

$$\omega = \frac{2\pi}{\tau} \,. \tag{3.9}$$

Thus, we call them the characteristic values *of the CLS.*

(2) *Let $\beta = \sum_{i\in\mathcal{I}} \beta_i$ and $\alpha = \sum_{i\in\mathcal{I}} \tau_i \alpha_i$. If $\beta = 0$, then stability results if $\alpha < 0$; that is, the time-average of the eigenvalues is negative. Likewise, if $\lambda_i = 0$ for all $i \in \mathcal{I}$, then stability depends only on β, which is independent of the eigenvalues of the individual matrices A_i.*

The previous remarks warrant a further examination of the constituent terms of μ. First, if the matrices $\{A_1, \dots, A_{k_0}\}$ commute pairwise, i.e., they form an Abelian Lie algebra, and they are of simple structure, i.e., they correspond to Case 1 or Case 2, then they can be simultaneously diagonalized and we obtain $\beta = 0$ [7, p. 224]. Next, let us say that a Lie Algebra $\mathcal{L}$ of $\mathbb{R}^{2\times 2}$ which contains the identity matrix has the *stable property* if any time-switched linear system whose dynamics are defined over any finite collection of Hurwitz elements of $\mathcal{L}$ is asymptotically stable. It is shown in [8] that solvable Lie subalgebras of $\mathbb{R}^{2\times 2}$ have the stable property (see also [9] for further extensions of these results). We make some connections between these Lie algebraic criteria for the stability of time-switched systems and our results on CLSs. Let $\beta(\Sigma)$ be the parameter β associated with a CLS Σ.

Theorem 8 *Let Σ be a CLS whose dynamics are governed by $\{A_i\,,\ i = 1, \dots, k_0\}$ and suppose that each A_i is either of the form*

$$A_i = P_i(\lambda_i I + \omega_i J)P_i^{-1}\,, \quad \omega_i \neq 0\,, \tag{3.10}$$

or

$$A_i = P_i(\lambda_i I + N)P_i^{-1}\,, \qquad N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\,,$$

and satisfies $(0,1)A_i\binom{1}{0} > 0$, *(i.e., trajectories flow counterclockwise). If the Lie algebra* $\mathcal{L}$ *generated by* $\{I\ ,\ A_i\ ,\ i = 1,\ldots,k_0\}$ *has the stable property, then* $\beta = 0$.

PROOF. First, suppose $\beta > 0$. We construct a set of Hurwitz matrices $\{A'_1,\ldots,A'_{k_0} \mid A'_i \in \mathcal{L}\}$ defining a new CLS which retains the same value of β. Let

$$A'_i = -(\lambda_i + \varepsilon)I + A_i = P_i(-\varepsilon I + \omega_i J)P_i^{-1}\,,$$

or

$$A'_i = -(\lambda_i + \varepsilon)I + A_i = P_i(-\varepsilon I + N)P_i^{-1}\,,$$

where $\varepsilon > 0$. First we note that $A'_i \in \mathcal{L}$ because I and any multiple of it belong to $\mathcal{L}$. Let Σ' be the CLS obtained from Σ by substituting A_i with A'_i, $i = 1,\ldots,k_0$. Note that, by using (3.3) and (3.6), $\beta(\Sigma') = \beta(\Sigma)$. Now we can choose ε such that each A'_i is Hurwitz but $\mu = -\varepsilon + \frac{\beta}{\tau} > 0$, since $\beta > 0$. According to Theorem 6, the CLS is unstable, contradicting the hypothesis that $\mathcal{L}$ has the stable property.

Instead, suppose $\beta < 0$. Replace each A_i by $\overline{A}_i \triangleq (\lambda_i - \varepsilon)I - A_i$ to form a CLS $\overline{\Sigma}$. The trajectories of $\overline{\Sigma}$ encircle the origin in the clockwise direction, and it can be easily verified that $\beta(\overline{\Sigma}) = -\beta(\Sigma) + \varepsilon > 0$, for $\varepsilon > 0$ sufficiently small. Repeating the argument above we arrive at the same contradiction. □

We have seen that if the Lie algebra generated by the subsystem matrices of a CLS is solvable, implying it has the stable property, then under the conditions of Theorem 8, $\beta = 0$. However, the converse statement is not true as the following example illustrates.

Example 9 *Consider the CLS with two subsystems*

$$A_1 = \begin{bmatrix} -1 & -2 \\ 0.5 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & -0.5 \\ 2 & -1 \end{bmatrix}.$$

The switching boundaries are $v_1 = [1\ 0]^{\mathsf{T}}$, $v_2 = [0\ 1]^{\mathsf{T}}$, $v_3 = -v_1$, *and* $v_4 = -v_2$. *Let* A_1 *be associated with* $\mathcal{K}_1 = \operatorname{cone}\{v_1, v_2\}$ *and* $\mathcal{K}_2 = \operatorname{cone}\{v_2, v_3\}$, *and* A_2 *be associated with* $\mathcal{K}_3 = \operatorname{cone}\{v_3, v_4\}$ *and* $\mathcal{K}_4 = \operatorname{cone}\{v_4, v_1\}$. *The eigenvalues of* A_1 *and* A_2 *are both* $-1 \pm j$ *and*

$$P_1 = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

With this data we find $\beta = 0$. *Let* $H_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $H_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, *and* $H_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. *They form a basis for* $sl(2)$, *the Lie algebra of the special linear group* $SL(2)$

of 2×2 *matrices, which is not solvable. Then we observe that* $A_1 = -I + \frac{1}{2}H_3 - 2H_2$, $A_2 = -I + 2H_3 + \frac{1}{2}H_2$, *and* $[A_1, A_2] = -3.75H_1$. *Therefore, the Lie algebra generated by* $\{A_1, A_2\}$ *is not solvable.*

Next we show that if $\beta = 0$ over all switching boundaries, then for a particular class of matrices, the generated Lie algebra is Abelian. Let $\mathcal{A}$ be a collection of matrices and $\mathfrak{S}$ denote the class of all CLS $\varSigma = \{(\varSigma_j, \mathcal{K}_j)\,,\ j = 1, \dots, \ell\}$, $\ell \geq 3$, where $\{\mathcal{K}_i\}$ is some conic partition of $\mathbb{R}^2$ and the dynamics $\varSigma_j$ on $\mathcal{K}_j$ are governed by $\dot{x} = A_j x$ with $A_j \in \mathcal{A}$.

Theorem 10 *Suppose* $\mathcal{A} = \{\tilde{A}_i\,,\ i = 1, \dots, k_0\}$ *is a collection of matrices of the form* (3.10) *and suppose* $(0,1)\tilde{A}_i \binom{1}{0} > 0$, *for all* $i = 1, \dots, k_0$ *(i.e., trajectories flow counterclockwise). If* $\beta(\varSigma) \leq 0$ *for all* $\varSigma \in \mathfrak{S}$, *then the matrices in* $\mathcal{A}$ *commute. Hence,* $\beta(\varSigma) = 0$ *for all* $\varSigma \in \mathfrak{S}$.

PROOF. First note that $\tilde{A}_i$ and $\tilde{A}_j$ commute if and only if $P_i^{-1}P_j = \varrho T$ where T is an orthonormal matrix and $\varrho > 0$, or equivalently iff $\left(P_i^{-1}P_j\right)^{\mathsf{T}} P_i^{-1}P_j$ is a multiple of the identity. We argue by contradiction. Suppose that for some $i, j \in \{1, \dots, k_0\}$, $\left(P_i^{-1}P_j\right)^{\mathsf{T}} P_i^{-1}P_j$ is not a multiple of the identity. Since it is symmetric and positive definite, it has distinct positive eigenvalues $\mu_1 < \mu_2$. Let $\tilde{v}_1$ be a unit eigenvector associated with μ_1. Then for any vector v

$$\|P_i^{-1}P_j v\|^2 = v^{\mathsf{T}}\left(P_i^{-1}P_j\right)^{\mathsf{T}} P_i^{-1}P_j v \geq \mu_1 \|v\|^2\,, \tag{3.11}$$

and the inequality is strict unless $v \in \operatorname{span}\{\tilde{v}_1\}$. Then noting that $P_j^{-1}JP_j\tilde{v}_1 \notin \operatorname{span}\{\tilde{v}_1\}$ and using (3.11) we have

$$\|P_i^{-1}JP_j\tilde{v}_1\| = \|P_i^{-1}P_j(P_j^{-1}JP_j\tilde{v}_1)\| > \sqrt{\mu_1}\|P_j^{-1}JP_j\tilde{v}_1\|\,. \tag{3.12}$$

Let $v_1 \triangleq P_j\tilde{v}_1$ and $v_k \triangleq J^{k-1}v_1$, for $k = 2, 3, 4$. Consider the partition generated by $\{v_i\}$ and associate $\tilde{A}_j$ to $\mathcal{K}_1 = \operatorname{cone}\{v_1, Jv_1\}$ and $\mathcal{K}_3 = \operatorname{cone}\{-v_1, -Jv_1\}$ and $\tilde{A}_i$ to $\mathcal{K}_2$ and $\mathcal{K}_4$. Then, by (3.12)

$$\begin{aligned}
\frac{\|P_j^{-1}v_1\|}{\|P_j^{-1}v_2\|}\frac{\|P_i^{-1}v_2\|}{\|P_i^{-1}v_3\|}\frac{\|P_j^{-1}v_3\|}{\|P_j^{-1}v_4\|}\frac{\|P_i^{-1}v_4\|}{\|P_i^{-1}v_1\|} &= \left(\frac{\|P_j^{-1}v_1\|}{\|P_j^{-1}v_2\|}\frac{\|P_i^{-1}v_2\|}{\|P_i^{-1}v_1\|}\right)^2 \\
&= \left(\frac{\|\tilde{v}_1\|}{\|P_j^{-1}JP_j\tilde{v}_1\|}\frac{\|P_i^{-1}JP_j\tilde{v}_1\|}{\|P_i^{-1}P_j\tilde{v}_1\|}\right)^2 \\
&> 1\,,
\end{aligned}$$

contradicting the hypothesis that $\beta(\varSigma) \leq 0$, for all $\varSigma \in \mathfrak{S}$. □

4 Controllability

Consider a controlled CLS Σ whose dynamics are specified by $\dot{x}(t) = A_i x(t) + b_i u(t)$ on $\mathcal{K}_i$, where $A_i \in \mathbb{R}^{2\times 2}$, $b_i \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$. As before, Σ_i denotes the restriction of Σ on $\mathcal{K}_i$. For $i \in \mathcal{I}$, define $\mathcal{B}_i = \operatorname{span}\{b_i\}$. We present a rather complete characterization of controllability of this system on $\mathbb{R}^2_* \triangleq \mathbb{R}^2 \setminus \{0\}$, the punctured plane which does not include the origin. The punctured plane is also used in the analysis of controllability of bilinear systems [10]. We can also develop a controllability theory for the full plane but this requires a more complicated analysis of trajectories that can cross through 0, and studying the well-posedness of trajectories that pass through a vertex of a partition.

Let $\mathcal{U}$ be a set of controls. if x', x'' are two points in $\mathbb{R}^2_*$, we say that x' can be steered to x'' over $\mathcal{U}$, and denote this by $x' \rightsquigarrow x''$, if there exists a $u \in \mathcal{U}$ and $T > 0$, such that the controlled system admits a unique solution in $\mathbb{R}^2_*$ satisfying $x(0) = x'$ and $x(T) = x''$. Solutions are meant in the sense of Filippov. If $D \subset \mathbb{R}^2_*$, then $x' \rightsquigarrow D$ means that $x' \rightsquigarrow x''$ for all $x'' \in D$. We say that Σ is *completely controllable* on $\mathbb{R}^2_*$ if $x' \rightsquigarrow \mathbb{R}^2_*$ for all $x' \in \mathbb{R}^2_*$. Also, we say that Σ_i is *completely controllable* if any two points in $\mathcal{K}_i$ can be joined through a trajectory in $\mathcal{K}_i$.

Difficulties with existence and uniqueness of solutions for discontinuous systems are well known [11–13], and several solution concepts have been proposed to overcome them [14, 15]. Here we highlight by way of an example the difference between open-loop and closed-loop controls with respect to uniqueness of solutions of discontinuous systems. The example illustrates that even if uniqueness of solutions is obtained via feedback, the corresponding solution in the sense of Caratheodory may not be unique. With this example as motivation, we present controllability results for open-loop and closed-loop controls independently, in each case requiring uniqueness of solutions.

Consider the one-dimensional system

$$\dot{x} = \begin{cases} u & \text{if } x < 0 \\ -u & \text{if } x > 0\,. \end{cases} \tag{4.13}$$

Suppose $x(0) = -1$. Clearly there is no continuous feedback control u which can steer the system to 1. On the other hand under the feedback $u = 1$, if $x < 0$ and $u = -1$, if $x > 0$, the closed loop system has a unique trajectory $x(t) = t - 1$, and hence -1 is steered to 1 on $[0, 2]$. Along this trajectory the control u takes the values $u(t) = 1$, for $t \in [0, 1]$, and $u(t) = -1$, for $t \in [1, 2]$. However, using this u as an open-loop control in (4.13) we observe that there is loss of uniqueness of the solution, and as a result -1 cannot be steered to 1 by this open loop control.

Apropos the above discussion, we consider two classes of control inputs: a) the set of all bounded measurable feedback controls $u : \mathbb{R}^2_* \to \mathbb{R}$, which is denoted by $\mathcal{U}_m$, and b) the set of all piecewise-continuous open-loop controls $u : [0, \infty) \to \mathbb{R}$, denoted by $\mathcal{U}$.

In Section 4.1 we study controllability over $\mathcal{U}$ of the subsystems Σ_i, $i \in \mathcal{I}$. We also establish that the reachable sets of Σ_i over $\mathcal{U}$ are also reachable over the class of constant gain linear feedback controls. In Section 4.2 we study the reachability from cone to cone, over $\mathcal{U}_m$. In Section 4.3 we combine these results to obtain necessary and sufficient conditions for controllability of Σ over $\mathcal{U}_m$. A slight strengthening of these conditions renders them necessary and sufficient for controllability of Σ over $\mathcal{U}$, as shown in Section 4.4.

In what follows we work with a refinement of the original partition which enables a simplification of the results on reachability within cones. If $\mathcal{B}_i \cap \mathcal{K}_i \neq \varnothing$, we divide $\mathcal{K}_i$ into two cones along $\mathcal{B}_i$. Similarly, if $A_i^{-1}\mathcal{B}_i$ is one dimensional and $A_i^{-1}\mathcal{B}_i \cap \mathcal{K}_i \neq \varnothing$, we divide $\mathcal{K}_i$ into two cones along $A_i^{-1}\mathcal{B}_i$. We retain the same notation for the CLS on the refinement of this partition.

4.1 Reachability within Cones

Controllability of Σ depends heavily on the reachable sets of Σ_i. Thus, in this section, the reachable sets of Σ_i are analyzed.

Let $\varphi_i(t, x_0; u)$ denote the trajectory $x(t)$, $t \geq 0$, of $\dot{x} = A_i x + b_i u$, satisfying $x(0) = x_0$. If $b_i \neq 0$, let b_i^* denote the unit vector which is orthogonal to b_i and satisfies $x^\mathsf{T} b_i^* > 0$ for all $x \in \mathcal{K}_i$. If $b_i = 0$, set $b_i^* = 0$. Also, let $\mathcal{K}_{i*} \triangleq \bar{\mathcal{K}}_i \setminus \{0\}$. For $x \in \mathcal{K}_{i*}$ define

$$\mathrm{Reach}_{\Sigma_i}(x) \triangleq \Big\{\varphi_i(t, x; u) : t \geq 0\,,\ u \in \mathcal{U}\,,\ \varphi_i(s, x; u) \in \mathcal{K}_{i*}\,,\ \forall s \in [0, t]\Big\}\,.$$

To assist in the taxonomy of $\mathrm{Reach}_{\Sigma_i}$, define, for $b_i \neq 0$,

$$W^+(x, b_i) \triangleq \{z \in \mathbb{R}^2 : (b_i^*)^\mathsf{T} z > (b_i^*)^\mathsf{T} x\} \cup \{x\}$$
$$W^-(x, b_i) \triangleq \{z \in \mathbb{R}^2 : (b_i^*)^\mathsf{T} z < (b_i^*)^\mathsf{T} x\} \cup \{x\}\,.$$

Lemma 11 *Assume that if $\mathcal{K}_{i*} \cap \mathcal{B}_i \neq \varnothing$ then $A_i b_i \notin \mathcal{B}_i$. For $x \in \mathcal{K}_{i*}$ the following hold:*

(A) If $b_i = 0$, then $\mathrm{Reach}_{\Sigma_i}(x) = \Big\{e^{A_i t}x : t \geq 0\,,$ and $e^{A_i t'}x \in \mathcal{K}_{i*}\,,\ \forall t' \in [0, t]\Big\}$.
(B) If $b_i \neq 0$ and $\mathrm{range}(A_i) \subset \mathcal{B}_i$, then $\mathrm{Reach}_{\Sigma_i}(x) = (x + \mathcal{B}_i) \cap \mathcal{K}_{i*}$.

(C) If $b_i \neq 0$, *and* $\operatorname{range}(A_i) \not\subset \mathcal{B}_i$, *then*

$$\operatorname{Reach}_{\Sigma_i}(x) = \begin{cases} W^+(x, b_i) \cap \mathcal{K}_{i*}\,, & \textit{if } (v_{i+1} + v_i)^\mathsf{T} A_i^\mathsf{T} b_i^* > 0 \\ W^-(x, b_i) \cap \mathcal{K}_{i*}\,, & \textit{if } (v_{i+1} + v_i)^\mathsf{T} A_i^\mathsf{T} b_i^* < 0\,. \end{cases}$$

PROOF. Cases (A) and (B) are obvious. For case (C) first note that since $A_i x \notin \mathcal{B}_i$ for all $x \in \mathcal{K}_i$, we have $x^\mathsf{T} A_i^\mathsf{T} b_i^* \neq 0$. Suppose, without loss of generality, that $x^\mathsf{T} A_i^\mathsf{T} b_i^* > 0$, for all $x \in \mathcal{K}_i$. It follows that if $\varphi_i(s, x; u) \in \operatorname{Reach}_{\Sigma_i}(x)$, where $x \in \mathcal{K}_{i*}$, $s \in [0, t]$, and $t > 0$, then $(b_i^*)^\mathsf{T} \dot{\varphi}_i(s, x; u) \geq 0$, for almost all $s \in [0, t]$. Suppose $\varphi_i(t, x; u) \neq x$. We claim that $(b_i^*)^\mathsf{T} \varphi_i(t, x; u) > (b_i^*)^\mathsf{T} x$. If not, then $(b_i^*)^\mathsf{T} \dot{\varphi}_i(s, x; u) = 0$ for almost all $s \in (0, t)$, from which it follows that $(b_i^*)^\mathsf{T} A_i \varphi_i(s, x; u) = 0$, for all $s \in [0, t]$, or equivalently that $A_i \varphi_i(s, x; u) \in \mathcal{B}_i$. This implies $\varphi_i(s, x; u) \notin \mathcal{K}_i$, so either $\varphi_i(s, x; u) \in \mathcal{V}_i$ or $\varphi_i(s, x; u) \in \mathcal{V}_{i+1}$, for all $s \in [0, t]$. Suppose, without loss of generality, the latter is the case. Then, $z \triangleq \varphi_i(t, x; u) - x \in \mathcal{V}_{i+1}$ is a nonzero vector in $\mathcal{K}_{i*}$ which satisfies $z \in \mathcal{B}_i$ (since by assumption $(b_i^*)^\mathsf{T} z = 0$) and $A_i z \in \mathcal{B}_i$. This contradicts the hypothesis of the lemma. Hence, $\operatorname{Reach}_{\Sigma_i}(x) \subset W^+(x, b_i) \cap \mathcal{K}_{i*}$.

To show the converse, let $x'' \in W^+(x', b_i) \cap \mathcal{K}_{i*}$, $x'' \neq x'$, and set $z = x'' - x'$. Suppose, without loss of generality that $b_i^* = J b_i$. If $A_i x' \notin \mathcal{B}_i$ and $A_i x'' \notin \mathcal{B}_i$ then if we let $u(t) = (b_i^\mathsf{T} J z)^{-1} z^\mathsf{T} J A_i x(t)$, we obtain

$$\dot{x}(t) = A_i x(t) + b_i u(t) = \frac{b_i^\mathsf{T} J A_i x(t)}{b_i^\mathsf{T} J z} z\,. \tag{4.14}$$

Since $\frac{b_i^\mathsf{T} J A_i x}{b_i^\mathsf{T} J z} > 0$ for all $x = \xi z + x'$, $\xi \in [0, 1]$, it follows by (4.14) that the solution $x(t)$ with $x(0) = x'$, satisfies $x(t'') = x''$ for some finite $t'' > 0$. Suppose that $A_i x'' \in \mathcal{B}_i$ and $A_i x' \notin \mathcal{B}_i$. Since, by construction of the partition, $A_i^{-1} \mathcal{B}_i \cap \mathcal{K}_i = \varnothing$, it must be the case that $x'' \in \mathcal{V}_i \cup \mathcal{V}_{i+1}$. Without loss of generality suppose $x'' \in \mathcal{V}_i$. If follows from the hypothesis that $\mathcal{V}_i \not\subset \mathcal{B}_i$ and thus the line $x' + \lambda b_i$, $\lambda \in \mathbb{R}$ intersects $\mathcal{V}_i$, i.e., $x' + \lambda_0 b_i \in \mathcal{V}_i$, for some $\lambda_0 \in \mathbb{R}$. Since $A_i x' \notin \mathcal{B}_i$, implying $x' \notin \mathcal{V}_i$, it follows that $\lambda_0 \neq 0$. We know that $x' + \lambda_0 b_i \neq x''$, since $x'' \in W^+(x', b_i)$. Let $\tilde{x}'' \in \mathcal{V}_i \cap W^+(x', b_i)$ be any point such that x'' lies in the open line segment joining $\tilde{x}''$ and $x' + \lambda_0 b_i$. Let $\zeta \in [0, \infty)$, $\tilde{z} = \tilde{x}'' - x'$, and consider the feedback control

$$u(t) = \frac{\tilde{z}^\mathsf{T} J A_i x(t)}{b_i^\mathsf{T} J \tilde{z}} - \zeta \lambda_0 b_i^\mathsf{T} J x(t)\,. \tag{4.15}$$

The closed-loop system resulting from (4.15) is

$$\dot{x}(t) = \frac{b_i^\mathsf{T} J A_i x(t)}{b_i^\mathsf{T} J \tilde{z}} \tilde{z} - \zeta b_i^\mathsf{T} J x(t) \lambda_0 b_i\,. \tag{4.16}$$

It follows from the foregoing that if $\zeta = 0$ then the trajectory $x(t)$ of (4.16) starting at $x(0) = x'$ converges asymptotically to $\tilde{x}''$, along the straight line joining these two points. Also, since $b_i^\mathsf{T} Jx = -(b_i^*)^\mathsf{T} x < 0$, for all $x \in \mathcal{K}_i$, and $b_i^\mathsf{T} Jx' \neq 0$, the vector field $b_i^\mathsf{T} Jx(t)\lambda_0 b_i$ results in a trajectory that joins x' and $x' + \lambda_0 b_i$ along a straight line in finite time. For $y \in \mathcal{K}_i$ let

$$\eta_1(y) = \mathcal{V}_i \cap \{y + \varrho\tilde{z} \mid \varrho \in \mathbb{R}\}, \qquad \eta_2(y) = \mathcal{V}_i \cap \{y + \varrho b_i \mid \varrho \in \mathbb{R}\},$$

and define $\mathit{\Gamma}_y = \operatorname{conv}\{y, \eta_1(y), \eta_2(y)\}$, where 'conv' denotes the convex hull. Let $\gamma_y(t, \zeta)$, with $t \geq 0$, denote the trajectory of (4.16), starting from y, i.e., $\gamma_y(0, \zeta) = y$, and set

$$\tau(y, \zeta) \triangleq \inf\{t \geq 0 : \gamma_y(t, \zeta) \in \mathcal{V}_1\}.$$

It is evident from the direction of the vector field of (4.16) that provided $\zeta > 0$, then $\tau(y, \zeta) < \infty$ and

$$\Big\{\gamma_y(t, \zeta) : t \in \Big(0, \tau(y, \zeta)\Big)\Big\} \subset \mathit{\Gamma}_y^o, \tag{4.17}$$

with $\mathit{\Gamma}_y^o$ denoting the interior of $\mathit{\Gamma}_y$. In particular, for $\zeta > 0$, $\gamma_{x'}(\tau(x', \zeta), \zeta)$ lies in the relative interior of $\operatorname{conv}\{\tilde{x}'', x' + \lambda_0 b_i\}$. Since the vector field of (4.16) is transversal to $\mathcal{V}_i$, $\tau(x', \zeta)$ is continuous in $\zeta \in (0, \infty)$, and in turn, the same holds for $\gamma_{x'}(\tau(x', \zeta), \zeta)$. Continuity of the solution of (4.16) with respect to ζ, combined with (4.17), shows that

$$\gamma_{x'}\Big(\tau(x', \zeta), \zeta\Big) \to \begin{cases} \tilde{x}'' & \text{as } \zeta \to 0 \\ x' + \lambda_0 b_i & \text{as } \zeta \to \infty\,. \end{cases}$$

Therefore, $\gamma_{x'}\Big(\tau(x', \zeta''), \zeta''\Big) = x''$, for some $\zeta'' \in (0, \infty)$.

If $A_i x'' \notin \mathcal{B}_i$ and $A_i x' \in \mathcal{B}_i$, the conclusion follows along the same lines, by using time reversal. If $A_i x'' \in \mathcal{B}_i$ and $A_i x' \in \mathcal{B}_i$, using an intermediate point $\hat{x} \in \mathcal{K}_i$ satisfying $A_i\hat{x} \in \mathcal{B}_i$ and $b_i^\mathsf{T} Jx' < b_i^\mathsf{T} J\hat{x} < b_i^\mathsf{T} Jx''$, the previous arguments show that $\hat{x} \in \operatorname{Reach}_{\mathit{\Sigma}_i}(x')$ and $x'' \in \operatorname{Reach}_{\mathit{\Sigma}_i}(\hat{x})$. $\square$

The proof of Lemma 11 shows that linear feedback control can be used to steer in $\operatorname{Reach}_{\mathit{\Sigma}_i}$ as stated in the following corollary.

Corollary 12 *Assume that if* $\mathcal{K}_{i*} \cap \mathcal{B}_i \neq \varnothing$ *then* $A_i b_i \notin \mathcal{B}_i$. *Also, suppose* $b_i \neq 0$ *and* $\operatorname{range}(A_i) \not\subset \mathcal{B}_i$. *Let* $x' \in \mathcal{K}_{i*}$ *and* $x'' \in \operatorname{Reach}_{\mathit{\Sigma}_i}(x')$ *such that* $\operatorname{span}\{A_i x', A_i x''\} \not\subset \mathcal{B}_i$. *Then there is a feedback control* $u = k_i^\mathsf{T} x$, *for some* $k_i \in \mathbb{R}^2$, *such that the trajectory* $x(t)$, *with* $x(0) = x'$, *satisfies* $x(t'') = x''$, *for some* $t'' > 0$ *and* $x(t) \in \mathcal{K}_i$ *for all* $t \in (0, t'')$.

4.2 Reachability between Cones

In this section we analyze the existence of controlled trajectories (over $\mathcal{U}_m$) starting in $\mathcal{K}_i$ and reaching $\mathcal{K}_{i+1}$, and vice versa. The main idea is to analyze the possible directions of flow of Σ_i and Σ_{i+1} along $\mathcal{V}_{i+1}$. We use the notation $\mathcal{K}_i \rightarrowtail \mathcal{K}_{i+1}$ to indicate that there exists a controlled trajectory $x(\cdot)$ in $\mathcal{K}_i \cup \mathcal{K}_{i+1}$, defined for $t \in [-\varepsilon, \varepsilon]$, with $\varepsilon > 0$ and satisfying $x(-\varepsilon) \in \mathcal{K}_i$, $x(\varepsilon) \in \mathcal{K}_{i+1}$. Analogously for $\mathcal{K}_{i+1} \rightarrowtail \mathcal{K}_i$. In order to indicate the direction (counterclockwise, or clockwise) that the boundary $\mathcal{V}_i$ can be crossed by controlled trajectories, we define the set $\mathcal{G}_i \subseteq \{1, -1\}$ with the property that $1 \in \mathcal{G}_i$ if $\mathcal{K}_i \rightarrowtail \mathcal{K}_{i+1}$, and $-1 \in \mathcal{G}_i$ if $\mathcal{K}_{i+1} \rightarrowtail \mathcal{K}_i$. Let

$$\beta_i^+ \triangleq n_i^\mathsf{T} b_i\,, \qquad \beta_i^- \triangleq n_{i+1}^\mathsf{T} b_i\,.$$

Then using (3.1) and the signum function, and allowing for discontinuous controls, we have

$$\mathcal{G}_i = \Big\{\operatorname{sgn}(\alpha_i^- + u\beta_i^-) : (\alpha_i^- + u\beta_i^-)(\alpha_{i+1}^+ + u'\beta_{i+1}^+) > 0\,,\ \exists u,\, u' \in \mathbb{R}\Big\}\,. \quad (4.18)$$

A more explicit characterization of $\mathcal{G}_i$ is provided by the following lemma.

Lemma 13 *For each $i \in \mathcal{I}$,*

(i) If $\beta_{i+1}^+\beta_i^- \neq 0$, then $\mathcal{G}_i = \{1, -1\}$.
(ii) If $\beta_{i+1}^+\beta_i^- = 0$, then

$$\mathcal{G}_i = \begin{cases} \{\operatorname{sgn}(\alpha_i^-)\} & \textit{if } \beta_{i+1}^+ = \beta_i^- = 0\,,\ \textit{and } \alpha_{i+1}^+\alpha_i^- > 0 \\ \{\operatorname{sgn}(\alpha_i^-)\} & \textit{if } \beta_{i+1}^+ \neq 0 \\ \{\operatorname{sgn}(\alpha_{i+1}^+)\} & \textit{if } \beta_i^- \neq 0 \\ \{\varnothing\} & \textit{otherwise.} \end{cases}$$

4.3 Main Result

In this section we gather the previous results on reachability within and between cones to obtain our main result on controllability. The essential idea is to analyze trajectories which encircle the origin either in a counterclockwise or clockwise sense. We compute the maximum and minimum growth around such a cycle. Necessary and sufficient conditions for controllability are obtained in terms of these growth factors—both shrinkage and expansion must be possible.

The existence of trajectories that encircle the origin is a necessary condition for controllability of Σ; for if not, either some $\mathcal{V}_i$ is invariant under any controlled trajectory or there is a subcollection of cones whose union is invariant under any controlled trajectory. Let $\mathcal{G} \triangleq \bigcap_{i\in\mathcal{I}} \mathcal{G}_i$. We require the following.

Condition 1 *$\mathcal{G} \neq \varnothing$.*

Note that under Condition 1 the hypothesis of Lemma 11 is satisfied for all $i \in \mathcal{I}$. For if not, then either $\alpha_i^+ = \beta_i^+ = 0$, or $\alpha_i^- = \beta_i^- = 0$, resulting in $\mathcal{G}_i = \varnothing$.

It is necessary to determine the growth around a cycle, as in Theorem 6. We define the inverse of the maximum possible growth in $\mathcal{K}_j$ as $\underline{\xi}_j(\kappa)$ and the minimum possible growth in $\mathcal{K}_j$ as $\overline{\xi}_j(\kappa)$. These growth factors can be computed explicitly using Lemma 11.

Definition 14 *Assume Condition 1. Define for $j \in \mathcal{I}$ and $\kappa \in \mathcal{G}$*

$$\underline{\xi}_j(\kappa) = \begin{cases} 0 & \text{if } (v_{j+1}+v_j)^\mathsf{T} A_j^\mathsf{T} b_j^* > 0 \\ \left(\frac{v_j^\mathsf{T} b_j^*}{v_{j+1}^\mathsf{T} b_j^*}\right)^{-\kappa} & \text{if } (v_{j+1}+v_j)^\mathsf{T} A_j^\mathsf{T} b_j^* \leq 0\,,\ b_j \neq 0 \\ e^{-\kappa\mu_j\tau_j} & \text{if } b_j = 0\,, \end{cases}$$

$$\overline{\xi}_j(\kappa) = \begin{cases} \left(\frac{v_j^\mathsf{T} b_j^*}{v_{j+1}^\mathsf{T} b_j^*}\right)^{\kappa} & \text{if } (v_{j+1}+v_j)^\mathsf{T} A_j^\mathsf{T} b_j^* \geq 0\,,\ b_j \neq 0 \\ 0 & \text{if } (v_{j+1}+v_j)^\mathsf{T} A_j^\mathsf{T} b_j^* < 0 \\ e^{\kappa\mu_j\tau_j} & \text{if } b_j = 0\,. \end{cases}$$

Here μ_j and τ_j are the trajectory growth rate and time to transverse $\mathcal{K}_j$ computed in Section 3.

Theorem 15 *For Σ to be completely controllable on $\mathbb{R}^2_*$, over $\mathcal{U}_m$, it is necessary and sufficient that*

(a) Condition 1 holds.
(b) For some $\kappa \in \mathcal{G}$ the following inequalities hold

$$\underline{\xi}(\kappa) \triangleq \prod_{j=1}^{\ell} \underline{\xi}_j(\kappa) < 1\,, \qquad \overline{\xi}(\kappa) \triangleq \prod_{j=1}^{\ell} \overline{\xi}_j(\kappa) < 1\,. \tag{4.19}$$

PROOF. Necessity of (a) has been discussed earlier. Note that if

$$(v_{j+1}+v_j)^\mathsf{T} A_j^\mathsf{T} b_j^* = 0\,, \tag{4.20}$$

and $b_j \neq 0$, then necessarily $\operatorname{range}(A_j) \subset \mathcal{B}_j$. Thus if (4.20) holds for all $j \in \mathcal{I}$, the reachable set from every point x is one-dimensional. It follows that if Σ is completely controllable, then $\underline{\xi}(\kappa)\overline{\xi}(\kappa) = 0$. To show that (b) is necessary, first observe that if $\mathcal{G} = \{1, -1\}$, then $\underline{\xi}(\kappa) = \underline{\xi}^{-1}(-\kappa)$, provided $\underline{\xi}(\kappa) \neq 0$, otherwise $\underline{\xi}(\kappa) = \underline{\xi}(-\kappa) = 0$. Similarly for $\overline{\xi}(\kappa)$. It follows from these arguments that if (b) does not hold, then we may suppose without loss of generality that $\mathcal{G} = \{1\}$, and $\prod_{j=1}^{\ell} \underline{\xi}_j(\kappa) \geq 1$. Consider the collection of points $z_i \in \mathcal{V}_i$ defined by $z_1 = v_1$ and

$$z_{j+1} = \begin{cases} e^{\mu_j \tau_j} z_j & \text{if } b_j = 0 \\ \frac{v_{j+1}^\mathsf{T} b_j^*}{v_j^\mathsf{T} b_j^*} z_j & \text{otherwise.} \end{cases}$$

Let $\gamma : [0, \ell] \to \mathbb{R}^2_*$, be the curve defined by

$$\gamma(s) = \begin{cases} e^{\mu_j (s-j+1)\tau_j} z_j & \text{if } b_j = 0\,,\ s \in [j-1, j] \\ z_{j+1} + (s-j)\Big(z_{j+1} - \frac{v_{j+1}^\mathsf{T} b_j^*}{v_j^\mathsf{T} b_j^*} z_j\Big) & \text{if } b_j \neq 0\,,\ s \in [j-1, j]. \end{cases}$$

According to the hypothesis $\|z_{\ell+1}\| \leq \|v_1\|$. Consider the Jordan curve consisting of $\{\gamma(s)\,,\ s \in [0, \ell]\}$ and the straight segment $[z_{\ell+1}, z_1] \subset \mathcal{V}_1$ and let $\mathcal{D}$ denote its interior. It follows by Lemma 11, that $\operatorname{Reach}_\Sigma(v_1) \subset \bar{\mathcal{D}}$, thus arriving at a contradiction.

Sufficiency: Assume (a)–(b). Without loss of generality suppose $1 \in \mathcal{G}$, and $\underline{\xi}(1) < 1$, $\overline{\xi}(1) < 1$. By Lemma 11, if $b_j \neq 0$ and $\operatorname{range}(A_j) \not\subset \mathcal{B}_j$, then $\operatorname{Reach}_{\Sigma_j}(v_j) \cap \mathcal{V}_{j+1}$ contains all points of the form $\varrho_j v_{j+1}$, where

$$\varrho_j \in \begin{cases} \left(\frac{v_j^\mathsf{T} b_j^*}{v_{j+1}^\mathsf{T} b_j^*}, \infty\right)\,, & \text{if } (v_{j+1} + v_j)^\mathsf{T} A_j^\mathsf{T} b_j^* > 0 \\ \left(0, \frac{v_j^\mathsf{T} b_j^*}{v_{j+1}^\mathsf{T} b_j^*}\right)\,, & \text{if } (v_{j+1} + v_j)^\mathsf{T} A_j^\mathsf{T} b_j^* < 0\,. \end{cases}$$

Otherwise, $\operatorname{Reach}_{\Sigma_j}(v_j) \cap \mathcal{V}_{j+1} = \{\varrho_j v_{j+1}\}$, where

$$\varrho_j = \begin{cases} \frac{v_j^\mathsf{T} b_j^*}{v_{j+1}^\mathsf{T} b_j^*}\,, & \text{if } b_j \neq 0\,, \text{ and } \operatorname{range}(A_j) \subset \mathcal{B}_j \\ e^{\mu_j \tau_j}\,, & \text{if } b_j = 0\,. \end{cases}$$

Then, by considering the trajectories that follow a complete cycle, we have

$$\operatorname{Reach}_\Sigma(v_1) \cap \mathcal{V}_1 \supset \begin{cases} \Big\{\varrho v_1 : \varrho \in (\overline{\xi}(1), \infty)\Big\} & \text{if } \underline{\xi}(1) = 0 \\ \Big\{\varrho v_1 : \varrho \in (0, \underline{\xi}^{-1}(1))\Big\} & \text{otherwise.} \end{cases} \tag{4.21}$$

Iterating (4.21) we obtain $\operatorname{Reach}_\Sigma(v_1) \cap \mathcal{V}_1 \supset \mathcal{V}_1$, and the result now easily follows. □

Remark 16 *If* $\underline{\xi}(\kappa)\overline{\xi}(\kappa) = 0$ *then* (4.19) *implies* $\underline{\xi}(\kappa)+\overline{\xi}(\kappa) < 1$. *On the other hand, if* $\underline{\xi}(\kappa)\overline{\xi}(\kappa) \neq 0$ *then* $\underline{\xi}(\kappa) = \overline{\xi}^{-1}(\kappa)$ *and* (4.19) *does not hold. It follows that* (4.19) *in Theorem 15 may be replaced by* $\underline{\xi}(\kappa) + \overline{\xi}(\kappa) < 1$.

Example 17 *In this example none of the individual pairs* (A_i, b_i) *are controllable, yet the CLS is completely controllable. Let* $\mathcal{K}_i$, $i = 1, 2, 3, 4$, *correspond to the four quadrants of the plane in counterclockwise order. We define*

$$A_1 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A_2 = 0, \quad b_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A_3 = A_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad b_3 = b_4 = 0\,.$$

An easy calculation yields $\mathcal{G} = \{-1\}$, $\underline{\xi}(-1) = 0$, *and* $\overline{\xi}(-1) = 0.5$.

Example 18 *In this example all of the individual pairs* (A_i, b_i) *are controllable and conditions (a)–(b) of Theorem 15 are satisfied, yet the CLS is not completely controllable. As in Example 17, let* $\mathcal{K}_i$, $i = 1, 2, 3, 4$, *correspond to the four quadrants of the plane in counterclockwise order. We define*

$$A_1 = A_3 = \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}, \quad A_2 = A_4 = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix},$$

$$b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad b_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Here, $\mathcal{G} = \{1, -1\}$, $\underline{\xi}(1) = \underline{\xi}(-1) = 0$, *and* $\overline{\xi}(1) = \overline{\xi}(-1) = 1$.

4.4 Controllability over $\mathcal{U}$

To study controllability of Σ over $\mathcal{U}$ (4.18) should be replaced by

$$\hat{\mathcal{G}}_i = \Big\{\mathrm{sgn}(\alpha_i^- + u\beta_i^-) : (\alpha_i^- + u\beta_i^-)(\alpha_{i+1}^+ + u\beta_{i+1}^+) > 0\,,\ \exists u \in \mathbb{R}\Big\}. \qquad (4.22)$$

Thus Lemma 13 should be replaced by

Lemma 19 *For each* $i \in \mathcal{I}$,

(i) If $\beta_{i+1}^+\beta_i^- > 0$, *then* $\hat{\mathcal{G}}_i = \{1, -1\}$.
(ii) If $\beta_{i+1}^+\beta_i^- = 0$, *then*

$$\hat{\mathcal{G}}_i = \begin{cases} \{\mathrm{sgn}(\alpha_i^-)\} & \text{if } \beta_{i+1}^+ = \beta_i^- = 0\,, \text{ and } \alpha_{i+1}^+\alpha_i^- > 0 \\ \{\mathrm{sgn}(\alpha_i^-)\} & \text{if } \beta_{i+1}^+ \neq 0 \\ \{\mathrm{sgn}(\alpha_{i+1}^+)\} & \text{if } \beta_i^- \neq 0 \end{cases}$$

(iii) If $\beta^+_{i+1}\beta^-_i < 0$*, and* $\det \begin{bmatrix} \alpha^+_{i+1} & \alpha^-_i \\ \beta^+_{i+1} & \beta^-_i \end{bmatrix} \neq 0$*, then*

$$\hat{\mathcal{G}}_i = \Bigl\{\operatorname{sgn}\Bigl(\alpha^-_j - \tfrac{\beta^-_j\alpha^+_{j+1}}{\beta^+_{j+1}}\Bigr)\Bigr\}\,.$$

(iv) In all other cases $\hat{\mathcal{G}}_i = \varnothing$*.*

Let $\hat{\mathcal{G}} \triangleq \bigcap_{i\in\mathcal{I}} \hat{\mathcal{G}}_i$. Then Theorem 15 still holds if in its statement $\mathcal{U}_m$ and $\mathcal{G}$ are replaced by $\mathcal{U}$ and $\hat{\mathcal{G}}$, respectively.

The following example demonstrates this difference.

Example 20 *In this example all of the individual pairs* (A_i, b_i) *are controllable and conditions (a)–(b) of Theorem 15 are satisfied, yet the CLS is only controllable if we allow discontinuous feedback controls. As in Example 17, let* $\mathcal{K}_i$*,* $i = 1, 2, 3, 4$*, correspond to the four quadrants of the plane in counterclockwise order. We define*

$$A_1 = A_3 = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}\,, \quad A_2 = \begin{pmatrix} -1 & 1 \\ -3 & 5 \end{pmatrix}\,, \quad A_4 = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}\,,$$

$$b_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,, \quad b_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\,, \quad b_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,, \quad b_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,.$$

An easy calculation yields $\alpha^+_1 = \alpha^+_4 = -1$*,* $\alpha^-_4 = \alpha^-_3 = -3$*,* $\beta^+_1 = \beta^-_3 = -1$*, and* $\beta^-_4 = \beta^+_4 = 1$*. Therefore* $\hat{\mathcal{G}} = \{-1\}$ *and* $\overline{\xi}(-1) = 2$*,* $\underline{\xi}(-1) = 0$*. On the other hand, since* $\mathcal{G} = \{-1, 1\}$ *and* $\overline{\xi}(1) = 0.5$*,* $\underline{\xi}(1) = 0$*, the system is controllable over* $\mathcal{U}_m$*.*

5 Stabilization

Theorem 21 *Suppose* Σ *is completely controllable over* $\mathcal{U}_m$*. Then it is stabilizable by piecewise-linear feedback of the form* $u = k_i^{\mathsf{T}} x$*, for* $x \in \mathcal{K}_i$*, where* $k_i \in \mathbb{R}^2$*,* $i \in \mathcal{I}$*.*

PROOF. Without loss of generality suppose $1 \in \mathcal{G}$, and $\overline{\xi}(1) < 1$. Let $i \in \mathcal{I}$ be arbitrary. By Lemma 11, if $b_i \neq 0$ and $\operatorname{range}(A_i) \not\subset \mathcal{B}_i$, then $\operatorname{Reach}_{\Sigma_i}(v_i) \cap \mathcal{V}_{i+1}$ contains all points of the form $\varrho_i v_{i+1}$, where

$$\varrho_i \in \begin{cases} \Bigl[\frac{v_i^{\mathsf{T}} b_i^*}{v_{i+1}^{\mathsf{T}} b_i^*}, \infty\Bigr)\,, & \text{if } (v_{i+1} + v_i)^{\mathsf{T}} A_i^{\mathsf{T}} b_i^* > 0 \\ \Bigl(0, \frac{v_i^{\mathsf{T}} b_i^*}{v_{i+1}^{\mathsf{T}} b_i^*}\Bigr]\,, & \text{if } (v_{i+1} + v_i)^{\mathsf{T}} A_i^{\mathsf{T}} b_i^* < 0\,. \end{cases}$$

Moreover, by Corollary 12, for any such ϱ_i, there exists a constant gain $k_i = k_i(\varrho_i)$, such that under the control $u = k_i^{\mathsf{T}} x$, the closed-loop trajectory in $\mathcal{K}_{i*}$ steers v_i to $\varrho_i v_{i+1}$. On the other hand, if $b_i \neq 0$ and $\mathrm{range}(A_i) \subset \mathcal{B}_i$, then $\mathrm{Reach}_{\Sigma_i}(v_i) = (v_i + \mathcal{B}_i) \cap \mathcal{K}_{i*}$. In this case, it easily follows that for some $\zeta_i \in \mathbb{R}$, the closed-loop trajectory starting at v_i and under the feedback control $u = \zeta_i b_i^*$, is a straight line segment in $\mathcal{K}_{i*}$ that joins v_i to $\frac{v_i^{\mathsf{T}} b_i^*}{v_{i+1}^{\mathsf{T}} b_i^*} v_{i+1}$. Hence we set $\varrho_i = \frac{v_i^{\mathsf{T}} b_i^*}{v_{i+1}^{\mathsf{T}} b_i^*}$. Lastly, if $b_i = 0$, in view of Lemma 11, set $\varrho_i = e^{\mu_i \tau_i}$. Since $\prod_{i \in \mathcal{I}} \overline{\xi}_i < 1$, it follows that the collection $\{\varrho_i,\ i \in \mathcal{I}\}$ may be selected such that $\prod_{i \in \mathcal{I}} \overline{\varrho}_i < 1$. Let $\tilde{\gamma}$ denote the segment of the closed-loop trajectory under a complete cycle. Clearly $\tilde{\gamma}$ steers v_1 to to $\left(\prod_{i \in \mathcal{I}} \varrho_i\right) v_1$, and it easily follows that the closed-loop trajectory converges asymptotically to the origin. Since, by linear scaling every $x \in \mathbb{R}^2_*$ satisfies $\lambda x \in \tilde{\gamma}$ for some $\lambda > 0$, it follows that the closed-loop system is asymptotically stable. □

Remark 22 *As seen in Example 18, even if every pair (A_i, b_i) is controllable, the system might not be stabilizable by state feedback. This connects directly to the stability analysis. Despite the fact that the eigenvalues of the closed loop system $A_i + b_i k_i^{\mathsf{T}}$ can be selected to have any negative values desired, thus making the coefficients α_i as negative as desired, this process also affects the gains β_i in a manner that might always result in an unstable system.*

References

[1] M. Pachter, D. H. Jacobson, The stability of planar dynamical systems linear-in-cones, IEEE Trans. Automat. Control 26 (2) (1981) 587–590.

[2] X. Xu, P. J. Antsaklis, Stabilization of second-order LTI switched systems, Internat. J. Control 73 (14) (2000) 1261–1279.

[3] U. Boscain, Stability of planar switched systems: the linear single input case, SIAM J. Control Optim. 41 (1) (2002) 89–112.

[4] D. Holcman, M. Margaliot, Stability analysis of second-order switched homogeneous systems, SIAM J. Control Optim. 41 (5) (2002) 1609–1625.

[5] M. Margaliot, Stability analysis of switched systems using variational principles: an introduction, Automatica, to appear.

[6] M. Pachter, D. H. Jacobson, Observability with a conic observation set, IEEE Trans. Automat. Control 24 (4) (1979) 632–633.

[7] F. R. Gantmacher, The Theory of Matrices, I, Chelsea, New York, 1960.

[8] D. Liberzon, J. P. Hespanha, A. S. Morse, Stability of switched systems: a Lie-algebraic condition, Systems Control Lett. 37 (3) (1999) 117–122.

[9] A. A. Agrachev, D. Liberzon, Lie-algebraic stability criteria for switched systems, SIAM J. Control Optim. 40 (1) (2001) 253–269.

[10] W. Boothby, A transitivity problem from control theory, J. of Differential Equations 17 (2) (1975) 296–307.

[11] J.-I. Imura, A. van der Schaft, Characterization of well-posedness of piecewise-linear systems, IEEE Trans. Automat. Control 45 (9) (2000) 1600–1619.

[12] A. Bacciotti, Some remarks on generalized solutions of discontinuous differential equations, Int. J. Pure Appl. Math. 10 (3) (2004) 257–266.

[13] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, A. I. Subbotin, Asymptotic controllability implies feedback stabilization, IEEE Trans. Automat. Control 42 (10) (1997) 1394–1407.

[14] A. F. Filippov, Differential equations whose right-hand side is discontinuous on intersecting surfaces, Differentsial′nye Uravneniya 15 (10) (1979) 1814–1823.

[15] A. F. Filippov, Differential equations with discontinuous righthand sides, Vol. 18 of Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1988, translated from the Russian.