TIME-VARYING AFFINE FEEDBACK FOR REACH CONTROL ON SIMPLICES

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Abstract. We study the reach control problem for affine systems on simplices, and the focus is on cases when it is known that the problem is not solvable by continuous state feedback. In previous work we used the reach control indices to construct a (discontinuous) piecewise affine control that solves the problem. Here we initiate an investigation of the extent to which time-varying feedbacks can be used. A particular time-varying feedback is proposed and shown to solve the problem.

1. Introduction

This paper studies the *reach control problem* in simplices. The problem is for an affine system defined on a simplex to leave the simplex through a prespecified facet in finite time. The problem was introduced in [7] and was further developed in [8, 9, 14, 3, 4, 5, 6]. In [3] it was shown that affine feedback and continuous state feedback are equivalent from the point of view of solvability of the reach control problem (RCP). In [4, 6] we developed reach control indices which expose how affine or continuous state feedbacks may fail - such feedbacks induce closed-loop equilibria in sub-simplices that are inherently starved of sufficient inputs. Fortunately, the reach control indices also give insight on how to overcome the problem of insufficient inputs. In [5, 6] we presented a subdivision procedure that triangulates the simplex into sub-simplices with sub-reach control problems that are solvable by affine feedback. The final outcome was that if the reach control problem is solvable by open-loop controls, then it is solvable by (discontinuous) piecewise affine feedback.

The objective of this paper is to explore whether other types of controls can be used to solve the problem, in the case when it is not solvable by continuous state feedback. We are especially interested in controls that are not discontinuous in order to circumvent issues about chattering due to measurement errors, and a natural choice is time-varying feedback. Here we present a method inspired by the information that is provided by the reach control indices. In particular, these indices indicate in which sub-simplices equilibria appear when affine feedback is used. This information is used to construct a compensator that dynamically shifts the set of equilibria so that trajectories can effectively "roll around" the equilibria in order to exit the simplex.

The paper is organized as follows. In Section 2 we explain the ideas of the paper using a 2D example. In Section 3 we formulate the reach control problem and we review those cases when the problem is not solvable by continuous state feedback, thereby setting the context for the rest of the paper. In Section 4 we review results on the reach control indices. In Section 5 we develop a flow-like condition that tells us about the general movement of

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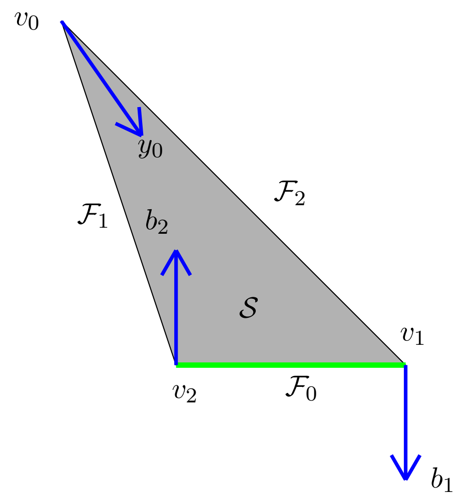


FIGURE 1. 2D example.

trajectories in the simplex. In Section 6 we present a time-varying compensator and we give the main result of the paper on solvability of RCP using this compensator. In Section 7 we apply the result to a pedagogical example arising from a materials transport problem. Proofs of supporting lemmas are found in the Appendix.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The complement of $\mathcal{K}$ is $\mathcal{K}^c := \mathbb{R}^n \setminus \mathcal{K}$, the closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. For a vector $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\mathscr{B}$ denotes the open unit ball, and $\overline{\mathscr{B}}$ denotes its closure. The notation $\mathrm{co}\{v_1, v_2, \dots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation $\mathrm{sp}\{y_1, y_2, \dots\}$ denotes the span of vectors $y_i \in \mathbb{R}^n$. Symbol $\mathbb{U}$ denotes a control type: we consider open-loop controls, continuous state feedback, affine feedback, or time-varying feedback.

2. CONTRIBUTION

We explain the contribution of the paper informally and intuitively by way of an example. Consider the simplex $\mathcal{S} = \mathrm{co}\{v_0, v_1, v_2\}$ and facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$, as in Figure 1. Consider a single-input control system $\dot{x} = Ax + bu + a$ defined on $\mathcal{S}$. The *reach control problem* is to find a state feedback $u(x)$ such that all closed-loop trajectories initialized in $\mathcal{S}$ leave $\mathcal{S}$ in finite time through the *exit facet* $\mathcal{F}_0$.

Suppose that $v_0 = (0, 0)$, $b = (0, 1)$, and that $Ax + a \in \mathrm{Im}(b)$ along a line $\mathcal{O}$ through v_1 and v_2. Let $\beta \in \mathrm{Ker}(b^T)$ be such that $\beta \cdot (Ax + a) = 0$ on $\mathcal{O}$, $\beta \cdot (Ax + a) > 0$ above $\mathcal{O}$, and $\beta \cdot (Ax + a) < 0$ below $\mathcal{O}$. Clearly closed loop equilibria can only appear on the set $\mathcal{G} := \mathcal{S} \cap \mathcal{O}$. The procedure to solve the control problem by continuous state feedback is to select control values u_i at the vertices v_i such that the velocity vectors $Av_i + bu_i + a$ point inside $\mathrm{cone}(\mathcal{S})$, the cone with apex at v_0 determined by $\mathcal{S}$; otherwise trajectories may leave $\mathcal{S}$ through $\mathcal{F}_1$ or $\mathcal{F}_2$, which is disallowed. The controller $u(x)$ is formed as a continuous interpolation of the control values at the vertices. Label the vertex velocity vectors as $y_0 = Av_0 + bu_0 + a$, and $b_i = Av_i + bu_i + a$, $i = 1, 2$, as shown in the figure. Importantly, $b_i \in \text{Im}(b)$ because v_1 and v_2 are in $\mathcal{O}$. Now it is obvious that this control problem cannot be solved by any continuous state feedback. For at v_1, b_1 has to point up to be inside cone($\mathcal{S}$), but at v_2, b_2 has to point down. If we continuously interpolate along $\mathcal{F}_0$ from v_1 to v_2, the continuous vector field, always in Im(b) along $\mathcal{F}_0$, must pass through zero (by the Intermediate Value Theorem). The defect is that there are two vertices v_1 and v_2 that "share" the only control direction available, b.

On the other hand, the problem is trivially solved by open-loop controls. Take any initial condition $x_0 \in \mathcal{S}$. For the sake of argument, say $x_0 \notin \mathcal{F}_0$. Pick any final point x_f close to $\mathcal{F}_0$ such that the slope of the line through x_0 and x_f is negative. Because $\beta \cdot (Ax + a) > 0$ for $x \in \mathcal{S} \setminus \mathcal{F}_0$, at every point z along the line $\overline{x_0 x_f}$ there exists a control value u_z such that $Az + bu_z + a$ is a non-zero velocity vector tangent to the line. This determines an open loop control that drives the state along the line from x_0 to x_f in finite time. Then to drive the state out of $\mathcal{S}$ from x_f sufficiently close to $\mathcal{F}_0$, use a sufficiently large negative input. A formal argument is provided by Theorem 6 of [11]. Our concern is with the gap between solvability by open loop controls and solvability by continuous state feedbacks, and we ask ourselves: is there any other type of feedback that can solve this problem? The only known result is the class of discontinuous piecewise affine feedbacks [5, 6]. What about continuous controllers?

Consider this situation. Suppose we devise an affine controller called $u^0(x)$ that assigns the same vertex velocity vectors as $u(x)$, except $u^0(x)$ places an equilibrium at v_1. Then we devise a second affine controller $u^\infty(x)$ that assigns the same vertex velocity vectors as $u(x)$, except $u^\infty(x)$ places an equilibrium at v_2. Finally, define a controller that interpolates between these two:

$$u(x, \alpha) := (1 - \alpha)u^0(x) + \alpha u^\infty(x)\,. \tag{1}$$

By playing with $\alpha \in [0, 1]$, we can "slide" the equilibrium along $\mathcal{F}_0$. See Figure 2. Therefore, we have introduced a new degree of freedom, so to speak, that is not available with continuous state feedback.

How do we exploit this new degree of freedom to solve the reach control problem by a continuous feedback? Here is one idea. Recall that $\beta \cdot (Ax + a) > 0$ for $x \in \mathcal{S} \setminus \mathcal{F}_0$ and that $\beta \in \text{Ker}b^T$. This means that in $\mathcal{S} \setminus \mathcal{F}_0$, *independent of the choice of controller*, trajectories naturally drift to the right. Now suppose we can slide the equilibrium to the left along $\mathcal{F}_0$ using $u(x, \alpha)$. Then trajectories could be made to "roll around" the equilibrium as it drifts towards v_2. There is one caveat. Trajectories that start in $\mathcal{S}$ to the left of v_2 might not move quickly enough to the right and thereby get stuck at an equilibrium at v_2. This is circumvented if the equilibrium is slid slowly enough so that trajectories originating to the left of v_2 have enough time to drift to the right past v_2. Our main lever for controlling the speed of sliding is the dynamics of α, which we get to pick. Finally, the overall argument works because b cuts $\mathcal{F}_0$ transversally; in other words, a hyperplane $\mathcal{H}^*$ with normal vector β strongly separates v_1 and v_2. If b were parallel to $\mathcal{F}_0$, the argument would fail completely.

The contribution of the paper is to make mathematically rigorous for multi-input systems the informal ideas presented for this single input 2D system.

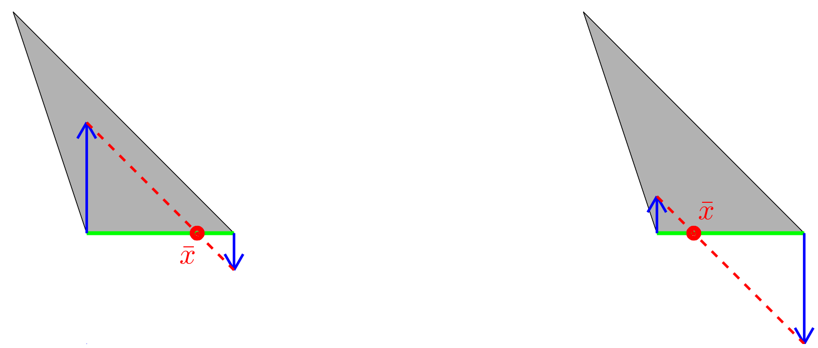


FIGURE 2. Possible locations for equilibria.

3. Reach Control Problem

Consider an n-dimensional simplex $\mathcal{S}$ with vertex set $V := \{v_0, v_1, \ldots, v_n\}$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ (the facet is indexed by the vertex it does not contain). Let h_i be the unit normal vector to the facet $\mathcal{F}_i$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet* of $\mathcal{S}$. Define the index sets $I := \{1, \ldots, n\}$ and $I_i := I \setminus \{i\}$ (note $I_0 = I$). Define the closed, convex cones

$$\mathcal{C}_i := \left\{ \, y \in \mathbb{R}^n \; : \; h_j \cdot y \leq 0, \; j \in I_i \, \right\}, \qquad i \in \{0, \ldots, n\}.$$

We'll write cone$(\mathcal{S}) := \mathcal{C}_0$ since $\mathcal{C}_0$ is the tangent cone to $\mathcal{S}$ at v_0. We consider the affine control system

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{S}, \tag{2}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and rank$(B) = m$. Let $\phi_u(t, x_0)$ denote the trajectory of (2) starting at x_0 under some control law u.

Problem 1 (Reach Control Problem (RCP))**.** *Consider system (2) defined on $\mathcal{S}$. Find a feedback control $u(x)$ such that for every $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\gamma > 0$ such that*

(i) $\phi_u(t, x_0) \in \mathcal{S}$ *for all* $t \in [0, T]$,
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, *and*
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ *for all* $t \in (T, T + \gamma)$.

RCP imposes that all closed-loop trajectories leave $\mathcal{S}$ in finite time through the exit facet $\mathcal{F}_0$, without first leaving $\mathcal{S}$ through another facet. Note that condition (iii) requires that the dynamics (2) are extended to a neighborhood of $\mathcal{S}$. The following conditions guarantee that closed-loop trajectories cannot leave $\mathcal{S}$ through a non-exit facet.

Definition 1. *We say the* invariance conditions are solvable *if for each $v_i \in V$ there exists $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a \in \mathcal{C}_i$. Equivalently,*

$$h_j \cdot (Av_i + Bu_i + a) \leq 0\,, \qquad i \in \{0, \ldots, n\}, \qquad j \in I_i\,. \tag{3}$$

The inequalities (3) are called the *invariance conditions*, and they are used to construct affine feedbacks [8]. For general state feedbacks, stronger conditions (also called invariance conditions) are needed.

Definition 2. *We say a state feedback $u(x)$ satisfies the invariance conditions if for all $j \in I$ and $x \in \mathcal{F}_j$,*

$$h_j \cdot (Ax + Bu(x) + a) \leq 0\,. \tag{4}$$

Definition 3. *A point $x_0 \in \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ and using control type $\mathbb{U}$, denoted by $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a control u of type $\mathbb{U}$ such that properties (i)-(iii) of Problem 1 hold. We write $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by control type $\mathbb{U}$ if for every $x_0 \in \mathcal{S}$, $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$ with control of type $\mathbb{U}$.*

For Problem 1 the following necessary and sufficient conditions have been established for the case of affine feedback.

Theorem 4. [9, 14] *Given the system* (2) *and an affine feedback $u(x) = Kx + g$, where $K \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$, the closed-loop system satisfies $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ if and only if*

- (a) *The invariance conditions* (3) *are satisfied with $u_0 = u(v_0), \ldots, u_n = u(v_n)$.*
- (b) *There is no closed-loop equilibrium in $\mathcal{S}$.*

Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Define $\mathcal{O} := \{\, x \in \mathbb{R}^n \;:\; Ax + a \in \mathcal{B}\}$ and $\mathcal{G} := \mathcal{S} \cap \mathcal{O}$. Notice that closed-loop equilibria can only appear in $\mathcal{G}$. In the remainder of the paper we make an important assumption concerning the placement of $\mathcal{O}$ with respect to $\mathcal{S}$. The reader is referred to [3] for the motivation for and a method of triangulation of the state space that achieves this assumption. See also [10].

Assumption 5. *Simplex $\mathcal{S}$ and system* (2) *satisfy the following condition: if $\mathcal{G} \neq \emptyset$, then $\mathcal{G}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \leq \kappa \leq n$.*

The primary conclusion of [3] is that under Assumption 5, RCP is solvable by affine feedback if and only if it is solvable by continuous state feedback. The goal of this paper is to solve RCP in cases where continuous state feedback cannot be used. Based on [3], the cases to be studied are captured by the following assumptions. Let $I_{\mathcal{G}} := \{1, \ldots, \kappa + 1\}$ be the vertex index set of $\mathcal{G}$.

Assumption 6. *Simplex $\mathcal{S}$ and system* (2) *satisfy the following conditions.*

- (A1) $\mathcal{G} = \mathrm{co}\{v_1, \ldots, v_{\kappa+1}\}$, *with $0 \leq \kappa < n$.*
- (A2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$.
- (A3) $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}$, $i \in I_{\mathcal{G}}$.
- (A4) *The maximum number of linearly independent vectors in any set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ (with only one vector for each $\mathcal{B} \cap \mathcal{C}_i, i \in I_{\mathcal{G}}$) is $\widehat{m}$ with $1 \leq \widehat{m} \leq \kappa$.*

Assumptions (A1)-(A2) exclude the application of results on solvability by affine feedback in [3]. Assumption (A3) is necessary for solvability by open-loop controls [5, 6]. Finally, (A4) introduces a new condition in terms of the variable $\widehat{m}$, which necessarily satisfies $\widehat{m} \leq \kappa+1$. When $\widehat{m} = \kappa + 1$, an affine feedback solves RCP [3]. The remaining cases when $\widehat{m} \leq \kappa$ are the topic of this paper.

In the sequel we make use of the following family of matrices. Let $1 \leq p \leq q \leq \kappa + 1$, $b_i \in \mathcal{B} \cap \mathcal{C}_i$, and define $H_{p,q} := [h_p \cdots h_q] \in \mathbb{R}^{n\times(q-p+1)}$, $Y_{p,q} := [b_p \cdots b_q] \in \mathbb{R}^{n\times(q-p+1)}$, and $M_{p,q} := H_{p,q}^T Y_{p,q}$. We say a matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \leq 0$ for all $i \neq j$ [2]. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in I_{\mathcal{G}}$, each $M_{p,q}$ is a $\mathscr{Z}$-matrix. Also under the condition that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$, certain matrices of the form $M_{p,q}$

will be shown to be nonsingular $\mathscr{M}$-matrices [6]. A complete characterization of nonsingular $\mathscr{M}$-matrices is found in [2], Ch. 6.

4. Reach Control Indices

Suppose Assumption 6 holds. We know from [3] that RCP is not solvable by continuous state feedback. Since $\widehat{m} \leq \kappa$, there exists $p \geq 1$ such that

$$\widehat{m} + p = \kappa + 1 \,.$$

Evidently there are not enough independent vectors in $\mathcal{B}$ to resolve all invariance conditions for vertices in $\mathcal{G}$. The reach control indices provide a means to bookkeep those vertices $v_i \in \mathcal{G}$ that share degrees of freedom in $\mathcal{B}$, and they capture the strong restrictions placed on $\mathcal{B}$ imposed by (3). We summarize the main results below.

Theorem 7 ([4, 6])**.** *Suppose Assumption 6 holds. Then there exist integers $r_1, \ldots, r_p \geq 2$ and an independent family of subspaces $\{\mathcal{B}_1, \ldots, \mathcal{B}_p\}$ such that w.l.o.g. (by reordering indices)*

$$\mathcal{B} \cap \mathcal{C}_i \;\subset\; \mathcal{B}_1 := \mathrm{sp}\{b_{m_1}, \ldots, b_{m_1+r_1-1}\} \,, \qquad i = m_1, \ldots, m_1 + r_1 - 1 \,, \tag{5a}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\mathcal{B} \cap \mathcal{C}_i \;\subset\; \mathcal{B}_p := \mathrm{sp}\{b_{m_p}, \ldots, b_{m_p+r_p-1}\} \,, \qquad i = m_p, \ldots, m_p + r_p - 1 \,, \tag{5b}$$

where $b_i \in \mathcal{B} \cap \mathcal{C}_i$ and

$$m_k := r_1 + \cdots + r_{k-1} + 1 \,, \qquad k = 1, \ldots, p \,. \tag{6}$$

Moreover, for each $k = 1, \ldots, p$, $\{b_{m_k}, \ldots, b_{m_k+r_k-2}\}$ are linearly independent and

$$b_{m_k+r_k-1} = c_{m_k} b_{m_k} + \cdots + c_{m_k+r_k-2} b_{m_k+r_k-2} \tag{7}$$

with $c_i < 0$.

We observe that due to (7) the lists (5a)-(5b) have the property that any vector in a list on the right is dependent on all the other vectors in its list. Also, if any vector is removed from a list, the remaining vectors are linearly independent. In particular, the kth list contains $r_k - 1$ linearly independent vectors in $\mathcal{B}$, so $\dim(\mathcal{B}_k) = r_k - 1$. The integers $\{r_1, \ldots, r_p\}$ are called the *reach control indices* of system (2) with respect to simplex $\mathcal{S}$. The reach control indices enable to discover a further necessary condition (D5) for solvability of RCP by open-loop control. This leads to the following assumptions to be used in the rest of the paper.

Assumption 8. *Simplex $\mathcal{S}$ and system* (2) *satisfy the following conditions.*

- (D1) $\mathcal{G} = \mathrm{co}\{v_1, \ldots, v_{\kappa+1}\}$, $0 \leq \kappa < n$.
- (D2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$.
- (D3) $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}\,, \quad i \in I_{\mathcal{G}}$.
- (D4) $\exists \widehat{m}$ *such that* (A4) *holds,* $\exists \ \{r_1, \ldots, r_p\}$ *such that* (5a)-(5b) *hold.*
- (D5) $\mathcal{B}_k \not\subset \mathcal{H}_0 := \{y \in \mathbb{R}^n \mid h_0 \cdot y = 0\}\,, \quad k = 1, \ldots, p$.

Conditions (D1)-(D3) are copied from (A1)-(A3) and define the problem setup. The necessity of (D3) and (D5) was proved in [6] for a slightly stronger version of RCP, and (D4) comes from Theorem 7. Below we summarize the algebraic consequences of the reach control indices, stated primarily in the language of $\mathscr{M}$-matrices [2]. Proofs are provided in the Appendix.

Let $H_{p,q} := [h_p \cdots h_q]$ and $Y_{p,q} := [b_p \cdots b_q]$, where the b_i's come from (5a)-(5b). Let $M_{p,q} := H_{p,q}^T Y_{p,q}$. Define

$$r := r_1 + \cdots + r_p$$

and for $k = 1, \dots, p$ define

$$I_{\mathcal{G}_k} := \{m_k, \dots, m_k + r_k - 1\}\,, \tag{8}$$

$$\mathcal{G}_k := \mathrm{co}\{v_{m_k}, \dots, v_{m_k+r_k-1}\}\,. \tag{9}$$

Lemma 9 ([4, 6])**.** *Suppose Assumption 8 holds. Then for each $k = 1, \dots, p$,*

$$h_j \cdot b_i = 0\,, \qquad i \in I_{\mathcal{G}_k}\,, \qquad j \in I \setminus I_{\mathcal{G}_k}\,. \tag{10}$$

Equivalently, $\mathcal{B}_k \perp \mathrm{sp}\{h_{m_1}, \dots, h_{m_{k-1}+r_{k-1}-1}, h_{m_{k+1}}, \dots, h_n\}$. Moreover,

$$\mathcal{B}_k \subset \mathrm{sp}\{v_{m_k} - v_0, \dots, v_{m_k+r_k-1} - v_0\}\,. \tag{11}$$

Lemma 10. *Suppose Assumption 8 holds. Then for each $k = 1, \dots, p$,*

(i) *Each principal submatrix of M_{m_k, m_k+r_k-1} is a nonsingular $\mathscr{M}$-matrix.*
(ii) *Matrix $M_{m_k, m_k+r_k-1} \in \mathbb{R}^{r_k \times r_k}$ is irreducible.*
(iii) *Matrix M_{m_k, m_k+r_k-1} is a singular $\mathscr{M}$-matrix.*

Lemma 11 ([4, 6])**.** *Suppose Assumption 8 holds. For each $k = 1, \dots, p$, there does not exist $\beta \in \mathcal{B}$ such that $\{b_{m_k}, \dots, b_{m_k+r_k-2}, \beta\}$ are linearly independent and*

$$h_j \cdot \beta \le 0\,, \qquad\qquad j \in I \setminus I_{\mathcal{G}_k}\,. \tag{12}$$

5. A Flow-like Condition

We begin our exploration of time-varying feedback to solve RCP in cases when it is not solvable by continuous state feedback. The development is divided into two parts. First we establish that a flow-like condition holds on $\mathcal{S}$ which has desirable properties relative to the sub-simplices $\mathcal{G}_k$, $k = 1, \dots, p$. By a *flow-like condition* we mean a condition of the form:

$$\xi \cdot (Ax + Bu(x) + a) \ge 0\,, \qquad\qquad x \in \mathcal{S}\,,$$

where $0 \neq \xi \in \mathbb{R}^n$. Geometrically this condition corresponds to a foliation of parallel hyperplanes $\mathcal{H}_c := \{x \in \mathbb{R}^n \mid \xi \cdot x = c\}$, $c \in \mathbb{R}$, with normal vector ξ such that closed-loop trajectories on $\mathcal{S}$ flow in one sense only with respect to the hyperplanes. Second, we propose a time-varying compensator whose role in essence is to dynamically shift the set of equilibria generated by affine feedback in a direction opposite to the direction indicated by the flow-like condition.

In this section the flow-like condition is developed. We give an overview of how it is derived. Consider each $\mathcal{G}_k$ and its associated set of velocity vectors in $\mathcal{B}_k$. By assumption (D5), $\mathcal{B}_k$ is not parallel to $\mathcal{H}_0$. Consequently one can find a vector β_k in $\mathrm{Ker}(B^T)$ (or a variant of it), with the property that there exists a hyperplane with normal vector β_k that strongly separates at least two vertices in $\mathcal{G}_k$. This is the content of Lemma 14. Simultaneously this β_k trivially contributes toward a flow-like condition on $\mathcal{G}$ since $\beta_k \cdot (Ax + Bu(x) + a) = 0$ for any $x \in \mathcal{G}$ and any control $u(x)$. By taking a linear combination of the β_k's, one gets both the vertex separation property and the flow-like condition using a single vector ξ^1. This is done in Lemma 15. The fact that the vertex separation property of β_i is not corrupted by the presence of β_j, $j \neq i$, in the expression for ξ^1 is a result of the special form of the β_k's. This form is derived in Lemma 12. It relies on the properties of the simplex summarized in Lemma 21 and on the properties of relevant $\mathscr{M}$-matrices summarized in Lemma 10.

So far we have discussed the strong separation of vertices in each $\mathcal{G}_k$ and a flow-like condition on $\mathcal{G}$ in terms of a vector ξ^1. There remains the question of a flow-like condition for $\mathcal{S} \setminus \mathcal{G}$ in terms of a vector ξ^2. This condition arrives automatically from convex analysis, as given in Lemma 16. Finally the flow-like conditions based on ξ^1 and ξ^2 are put together into one flow-like condition based on a vector ξ^* in Theorem 17. This ξ^* is adjusted to retain the vertex separation property, the main property upon which the time-varying compensator is based.

Define the matrix

$$\widehat{B} = \left[\, b_1 \cdots \; b_{m_1+r_1-2} \cdots \; b_{m_p} \cdots \; b_{m_p+r_p-2} \right] \in \mathbb{R}^{n \times (r-p)}\,, \tag{13}$$

and let $\widehat{\mathcal{B}} = \mathrm{Im}(\widehat{B})$. Note that the columns of $\widehat{B}$ are ordered according to Theorem 7 and that vectors $b_{m_k+r_k-1}$, $k = 1, \ldots, p$, do not appear in the columns of $\widehat{B}$. Also, velocity vectors associated with v_i, $i = r+1, \ldots, \kappa+1$ are not yet defined, so they do not appear. By Theorem 7, $\dim(\mathcal{B}_k) = r_k - 1$ and $\mathcal{B}_1, \ldots, \mathcal{B}_p$ form a family of independent subspaces, so $\mathrm{rank}(\widehat{B}) = r - p$. Recall the definitions of $I_{\mathcal{G}_k}$ and $\mathcal{G}_k$ from (8) and (9). The next result gives a specially selected vector used to strongly separate at least two vertices in each $\mathcal{G}_k$, $k = 1, \ldots, p$.

Lemma 12. *Suppose Assumption 8 holds. For each $k \in \{1, \ldots, p\}$ there exists $\beta_k \in \mathrm{Ker}(\widehat{B}^T)$ such that*

(i) $\beta_k = d_{m_k} h_{m_k} + \cdots + d_{m_k+r_k-1} h_{m_k+r_k-1}$, *with* $d_i < 0$.
(ii) $\beta_k \cdot (v_i - v_0) = 0, \quad i \in I \setminus I_{\mathcal{G}_k}$.
(iii) $\beta_k \cdot (v_i - v_j) = 0, \quad i, j \in I_{\mathcal{G}} \setminus I_{\mathcal{G}_k}$.

Proof. Let $k \in \{1, \ldots, p\}$. By Lemma 10(ii) and (iii), M_{m_k, m_k+r_k-1} is a singular, irreducible $\mathscr{M}$-matrix; therefore, so is $M^T_{m_k, m_k+r_k-1}$. By Theorem 6.4.16(2) of [2], there exists $d \prec 0$ such that $M^T_{m_k, m_k+r_k-1} d = 0$. Define

$$\beta_k := H_{m_k, m_k+r_k-1} d\,.$$

This gives the form (i). Next we show $\beta_k \in \mathrm{Ker}(\widehat{B}^T)$. First, we have

$$M^T_{m_k, m_k+r_k-1} d = Y^T_{m_k, m_k+r_k-1} H_{m_k, m_k+r_k-1} d = Y^T_{m_k, m_k+r_k-1} \beta_k = 0\,.$$

That is,

$$\beta_k \cdot b_i = 0\,, \qquad i = m_k, \ldots, m_k + r_k - 1\,.$$

Also from (10)

$$\beta_k \cdot b_i = 0\,, \qquad i = 1, \ldots, m_{k-1} + r_{k-1} - 1, m_{k+1}, \ldots, r\,.$$

We conclude $\beta \in \mathrm{Ker}(\widehat{B}^T)$. The statement (ii) follows from Lemma 21(ii). The statement (iii) follows from Lemma 21(iii). □

Lemma 13. *Suppose Assumption 8 holds. Consider $\beta_1, \ldots, \beta_p$ from Lemma 12. Then*

$$\mathrm{Ker}(\widehat{B}^T) = \mathrm{sp}\{\beta_1, \ldots, \beta_p, h_{r+1}, \ldots, h_n\}.$$

Proof. By construction $\beta_k \in \mathrm{sp}\{h_{m_k}, \ldots, h_{m_k+r_k-1}\}$, $k = 1, \ldots, p$, and so by Lemma 21(viii), $\{\beta_1, \ldots, \beta_p, h_{r+1}, \ldots, h_n\}$ are linearly independent. From (10), $\widehat{\mathcal{B}} \perp \mathrm{sp}\{h_{r+1}, \ldots, h_n\}$. Thus, $h_{r+1}, \ldots, h_n \in \mathrm{Ker}(\widehat{B}^T)$. Also from Lemma 12, $\beta_1, \ldots, \beta_p \in \mathrm{Ker}(\widehat{B}^T)$. Now $\mathrm{rank}(\widehat{B}^T) = r-p$, so $\dim(\mathrm{Ker}(\widehat{B}^T)) = n-r+p$. Thus, $\{\beta_1, \ldots, \beta_p, h_{r+1}, \ldots, h_n\}$ is a basis of $\mathrm{Ker}(\widehat{B}^T)$. □

The next result shows that for each $k = 1, \ldots, p$, vector β_k can be used to strongly separate at least two vertices in $\mathcal{G}_k$.

Lemma 14. *Suppose Assumption 8 holds. Consider $\beta_1, \ldots, \beta_p$ from Lemma 12. For each $k \in \{1, \ldots, p\}$, there exist $i_k, j_k \in I_{\mathcal{G}_k}$ such that*

$$\beta_k \cdot (v_{i_k} - v_{j_k}) \neq 0 \,. \tag{14}$$

Proof. Let $k \in \{1, \ldots, p\}$ and suppose by way of contradiction that for every $\beta \in \mathrm{Ker}(\widehat{B}^T)$ and $i, j \in I_{\mathcal{G}_k}$, $\beta \cdot (v_i - v_j) = 0$. This implies $(v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k}) \in \widehat{\mathcal{B}}$. Suppose w.l.o.g. that

$$v_{m_k+1} - v_{m_k} = b' + b''$$

where $b' \in \mathcal{B}_k$ and $0 \neq b'' \in \mathrm{sp}\{b_{m_1}, \ldots, b_{m_{k-1}+r_{k-1}-1}, b_{m_{k+1}}, \ldots, b_r\}$. Then by Lemma 21(iii) and by (10)

$$0 = h_j \cdot (v_{m_k+1} - v_{m_k}) = h_j \cdot b' + h_j \cdot b'' = h_j \cdot b'', \quad j = m_1, \ldots, m_{k-1} + r_{k-1} - 1, m_{k+1}, \ldots, n.$$

By construction $\{b_{m_k}, \ldots, b_{m_k+r_k-2}, b''\}$ are linearly independent. This contradicts Lemma 11. Thus, $b'' = 0$. This argument can be repeated for each $v_{m_k+i} - v_{m_k}$, $i = 1, \ldots, r_k - 1$ to get $(v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k}) \in \mathcal{B}_k$. By Lemma 21(vii), $\{(v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k})\}$ is a basis for $\mathcal{B}_k$, so $\mathcal{B}_k \subset \mathcal{H}_0$. This contradicts Assumption (D5).

We deduce that there exist $i_k, j_k \in I_{\mathcal{G}_k}$ and $\beta \in \mathrm{Ker}(\widehat{B}^T)$ such that $\beta \cdot (v_{i_k} - v_{j_k}) \neq 0$. By Lemma 13, $\beta \in \mathrm{Ker}(\widehat{B}^T)$ can be expressed as

$$\beta = \alpha_1 \beta_1 + \cdots + \alpha_p \beta_p + \alpha_{r+1} h_{r+1} + \cdots + \alpha_n h_n \,, \quad \alpha_i \in \mathbb{R} \,.$$

By Lemma 12(iii) and Lemma 21(iii)

$$\begin{aligned} 0 \neq \beta \cdot (v_{i_k} - v_{j_k}) &= (\alpha_1 \beta_1 + \cdots + \alpha_p \beta_p + \alpha_{r+1} h_{r+1} + \cdots + \alpha_n h_n) \cdot (v_{i_k} - v_{j_k}) \\ &= \alpha_k \beta_k \cdot (v_{i_k} - v_{j_k}) \end{aligned}$$

implying that $\beta_k \cdot (v_{i_k} - v_{j_k}) \neq 0$. □

In light of Lemma 14, we assume without loss of generality (by reordering the indices within each group $I_{\mathcal{G}_k}$) that for $k = 1, \ldots, p$

$$v_{m_k} \in \arg \max_{i \in I_{\mathcal{G}_k}} \beta_k \cdot v_i \,, \qquad v_{m_k+r_k-1} \in \arg \min_{i \in I_{\mathcal{G}_k}} \beta_k \cdot v_i \,. \tag{15}$$

We also define the sets

$$\begin{aligned} \mathcal{E}^0 \quad &:= \quad \mathrm{co}\big\{v_{m_1}, v_{m_2}, \ldots, v_{m_p}\big\} \\ \mathcal{E}^\infty \quad &:= \quad \mathrm{co}\big\{v_{m_1+r_1-1}, v_{m_2+r_2-1}, \ldots, v_{m_p+r_p-1}\big\} \,. \end{aligned}$$

In the next result we pick a single hyperplane that strongly separates the two sets $\mathcal{E}^0$ and $\mathcal{E}^\infty$.

Lemma 15. *Suppose Assumption 8 holds. Consider* $\beta_1, \ldots, \beta_p \in \mathrm{Ker}(\widehat{B}^T)$ *from Lemma 12. Define*

$$\xi^1 := \xi_1^1 \beta_1 + \cdots + \xi_p^1 \beta_p \,, \quad \xi_i^1 \in \mathbb{R} \,, \tag{16}$$

and

$$\mathcal{H} := \{ x \in \mathbb{R}^n : \xi^1 \cdot (x - v_0) = 1 \} \,. \tag{17}$$

There exist $\xi_1^1, \ldots, \xi_p^1 > 0$ *such that* $\mathcal{H}$ *strongly separates* $\mathcal{E}^0$ *and* $\mathcal{E}^\infty$*.*

Proof. Let $x \in \mathcal{E}^0$ and $y \in \mathcal{E}^\infty$. Then there exist $\alpha_i \geq 0$, $\sum_{i=1}^p \alpha_i = 1$, and $\gamma_i \geq 0$, $\sum_{i=1}^p \gamma_i = 1$ such that $x = \gamma_1 v_{m_1} + \cdots + \gamma_p v_{m_p}$ and $y = \alpha_1 v_{m_1+r_1-1} + \cdots + \alpha_p v_{m_p+r_p-1}$. For $k = 1, \ldots, p$ define

$$\Pi_k := \beta_k \cdot (v_{m_k} - v_0), \qquad \qquad \pi_k := \beta_k \cdot (v_{m_k+r_k-1} - v_0) \,.$$

By Lemma 12(i) we may write $\beta_k = d_{m_k} h_{m_k} + \cdots + d_{m_k+r_k-1} h_{m_k+r_k-1}$ with $d_i < 0$. By Lemma 21(iii)-(iv) we have that

$$\beta_k \cdot (v_i - v_0) = d_i h_i \cdot (v_i - v_0) > 0 \,, \qquad i \in I_{\mathcal{G}_k} \,.$$

In particular, $\pi_k > 0$. By Lemma 14 we know that $\Pi_k \neq \pi_k$; thus,

$$0 < \pi_k < \Pi_k \,, \qquad \qquad k = 1, \ldots, p \,.$$

Select $\xi_k^1 \in (\frac{1}{\Pi_k}, \frac{1}{\pi_k}) \neq \emptyset$ for $k = 1, \ldots, p$. Then using Lemma 12(ii)

$$\begin{aligned} \xi^1 \cdot (x - v_0) &= (\xi_1^1 \beta_1 + \cdots + \xi_p^1 \beta_p) \cdot (\gamma_1 (v_{m_1} - v_0) + \cdots + \gamma_p (v_{m_p} - v_0)) \\ &= \gamma_1 \xi_1^1 \beta_1 \cdot (v_{m_1} - v_0) + \cdots + \gamma_p \xi_p^1 \beta_p \cdot (v_{m_p} - v_0) \\ &= \gamma_1 \xi_1^1 \Pi_1 + \cdots + \gamma_p \xi_p^1 \Pi_p \\ &\geq \min_k \{\xi_k^1 \Pi_k\} > 1 \,. \end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned} \xi^1 \cdot (y - v_0) &= (\xi_1^1 \beta_1 + \cdots + \xi_p^1 \beta_p) \cdot (\alpha_1 (v_{m_1+r_1-1} - v_0) + \cdots + \alpha_p (v_{m_p+r_p-1} - v_0)) \\ &= \alpha_1 \xi_1^1 \beta_1 \cdot (v_{m_1+r_1-1} - v_0) + \cdots + \alpha_p \xi_p^1 \beta_p \cdot (v_{m_p+r_p-1} - v_0) \\ &= \alpha_1 \xi_1^1 \pi_1 + \cdots + \alpha_p \xi_p^1 \pi_p \\ &\leq \max_k \{\xi_k^1 \pi_k\} < 1 \,. \end{aligned} \tag{19}$$

Thus, $\mathcal{H}$ strongly separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$. □

The next result shows that the open-loop system naturally drifts in some fixed direction for points away from $\mathcal{G}$.

Lemma 16. *Suppose Assumption 8 holds. Let* $\mathcal{P} := \mathrm{co}\{v_0, v_{\kappa+2}, \ldots, v_n\}$*. There exists* $\xi^2 \in \mathrm{Ker}(B^T)$ *such that*

$$\xi^2 \cdot (Ax + a) > 0 \,, \qquad \qquad x \in \mathcal{P} \,. \tag{20}$$

Proof. Observe that $\mathcal{P}$ is compact and convex and that $\mathcal{P} \cap \mathcal{O} = \emptyset$. The image of $\mathcal{P}$ under the affine map $x \mapsto Ax + a$, denoted $\mathcal{C}_1 = A\mathcal{P} + a$ is also compact and convex. We observe that $\mathcal{C}_1 \cap \mathcal{B} = \emptyset$. For suppose not. Then there is a point $x \in \mathcal{P}$ such that $Ax + a \in \mathcal{B}$. Then $x \in \mathcal{O}$, by definition, which contradicts $\mathcal{P} \cap \mathcal{O} = \emptyset$. Note that both $\mathcal{C}_1$ and $\mathcal{B}$ are convex sets, and that $\mathcal{C}_1$ is bounded. By Corollary 11.4.2 of [13], there exists a hyperplane $\mathcal{H}_2$ separating $\mathcal{B}$ and $\mathcal{C}_1$ strongly. This implies $\mathcal{B}$ is parallel to $\mathcal{H}_2$ since $\mathcal{B}$ is a subspace.

Let ξ^2 be the normal vector to $\mathcal{H}_2$ pointing to the side containing $\mathcal{P}$. Then, $\xi^2 \in \mathrm{Ker}(B^T)$ and $\xi^2 \cdot (Ax + a) > 0$ for all $x \in \mathcal{P}$. $\square$

The following is the main result on a flow-like condition on $\mathcal{S}$.

Theorem 17. *Suppose Assumption 8 holds. Let $u(x,t)$ be a time-varying affine feedback such that for $t \geq 0$*

$$Av_i + Bu(v_i,t) + a \in \mathcal{C}_i\,, \quad i = 0, r+1, \dots, n \tag{21a}$$

$$Av_i + Bu(v_i,t) + a \in \widehat{\mathcal{B}}\,, \quad i = 1, \dots, r\,. \tag{21b}$$

Then there exists $0 \neq \xi^* \in \mathrm{Ker}(\widehat{B}^T)$ such that for $t \geq 0$

$$\xi^* \cdot (Ax + Bu(x,t) + a) \geq 0\,, \qquad x \in \mathcal{S}\,, \tag{22}$$

and such that $\mathcal{H}^*$ strongly separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$ where

$$\mathcal{H}^* := \{x \in \mathbb{R}^n \mid \xi^* \cdot (x - v_0) = 1\}\,. \tag{23}$$

Proof. Consider ξ^1 given by Lemma 15 and ξ^2 given by Lemma 16. Define

$$\xi^* := (1-\lambda)\xi^1 + \lambda\xi^2\,, \qquad \lambda \in (0,1). \tag{24}$$

Since $\beta_k = H_{m_k, m_k+r_k-1} d$ with $d \prec 0$ and by (21a) we have for each $k = 1, \dots, p$,

$$\beta_k \cdot (Av_i + Bu(v_i,t) + a) \geq 0\,, \qquad i = 0, r+1, \dots, n\,.$$

Therefore from (16)

$$\xi^1 \cdot (Av_i + Bu(v_i,t) + a) \geq 0\,, \qquad i = 0, r+1, \dots, n\,.$$

Using (20), for $i = 0, \kappa+2, \dots, n$

$$\xi^* \cdot (Av_i + Bu(v_i,t) + a) = (1-\lambda)\underbrace{\xi^1 \cdot (Av_i + Bu(v_i,t) + a)}_{\geq 0} + \lambda \underbrace{\xi^2 \cdot (Av_i + Bu(v_i,t) + a)}_{>0} > 0$$

where we use the fact that $\xi^2 \in \mathrm{Ker}(B^T)$. Similarly, for $i = r+1, \dots, \kappa+1$,

$$\xi^* \cdot (Av_i + Bu(v_i,t) + a) = (1-\lambda)\xi^1 \cdot (Av_i + Bu(v_i,t) + a) \geq 0\,,$$

where we use the fact that $\xi^2 \in \mathrm{Ker}(B^T)$ and $Av_i + a \in \mathcal{B}$, $i = r+1, \dots, \kappa+1$. Finally, since $\xi^1 \in \mathrm{Ker}(\widehat{B}^T)$, $\xi^2 \in \mathrm{Ker}(B^T)$, and $\mathrm{Ker}(B^T) \subset \mathrm{Ker}(\widehat{B}^T)$, then $\xi^* \in \mathrm{Ker}(\widehat{B}^T)$. Thus, by (21b), for $i = 1, \dots, r$

$$\xi^* \cdot (Av_i + Bu(v_i,t) + a) = 0\,.$$

By convexity of $Ax + Bu(x,t) + a$ in x, $\xi^* \cdot (Av_i + Bu(x,t) + a) \geq 0$ for all $t \geq 0$, $x \in \mathcal{S}$, and $\lambda \in (0,1)$, which proves (22).

Let $x \in \mathcal{E}^0$ and $y \in \mathcal{E}^\infty$, and let γ_i and α_i be as in the proof of Lemma 15. Using (18) and (19)

$$\xi^* \cdot (x - v_0) = (1-\lambda)\underbrace{\xi^1 \cdot (x - v_0)}_{>1} + \lambda\xi^2 \cdot (x - v_0)$$

and

$$\xi^* \cdot (y - v_0) = (1-\lambda)\underbrace{\xi^1 \cdot (y - v_0)}_{<1} + \lambda\xi^2 \cdot (y - v_0)\,.$$

Since functions $f_1(x) = \xi^1 \cdot (x - v_0)$ and $f_2(x) = \xi^2 \cdot (x - v_0)$ are continuous, they achieve a minimum and maximum on each compact set $\mathcal{E}^0$ and $\mathcal{E}^\infty$. This means we can select $\lambda \in (0, 1)$ close enough to 0 such that

$$\xi^* \cdot (y - v_0) < 1 < \xi^* \cdot (x - v_0)\,, \quad x \in \mathcal{E}^0, y \in \mathcal{E}^\infty\,. \tag{25}$$

With this choice of λ, $\mathcal{H}^*$ strongly separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$. □

6. Time-varying Compensator

The time-varying compensator will be constructed so as to exploit the flow-like condition (22) and the separation property of $\mathcal{H}^*$ in (23). First, we define two affine feedbacks $u^0(x)$ and $u^\infty(x)$ that place equilibria at $\mathcal{E}^0$ and $\mathcal{E}^\infty$, respectively. Then we define a compensator $u(x, \alpha)$ with additional state $\alpha \in \mathbb{R}$. This compensator simply interpolates between $u^0(x)$ and $u^\infty(x)$ as α varies from 0 to 1. By construction when $\alpha = 0$, all closed-loop equilibria are in $\mathcal{E}^0$. When $\alpha = 1$, they are in $\mathcal{E}^\infty$. Thus, as α varies from 0 to 1, the set of closed-loop equilibria crosses from one side of $\mathcal{H}^*$ to the other in a direction with decreasing ξ^* component. Informally, we can say that trajectories flow downstream according to (22) while equilibria flow upstream, so that no trajectory can be "stuck" at an equilibrium. Ultimately, this enables all trajectories to exit $\mathcal{S}$, as shown in Theorem 20.

Suppose the invariance conditions for $\mathcal{S}$ are solvable; thus, there exist inputs $u_0^0, \ldots, u_n^0 \in \mathbb{R}^m$ such that (3) hold. Let $y_i^0 := Av_i + Bu_i^0 + a$, for $i = 0, \ldots, n$. We choose $u_{m_1}^0, u_{m_2}^0, \ldots, u_{m_p}^0 \in \mathbb{R}^m$ such that

$$y_i^0 = 0\,, \qquad i \in \{m_1, m_2, \ldots, m_p\} \tag{26a}$$
$$y_i^0 = b_i\,, \qquad i \in I_{\mathcal{G}} \setminus \{m_1, m_2, \ldots, m_p\}\,, \tag{26b}$$

where $b_i \in \widehat{\mathcal{B}} \cap \mathcal{C}_i$, $i = 1, \ldots, r$, are provided by Theorem 7; and $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i = r+1, \ldots, \kappa+1$, are selected so that $\widehat{m}$ independent directions in $\mathcal{B}$ are associated with $\mathcal{G}$, as per (D4). Finally, construct the associated affine feedback

$$u^0(x) = K^0 x + g^0,$$

and let $\phi^0(t, x_0)$ denote trajectories of the closed-loop system. Note that the closed-loop system has equilibria at $v_{m_1}, \ldots, v_{m_p}$.

Next we define a symmetrical controller $u^\infty(x)$ which is identical to $u^0(x)$ except that it places equilibria at $v_{m_k + r_k - 1}$, $k = 1, \ldots, p$. Let $y_i^\infty := Av_i + Bu_i^\infty + a$, for $i = 0, \ldots, n$. First set $u_i^\infty = u_i^0$, $i = 0, r+1, \ldots, n$. Then we choose $u_1^\infty, \ldots, u_{\kappa+1}^\infty \in \mathbb{R}^m$ such that

$$y_i^\infty = b_i\,, \quad i \in \{1, \ldots, r\} \setminus \{m_1 + r_1 - 1, m_2 + r_2 - 1, \ldots, m_p + r_p - 1\} \tag{27a}$$
$$y_i^\infty = 0\,, \quad i \in \{m_1 + r_1 - 1, m_2 + r_2 - 1, \ldots, m_p + r_p - 1\}\,, \tag{27b}$$

where again $b_i \in \widehat{\mathcal{B}} \cap \mathcal{C}_i$, $i = 1, \ldots, r$, are provided by Theorem 7. Finally, construct the associated affine feedback

$$u^\infty(x) = K^\infty x + g^\infty,$$

and let $\phi^\infty(t, x_0)$ denote trajectories of the closed-loop system. Note that this closed-loop system has equilibria at $v_{m_k + r_k - 1}$, $k = 1, \ldots, p$. The next result argues that, away from equilibria, trajectories do exit $\mathcal{S}$.

Lemma 18. *There exist $\xi^0, \xi^\infty \in \mathbb{R}^n$ such that*

$$\xi^0 \cdot (Ax + Bu^0(x) + a) \;>\; 0\,, \quad x \in \mathcal{S} \setminus \mathcal{E}^0\,, \tag{28}$$
$$\xi^\infty \cdot (Ax + Bu^\infty(x) + a) \;>\; 0\,, \quad x \in \mathcal{S} \setminus \mathcal{E}^\infty\,. \tag{29}$$

Proof. We consider only (28), since the proof for (29) is completely analogous. First, we claim

$$Ax + Bu^0(x) + a \neq 0\,, \qquad \forall x \in \mathcal{S} \setminus \mathcal{E}^0\,.$$

Suppose not. Then there exists $\overline{x} \in \mathcal{S} \setminus \mathcal{E}^0$ such that

$$A\overline{x} + Bu^0(\overline{x}) + a = 0\,.$$

Since necessarily $\overline{x} \in \mathcal{G}$, there exist $\lambda_1, \ldots, \lambda_{\kappa+1} \geq 0$ with $\sum \lambda_i = 1$ and not all λ_i, $i \in I_{\mathcal{G}} \setminus \{m_1, \ldots, m_p\}$ equal to zero such that

$$\overline{x} = \sum_{i=1}^{\kappa+1} \lambda_i v_i\,.$$

By convexity of $y^0(x) := Ax + Bu^0(x) + a$ and the fact that $y^0(v_i) = 0$ for $i = m_1, \ldots, m_p$, we have

$$0 = y^0(\overline{x}) = \sum_{i=1}^{\kappa+1} \lambda_i y^0(v_i) = \sum_{i \in I_{\mathcal{G}} \setminus \{m_1, \ldots, m_p\}} \lambda_i b_i\,,$$

with not all λ_i's equal to zero. The set $\{b_i \mid i \in I_{\mathcal{G}} \setminus \{m_1, \ldots, m_p\}\}$ consists of $\kappa + 1 - p = \widehat{m}$ vectors in $\mathcal{B}$. The first $r - p$ vectors are selected from the r vectors in (5a)-(5b), except that one vector has been removed from each group $\{b_{m_k}, \ldots, b_{m_k + r_k - 1}\}$, $k = 1, \ldots, p$. The remaining $r - p$ vectors are linearly inpendendent according to Theorem 7. The last $\widehat{m} - (r - p)$ vectors in the set $\{b_i \mid i \in I_{\mathcal{G}} \setminus \{m_1, \ldots, m_p\}\}$ are selected to fulfill (A4). In sum, $\{b_i \mid i \in I_{\mathcal{G}} \setminus \{m_1, \ldots, m_p\}\}$ are linearly independent. Thus, we reach a contradiction.

Now let $\mathcal{P} := \mathbf{0}$ and $\mathcal{S}' := \text{co}\{v_i \mid i \in \{0, \ldots, n\} \setminus \{m_1, \ldots, m_p\}\}$. Since $y^0(x)$ is affine, $\mathcal{P}' := \{y^0(x) \mid x \in \mathcal{S}'\}$ is compact and convex. By the argument above $\mathcal{P} \cap \mathcal{P}' = \emptyset$, so by Corollary 11.4.2 of [13], there exists $\xi^0 \in \mathbb{R}^n$ such that

$$\xi^0 \cdot (Ax + Bu^0(x) + a) > 0\,, \qquad x \in \mathcal{S}'\,.$$

Let $x \in \mathcal{S} \setminus \mathcal{E}^0$. That is, there exist $\lambda_0, \ldots, \lambda_n \geq 0$ with $\sum \lambda_i = 1$ and not all λ_i, $i \in \{0, \ldots, n\} \setminus \{m_1, \ldots, m_p\}$ equal to zero such that

$$x = \sum_{i=0}^{n} \lambda_i v_i\,.$$

By convexity of $y^0(x)$ we have

$$\xi^0 \cdot y^0(x) = \sum_{i=0}^{n} \lambda_i \xi^0 \cdot y^0(v_i)\,.$$

Since not all λ_i, $i \in \{0, \ldots, n\} \setminus \{m_1, \ldots, m_p\}$ are equal to zero, $\xi^0 \cdot y^0(v_i) > 0$ for $v_i \in \mathcal{S}'$, and $\xi^0 \cdot y^0(v_i) = 0$ for $v_i \in \mathcal{E}^0$, and we get $\xi^0 \cdot y^0(x) > 0$, as desired. ☐

Now we extend the state x by an additional state $\alpha \in \mathbb{R}$ with dynamics

$$\dot{\alpha} = -c\alpha + c\,, \qquad \alpha(0) = 0\,, \tag{30}$$

where $c > 0$ is a to-be-determined constant. Construct the extended state vector $x_e := (x, \alpha)$ and define a multiaffine feedback

$$u(x, \alpha) := (1 - \alpha)u^0(x) + \alpha u^\infty(x)\,. \tag{31}$$

Clearly the role of $u(x, \alpha)$ is to interpolate from $u^0(x)$ to $u^\infty(x)$ as α varies from 0 to 1. Define the closed-loop system

$$y(x, \alpha) := Ax + Bu(x, \alpha) + a\,.$$

The next result guarantees that using $u(x, \alpha)$, closed-loop trajectories cannot exit from restricted facets $\mathcal{F}_1, \ldots, \mathcal{F}_n$.

Lemma 19. *For each $\alpha \in [0, 1]$, $y(x, \alpha)$ satisfies the invariance conditions* (4).

Proof. Let $y^0(x) := Ax + Bu^0(x) + a$ and $y^\infty(x) := Ax + Bu^0(x) + a$. We have that

$$\begin{aligned} y(x, \alpha) &= (1 - \alpha)(Ax + a + Bu^0(x)) + \alpha(Ax + a + Bu^\infty(x)) \\ &= (1 - \alpha)y^0(x) + \alpha y^\infty(x)\,. \end{aligned}$$

By construction, using $u^0(x)$ or $u^\infty(x)$ the invariance conditions (3) are satisfied. That is, $h_j \cdot y^0(v_i) \leq 0$ and $h_j \cdot y^\infty(v_i) \leq 0$ for $i \in \{0, \ldots, n\}$ and $j \in I_i$. Therefore, for each $\alpha \in [0, 1]$,

$$h_j \cdot y(v_i, \alpha) = (1 - \alpha)h_j \cdot y^0(v_i) + \alpha h_j \cdot y^\infty(v_i) \leq 0\,, \qquad i \in \{0, \ldots, n\}\,, \qquad j \in I_i\,.$$

Finally, by the convexity of $y(x, \alpha)$ in x, the invariance conditions (4) hold for all $x \in \mathcal{S}$. □

The following is the main result of the paper.

Theorem 20. *Suppose Assumption 8 holds and suppose the invariance conditions* (3) *for $\mathcal{S}$ are solvable. There exists $c > 0$ sufficiently small such that $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using feedback $u(x, \alpha)$ defined in* (30)-(31).

Proof. To construct $u(x, \alpha)$ in (31), we define $u^0(x)$ and $u^\infty(x)$. For $i \in I_{\mathcal{G}}$, assign $u^0(v_i)$ and $u^\infty(v_i)$ according to (26a)-(26b) and (27a)-(27b), respectively. For $i \in \{0, \kappa+2, \ldots, n\}$, assign $u^0(v_i) = u^\infty(v_i)$ so that $Av_i + Bu^0(v_i) + a \in \mathcal{C}_i$. Let $u^0(x)$ and $u^\infty(x)$ be the associated affine feedbacks. By construction $u^0(x)$ and $u^\infty(x)$ satisfy (21a)-(21b) of Theorem 17. Now consider $\mathcal{H}^*$ given by (23). Define the compact, convex sets $\mathcal{P}^- := \{x \in \mathcal{S} \mid \xi^* \cdot x \leq 1\}$ and $\mathcal{P}^+ := \{x \in \mathcal{S} \mid \xi^* \cdot x \geq 1\}$. From (25), $\mathcal{E}^0 \subset \mathcal{P}^+$, $\mathcal{E}^\infty \subset \mathcal{P}^-$, $\mathcal{P}^- \subset \mathcal{S} \setminus \mathcal{E}^0$, and $\mathcal{P}^+ \subset \mathcal{S} \setminus \mathcal{E}^\infty$.

First we discuss the behavior of trajectories using $u^0(x)$ and $u^\infty(x)$. By Lemma 18, a flow condition holds on $\mathcal{P}^-$ using $u^0(x)$. Since $\mathcal{P}^-$ is compact, by a standard argument all trajectories $\phi^0(t, x_0)$ exit $\mathcal{P}^-$ in finite time. Since the invariance conditions hold using $u^0(x)$, trajectories only exit $\mathcal{P}^-$ via $\mathcal{F}_0$ or $\mathcal{H}^*$. Similarly, by Lemma 18, a flow condition holds on $\mathcal{P}^+$ using $u^\infty(x)$. Since $\mathcal{P}^+$ is compact, all trajectories $\phi^\infty(t, x_0)$ exit $\mathcal{P}^+$ in finite time. Since the invariance conditions hold using $u^\infty(x)$, trajectories only exit $\mathcal{P}^+$ via $\mathcal{F}_0$ or $\mathcal{H}^*$. Because $u^\infty(x)$ satisfies (22), trajectories $\phi^\infty(t, x_0)$ cannot exit through $\mathcal{H}^*$. Hence all trajectories $\phi^\infty(t, x_0)$ starting in $\mathcal{P}^+$ exit $\mathcal{S}$ in finite time.

Now we consider the controller $u(x, \alpha)$ with associated trajectories $\phi(t, x_0)$. Abusing notation, it can be rewritten as a time-varying affine feedback

$$u(x, t) = \mathrm{e}^{-ct}u^0(x) + (1 - \mathrm{e}^{-ct})u^\infty(x)\,.$$

By Lemma 19, $u(x, t)$ satisfies (21a) and by construction it satisfies (21b); therefore (22) holds. For $c > 0$ sufficiently small, $u(x, t)$ is sufficiently close to $u^0(x)$ for a sufficiently long

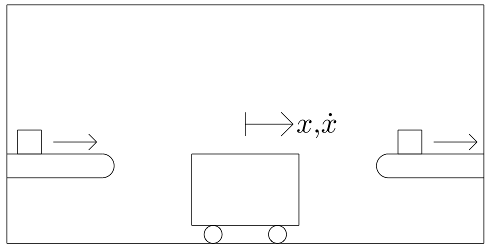


Figure 3. Schematic of material transfer system.

time interval $[0,\tau_1]$ so that (28) holds on $\mathcal{P}^-$ and all trajectories $\phi(t,x_0)$ initialized in $\mathcal{P}^-$ either exit $\mathcal{S}$ or enter $\mathcal{P}^+$ in a finite time $\tau < \tau_1$. Trajectories initialized in $\mathcal{P}^+$ either exit $\mathcal{S}$ or they remain in $\mathcal{P}^+$ (since they cannot cross over to $\mathcal{P}^-$ by (22)).

There exists $\tau_2 > \tau_1$ when all trajectories remaining in $\mathcal{S}$ are in $\mathcal{P}^+$ and $u(x,t)$ is sufficiently close to $u^\infty(x)$ such that the flow condition (29) takes effect on $\mathcal{P}^+$. Thus, all trajectories must exit $\mathcal{P}^+$, and they do so through $\mathcal{F}_0$ and not $\mathcal{H}^*$, again, because of (22). Thus, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using feedback $u(x,t)$ (equivalently $u(x,\alpha)$). □

7. Example

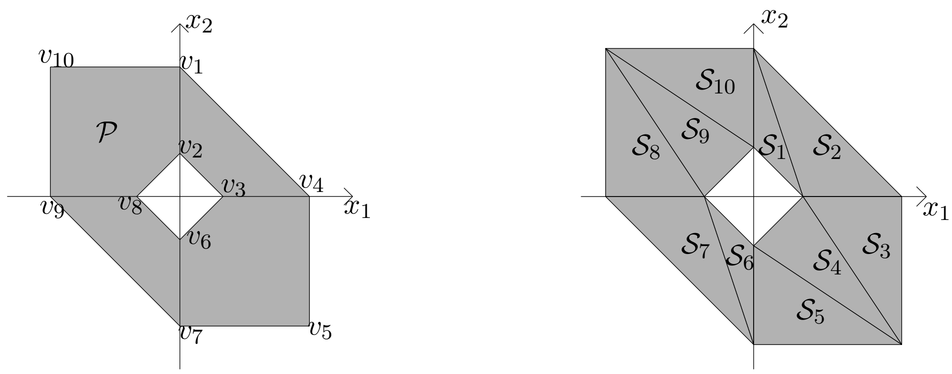
We apply the time-varying compensation technique to a simplified model of a material transfer system. Consider the conveyor system shown in Figure 3. Its objective is to move goods in a production facility between two locations autonomously, in this case two conveyors, and it is constrained to move on a linear track. The dynamics of the cart are given by

$$\dot{x} = Ax + a + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\,. \tag{32}$$

7.1. Specifications.

We impose the following specifications. First, we have a safety specification limiting the speed of the cart: $|x_2| \leq 3$m/s. Second, we want to limit the motion of the cart such that it does not contact the conveyor: $|x_1| \leq 3$m. These specifications constrain the state space to a square region. Also, the cart cannot have a non-zero speed when adjacent to the conveyors, otherwise the cart would cause damage to the system. Thus, we add the specification: $|x_1 + x_2| \leq 3$. The last specification is a liveness requirement that the cart move from the left to right conveyor and back again. In other words, the cart should have non-zero speed around $x = 0$ and therefore we exclude this region with: $\|x\|_1 \geq 1$. We summarize our constraints as follows: (1) Safety: $|x_1| \leq 3$, $|x_2| \leq 3$, $|x_1 + x_2| \leq 3$, and (2) Liveness: $|x_1| + |x_2| \geq 1$. These specifications generate a polytope $\mathcal{P}$ (in the sense of algebraic topology) with the following vertices: $v_1 = (0,3)$, $v_2 = (0,1)$, $v_3 = (1,0)$, $v_4 = (3,0)$, $v_5 = (3,-3)$, $v_6 = (0,-1)$, $v_7 = (0,-3)$, $v_8 = (-1,0)$, $v_9 = (-3,0)$ and $v_{10} = (-3,3)$. $\mathcal{P}$ and its vertices are shown in Figure 7.1.

We triangulate $\mathcal{P}$ using only the vertices of $\mathcal{P}$ to obtain ten simplices $\mathcal{S}_1 = \text{co}\{v_2, v_1, v_3\}$, $\mathcal{S}_2 = \text{co}\{v_1, v_4, v_3\}$, $\mathcal{S}_3 = \text{co}\{v_4, v_5, v_3\}$, $\mathcal{S}_4 = \text{co}\{v_3, v_5, v_6\}$, $\mathcal{S}_5 = \text{co}\{v_5, v_7, v_6\}$, $\mathcal{S}_6 =$



(a) (b)

$\text{co}\{v_6, v_7, v_8\}$, $\mathcal{S}_7 = \text{co}\{v_7, v_9, v_8\}$, $\mathcal{S}_8 = \text{co}\{v_9, v_{10}, v_8\}$, $\mathcal{S}_9 = \text{co}\{v_8, v_{10}, v_2\}$, and $\mathcal{S}_{10} = \text{co}\{v_{10}, v_1, v_2\}$. The triangulation of $\mathcal{P}$ is shown in Figure 7.1.

We design a control law that solves RCP for each simplex; thus, each controller is only valid in a certain portion of the state space. To handle the switching between the controllers, we employ a discrete supervisory controller modeled as a discrete event system [12]. The states of the DES coincide with membership of the continuous time state in a particular simplex $\mathcal{S}_i$, $i = 1, \dots, 10$. The transitions between the states of the DES correspond to *events*, which occur when the continuous state crosses exit facets. The exit facet for each simplex is given by $\mathcal{F}_0^i := \mathcal{S}_i \cap \mathcal{S}_{i+1}$, $i = 1, \dots, 9$, and for simplex $\mathcal{S}_{10}$ the exit facet is $\mathcal{F}_0^{10} := \mathcal{S}_{10} \cap \mathcal{S}_1$.

7.2. **Affine Feedback Design.** By examining (32) we find that $\mathcal{O} = \{x \mid x_2 = 0\}$ and we define $\mathcal{G}^i := \mathcal{S}_i \cap \mathcal{O}$, $i = 1, \dots, 10$. It can be verified that the invariance conditions (3) are solvable for each $\mathcal{S}_i$. By inspection, $\mathcal{G}^5 = \emptyset$ and $\mathcal{G}^{10} = \emptyset$. Therefore, by Theorem 6.1 of [3], $\mathcal{S}^5 \xrightarrow{\mathcal{S}^5} \mathcal{F}_0^5$ $\mathcal{S}^{10} \xrightarrow{\mathcal{S}^{10}} \mathcal{F}_0^{10}$ by affine feedback. Also, $\mathcal{G}^i \neq \emptyset$ and $\mathcal{B} \cap \text{cone}(\mathcal{S}_i) \neq \mathbf{0}$ for $i \in \{1, 3, 4, 6, 8, 9\}$. Therefore, by Theorem 6.2 of [3], $\mathcal{S}^i \xrightarrow{\mathcal{S}^i} \mathcal{F}_0^i$ for $i \in \{1, 3, 4, 6, 8, 9\}$. The affine feedbacks for these simplices (see [8] for a procedure) are as follows:

$$
\begin{aligned}
u_1(x) &= \begin{bmatrix} -\frac{5}{2} & -\frac{5}{2} \end{bmatrix} x + \frac{7}{2} \\
u_3(x) &= \begin{bmatrix} 0 & -\frac{2}{3} \end{bmatrix} x - 1 \\
u_4(x) &= \begin{bmatrix} \frac{2}{5} & -\frac{2}{5} \end{bmatrix} x - \frac{7}{5} \\
u_5(x) &= \begin{bmatrix} -1 & -\frac{5}{2} \end{bmatrix} x - \frac{7}{2} \\
u_6(x) &= \begin{bmatrix} -\frac{5}{2} & -\frac{5}{2} \end{bmatrix} x - \frac{7}{2} \\
u_8(x) &= \begin{bmatrix} 0 & -\frac{2}{3} \end{bmatrix} x + 1 \\
u_9(x) &= \begin{bmatrix} \frac{2}{5} & -\frac{2}{5} \end{bmatrix} x + \frac{7}{5} \\
u_{10}(x) &= \begin{bmatrix} -1 & -\frac{5}{2} \end{bmatrix} x + \frac{7}{2}.
\end{aligned}
$$

7.3. **Time-varying Feedback Design.** It remains to design control laws for $\mathcal{S}_2$ and $\mathcal{S}_7$. For these simplices, $\mathcal{G}^i \neq \emptyset$, $\mathcal{B} \cap \text{cone}(\mathcal{S}_i) = \mathbf{0}$, and one can check that Assumption 8 is satisfied. By the results of [3], RCP is not solvable by continuous state feedback on $\mathcal{S}_2$ and $\mathcal{S}_7$. However, by Theorem 20, it is solvable by the proposed time-varying feedback. From the list of affine control laws above we observe there is a symmetry between the control pairs $(u_1(x), u_6(x))$, $(u_3(x), u_8(x))$, $(u_4(x), u_9(x))$, and $(u_5(x), u_{10}(x))$; namely that $g_i = -g_j$ and $K_i = K_j$ for these pairs. Thus, we only consider the control design for $\mathcal{S}_2$, since that for $\mathcal{S}_7$ will be symmetrical.

We have $\mathcal{S}_2 = \text{co}\{v_0, v_1, v_2\}$ where $v_0 = (0, 3)$, $v_1 = (3, 0)$, and $v_2 = (1, 0)$. The normal vectors for $\mathcal{S}_2$ are $h_0 = (0, -1)$, $h_1 = (-3, -1)$, and $h_2 = (1, 1)$. Since $\mathcal{O} = \{x \mid x_2 = 0\}$, $\mathcal{G}^2 = \text{co}\{v_1, v_2\}$. We observe that $\kappa = 1$, $\widehat{m} = 1$, and $\widehat{\mathcal{B}} = \mathcal{B}$. To design $u(x, \alpha)$ must find $u^0(x)$ and $u^\infty(x)$, and these depend on the flow-like condition provided by Theorem 17. This presents a dilemma that Theorem 17 assumes the apriori existence of a time-varying controller satisfying (21a)-(21b). A convenient workaround is to carry out the design in two steps. First apply a preliminary feedback transformation to (32) so that (21a) is met. In this example, we want the invariance conditions at v_0 to be satisfied. We select

$$u = K_1 x + g_1 + G_1 w = \begin{bmatrix} 0 & -2 \end{bmatrix} x + w,$$

and obtain the transformed system

$$\dot{x} = \tilde{A}x + \tilde{a} + \tilde{B}w = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w.$$

At v_1 we select $w_1 = -1$ which generates $b_1 = (0, -1)$ and at v_2 we select $w_2 = 1$ which generates $b_2 = (0, 1)$. Figure 4 presents $\mathcal{S}_2$ and these geometric properties graphically.

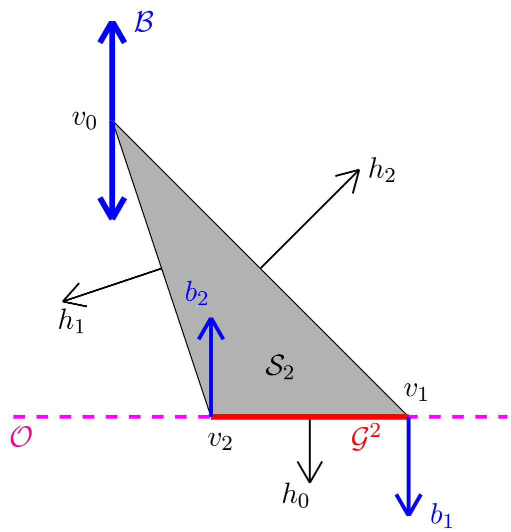


FIGURE 4. Simplex $\mathcal{S}_2$.

Now that all the velocity vectors at vertices are selected, we can find the flow-like condition on $\mathcal{S}_2$. We have that $\text{Ker}\hat{B}^T = \text{sp}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. According to Lemma 14, we seek $\beta_1 \in \text{Ker}\hat{B}^T$

such that $\beta_1 \cdot (v_1 - v_2) \neq 0$. As per Lemma 12, we find $\beta_1 = -\frac{1}{2}h_1 - \frac{1}{2}h_2 = (1, 0)$. We also verify (21a); namely $\beta_1 \cdot (\tilde{A}v_0 + \tilde{a} > 0$, so it is not necessary to find ξ^2 according to Lemma 16. Therefore we can directly choose $\xi^* = \beta_1$. Clearly $\xi^* \cdot (\tilde{A}x + \tilde{a}) \geq 0$ for all $x \in \mathcal{S}$, so ξ^* provides the flow-like condition. We can now order the vertices in $\mathcal{S}_2$. We have $\pi_1 = \xi^* \cdot v_2 = 1$ and $\Pi_1 = \xi^* \cdot v_1 = 3$.

We design $u^0(x)$ such that $\tilde{A}v_1 + Bu_1^0 + \tilde{a} = 0$ and $\tilde{A}v_2 + Bu_2^0 + \tilde{a} = b_2$. This gives

$$u^0(x) = K^0 x + g^0 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} x + \frac{3}{2}. \tag{33}$$

We design $u^\infty(x)$ such that $\tilde{A}v_1 + Bu_1^\infty + \tilde{a} = b_1$ and $\tilde{A}v_2 + Bu_2^\infty + \tilde{a} = 0$. This gives

$$u^\infty(x) = K^\infty x + g^\infty = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{6} \end{bmatrix} x + \frac{1}{2}. \tag{34}$$

We now extend the state with a new state α_2 and we obtain

$$u(x, \alpha_2) = (1 - \alpha_2)u^0(x) + \alpha_2 u^\infty(x)$$

where $\alpha_2(t) = 1 - e^{-ct}$ and $c > 0$ is a parameter to be selected. Since we have used a feedback transformation, we must determine our actual input u to be

$$\begin{aligned} u_2(x, \alpha_2) &= K_1 x + g_1 + G_1 u(x, \alpha_2) \\ &= (1 - \alpha_2)\left(\begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \end{bmatrix} x + \frac{3}{2}\right) + \alpha_2 \left(\begin{bmatrix} -\frac{1}{2} & -\frac{13}{6} \end{bmatrix} x + \frac{1}{2}\right). \end{aligned}$$

Using the symmetry of $u_2(x, \alpha_2)$ and $u_7(x, \alpha_7)$ we find that

$$u_7(x, \alpha_7) = (1 - \alpha_7)\left(\begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \end{bmatrix} x - \frac{3}{2}\right) + \alpha_7 \left(\begin{bmatrix} -\frac{1}{2} & -\frac{13}{6} \end{bmatrix} x - \frac{1}{2}\right),$$

where α_7 is the added state for $\mathcal{S}_7$.

7.4. **Simulation.** The closed loop system was simulated using Matlab. Figures 5-7 present three simulations for $c = 100$, $c = 1$, and $c = 0.01$ using time-varying feedback. The plots contain an overlay of the boundaries of the simplices in black. In blue is a plot of the closed-loop vector field using Matlab's `streamslice`. For $\mathcal{S}_2$ and $\mathcal{S}_7$ where time-varying affine feedback is used, the `streamslice` was generated with $\alpha = 0$. The red and green represent the trajectories from ten initial conditions corresponding to the vertices of $\mathcal{P}$. Each trajectory has a red and green portion where the red portion represents the trajectory traversing the initial fifteen simplices it encounters and the green portion represents the final fifteen simplices. The green portion illustrates the location of the closed-loop limit cycle generated in this affine system using time-varying affine feedbacks. The only noticeable difference between the simulations in Figures 5 and 6 is the slightly larger width of the limit cycle for the case when $c = 1$. However, there is a significant difference in the simulation in Figure 7 where the limit cycle no longer has a pleasing convex shape. It can be seen that the trajectory roughly follows the `streamslice` when $\alpha = 0$. This is due to the relatively slow speed of movement of the (virtual) equilibria in $\mathcal{S}_2$ and $\mathcal{S}_7$. The closed-loop vector field changes very slowly as compared to the system state which causes the system state to approach the slow moving equilibria near $v_{m_i+r_i-1}$ for $i = 1, 2$.

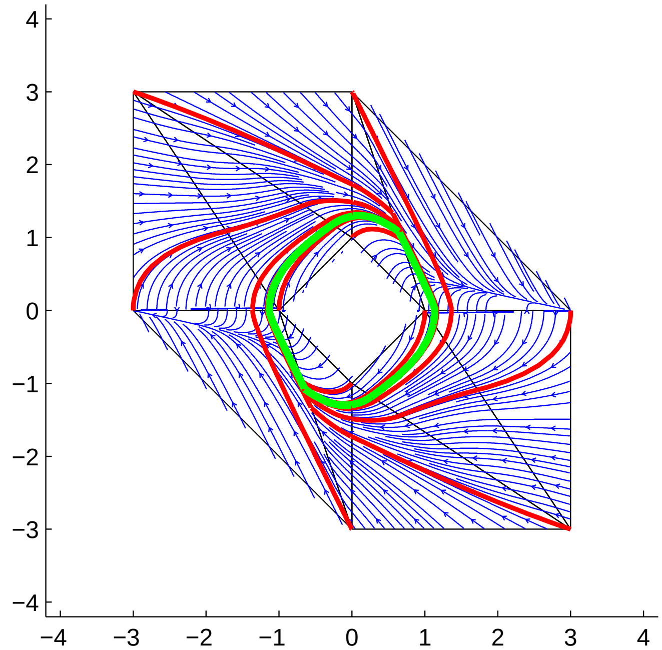


FIGURE 5. Simulation of material transfer system with $c = 100$ for time-varying affine feedbacks.

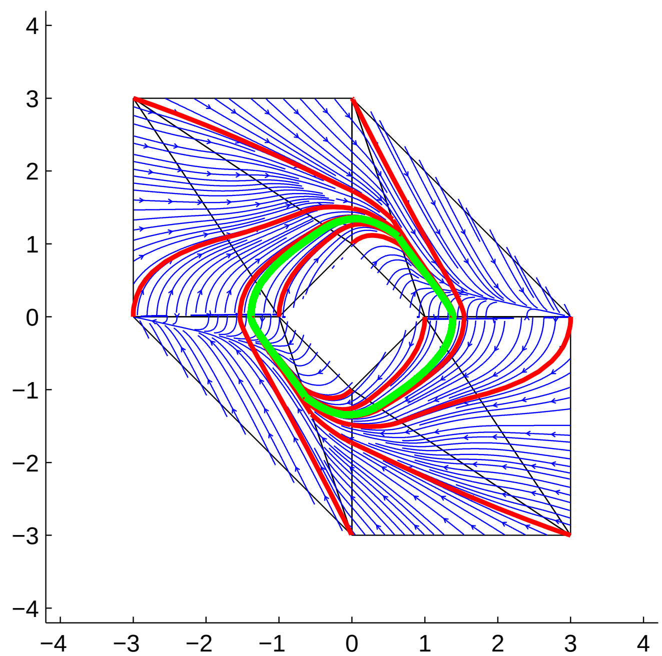


FIGURE 6. Simulation of material transfer system with $c = 1$ for time-varying affine feedbacks.

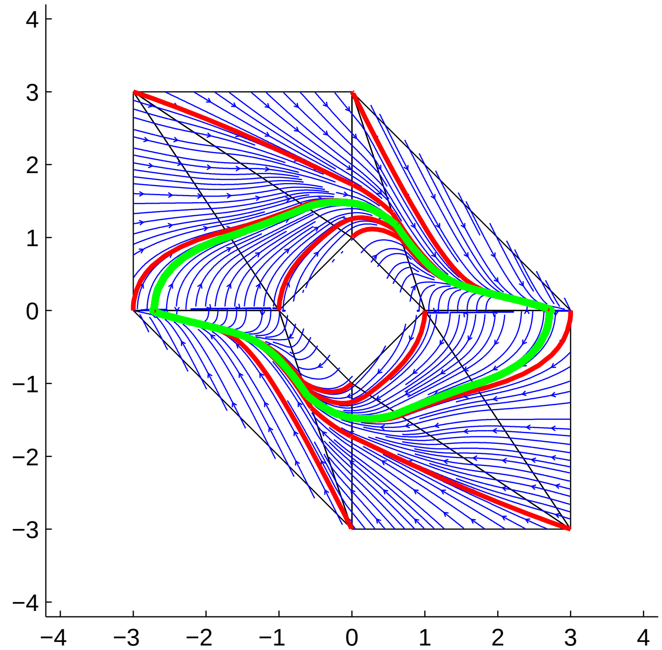


FIGURE 7. Simulation of material transfer system with $c = 0.01$ for time-varying affine feedbacks.

8. CONCLUSION

The paper studies the reach control problem on simplices, and we investigate cases when the problem is not solvable by continuous state feedback. It is shown that the class of time-varying affine feedbacks is sufficient to solve the problem in all cases of interest. As seen from the form of (30), our controller is among the simplest in the class of time-varying feedbacks. An area of future research is to devise more elaborate time-varying affine feedbacks that additionally address speed of response requirements for reach control.

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Appendix A.

Lemma 21. *Let $\mathcal{S}$ be a simplex. Then the following hold:*

(i) *If $x \in \mathrm{co}\{v_1, \dots, v_k\}$, then $x \in \mathcal{F}_j$, for $k+1 \le j \le n$.*
(ii) *$h_j \cdot (v_i - v_0) = 0$ for all $1 \le i, j \le n$ and $j \ne i$.*
(iii) *$h_j \cdot (v_i - v_k) = 0$ for all $0 \le i, k \le n$ and $j \ne i, k$.*
(iv) *$h_i \cdot (v_i - v_0) < 0$, for all $1 \le i \le n$.*
(v) *$h_j \cdot (v_i - x) > 0$ for all $x \in \mathcal{S} \setminus \mathcal{F}_j$ and $1 \le i, j \le n$ and $i \ne j$.*
(vi) *$h_0 \cdot (v_i - v_0) > 0$ for all $1 \le i \le n$.*
(vii) *The vectors $\{v_1 - v_0, \dots, v_n - v_0\}$ are a basis for $\mathbb{R}^n$.*
(viii) *The vectors $\{h_1, \dots, h_n\}$ are a basis for $\mathbb{R}^n$.*
(ix) *There exist $\gamma_1 > 0, \dots, \gamma_n > 0$ such that $h_0 = -\gamma_1 h_1 - \cdots - \gamma_n h_n$.*

Proof of Lemma 9. We prove (10) only for $k = 1$; the other cases follow by reordering indices. Let b_{r_1} be as in (7). Since $b_{r_1} \in \mathcal{B} \cap \mathcal{C}_{r_1}$,

$$h_j \cdot b_{r_1} = h_j \cdot \big(c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}\big) \le 0\,, \qquad j \in I \setminus \{1, \dots, r_1\}\,.$$

Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$ and $c_i < 0$, every term in the sum is non-negative. It follows

$$h_j \cdot b_i = 0\,, \qquad\qquad i = 1, \dots, r_1\,, \qquad j = r_1 + 1, \dots, n\,.$$

The equation following (10) is simply a restatement of (10). Finally, applying Lemma 21(iii), we can decompose the state space as

$$\mathbb{R}^n = \underbrace{\mathrm{sp}\{v_{m_k} - v_0, \dots, v_{m_k + r_k - 1} - v_0)\}}_{\mathbb{R}^{r_k}} \overset{\perp}{\oplus} \underbrace{\mathrm{sp}\{h_{m_1}, \dots, h_{m_{k-1}+r_{k-1}-1}, h_{m_{k+1}}, \dots, h_n\}}_{\mathbb{R}^{n-r_k}}\,.$$

Combined with (10), we obtain (11). □

Proof of Lemma 10. First, we prove (i) only for $k = 1$ and the submatrix M_{1,r_1-1}. The other cases follow by reordering indices. First, we know M_{1,r_1-1} is a $\mathscr{Z}$-matrix because $h_j \cdot b_i \le 0$, $j \ne i$, so the off-diagonal entries are non-positive. Second, we show $M_{1,r_1-1} = H^T_{1,r_1-1} Y_{1,r_1-1}$ is nonsingular. Suppose there exists $c \in \mathbb{R}^{r_1-1}$ such that $H^T_{1,r_1-1} Y_{1,r_1-1} c = 0$. Let $y := Y_{1,r_1-1} c$. Then $h_j \cdot y = 0$, $j = 1, \dots, r_1 - 1$. Also by (10), $h_j \cdot y = 0$, $j = r_1 + 1, \dots, n$. Thus, either $y \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S})$ or $-y \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S})$. By Assumption (D2), $y = 0$. However,

$y = c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}$ and $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent, so $c = 0$. We conclude that M_{1,r_1-1} is nonsingular.

Now we show M_{1,r_1-1} satisfies case (Q_{50}) of Theorem 6.2.3 of [2]. Suppose there exists $c \in \mathbb{R}^{r_1-1}$ with $c \neq 0$ and $c \succeq 0$ such that $M_{1,r_1-1}c \preceq 0$. Define the vector $\bar{y} = Y_{1,r_1-1}c \in \mathcal{B}$. Note that $\bar{y} \neq 0$ because $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent. Then $M_{1,r_1-1}c = H^T_{1,r_1-1}Y_{1,r_1-1}c = H^T_{1,r_1-1}\bar{y} \preceq 0$ implies $h_j \cdot \bar{y} \leq 0$ for $j = 1, \ldots, r_1 - 1$. Also, since $c_i \geq 0$ and $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $h_j \cdot \bar{y} = \sum_{i=1}^{r_1-1} c_i (h_j \cdot b_i) \leq 0$, $j = r_1, \ldots, n$. This implies $0 \neq \bar{y} \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, a contradiction. Therefore, M_{1,r_1-1} has the property that the only solution of the inequalities $c \succeq 0$ and $M_{1,r_1-1}c \preceq 0$ is $c = 0$. In sum, M_{1,r_1-1} is a nonsingular $\mathscr{Z}$-matrix satisfying Theorem 6.2.3, case (Q_{50}) of [2], so M_{1,r_1-1} is a nonsingular $\mathscr{M}$-matrix.

Second, we prove (ii). Suppose not. Then by the definition of reducibility there exists a permutation matrix P such that

$$P M_{m_k, m_k+r_k-1} P^T = \begin{bmatrix} M_1 & 0 \\ \star & M_2 \end{bmatrix}$$

where $M_1 \in \mathbb{R}^{\rho \times \rho}$ and $M_2 \in \mathbb{R}^{(r_k-\rho)\times(r_k-\rho)}$ for some $1 \leq \rho \leq r_k - 1$. Without loss of generality suppose we have reordered the indices $\{m_k, \ldots, m_k + r_k - 1\}$ in accordance with the permutation matrix P. Then

$$M_{m_k, m_k+r_k-1} = \begin{bmatrix} M_{m_k, m_k+\rho-1} & H^T_{m_k, m_k+\rho-1} Y_{m_k+\rho, m_k+r_k-1} \\ \star & M_{m_k+\rho, m_k+r_k-1} \end{bmatrix}$$

and $H^T_{m_k, m_k+\rho-1} Y_{m_k+\rho, m_k+r_k-1} = 0$. The latter gives

$$h_j \cdot b_i = 0\,, \qquad i = m_k + \rho, \ldots, m_k + r_k - 1\,, \quad j = m_k, \ldots, m_k + \rho - 1\,. \tag{35}$$

Combining (35) with (10) we get

$$h_j \cdot b_i = 0\,, \qquad i = m_k + \rho, \ldots, m_k + r_k - 1\,, \quad j \in I \setminus \{m_k + \rho, \ldots, m_k + r_k - 1\}\,. \tag{36}$$

Consider $M_{m_k+\rho, m_k+r_k-1}$. By Lemma 10(i) it is a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3 case (I_{28}) of [2], there exists $c \in \mathbb{R}^{r_k-\rho}$, $c \neq 0$, such that $c \preceq 0$ and $M_{m_k+\rho, m_k+r_k-1}c \prec 0$. Let $y := Y_{m_k+\rho, m_k+r_k-1}c$. Note that $y \neq 0$ since $\{b_{m_k+\rho}, \ldots, b_{m_k+r_k-1}\}$ are linearly independent for any $\rho \geq 1$ by Theorem 7. Then we have $M_{m_k+\rho, m_k+r_k-1}c = H^T_{m_k+\rho, m_k+r_k-1} Y_{m_k+\rho, m_k+r_k-1} c = H^T_{m_k+\rho, m_k+r_k-1} y \prec 0$. That is,

$$h_j \cdot y < 0\,, \qquad j = m_k + \rho, \ldots, m_k + r_k - 1\,.$$

Also from (36)

$$h_j \cdot y = 0\,, \qquad j \in I \setminus \{m_k + \rho, \ldots, m_k + r_k - 1\}\,.$$

We conclude $0 \neq y \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, a contradiction of Assumption 8.

Finally, we prove (iii). By Lemma 21(viii), $\text{rank}(H_{m_k, m_k+r_k-1}) = r_k$. Also we know $\text{rank}(Y_{m_k, m_k+r_k-1}) = r_k - 1$. Therefore, $\text{rank}(M_{m_k, m_k+r_k-1}) \leq r_k - 1$, so it is clearly singular. Now we prove that M_{m_k, m_k+r_k-1} is an $\mathscr{M}$-matrix. By Lemma 10(i), each principal submatrix of M_{m_k, m_k+r_k-1} (formed by removing the ith row and column from M_{m_k, m_k+r_k-1}) is a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3 case (D_{16}) of [2], every real eigenvalue of a nonsingular $\mathscr{M}$-matrix is positive. Therefore, every real eigenvalue of each principal submatrix of M_{m_k, m_k+r_k-1} is positive. By Theorem 6.4.6 case (A_2) of [2], M_{m_k, m_k+r_k-1} is an $\mathscr{M}$-matrix. $\square$

Proof of Lemma 11. We prove the result only for $k = 1$; the other cases follow by reordering indices. By Lemma 9(i), M_{1,r_1-1} is a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3 case (I_{28}) of [2], there exists $c' = (c'_1, \ldots, c'_{r_1-1})$ such that $c' \preceq 0$ and $M_{1,r_1-1}c' \prec 0$. Define $b'_{r_1} := Y_{1,r_1-1}c'$. The vector $H^T_{1,n} b'_{r_1} \in \mathbb{R}^n$ has the sign pattern:

$$(-, \ldots, -, *, 0, \ldots, 0) \tag{37}$$

where the $*$ appears in the (r_1)th component and the zero components are due to (10). In particular $b'_{r_1} \in \mathcal{B} \cap \mathcal{C}_{r_1}$ and the first $r_1 - 1$ invariance conditions are strictly negative. Now suppose we find a non-zero vector $\beta \in \mathcal{B}$ such that (12) holds and $\{b_1, \ldots, b_{r_1-1}, \beta\}$ are linearly independent. Then for $\alpha > 0$ we can form

$$b''_{r_1} := b'_{r_1} + \alpha\beta \,.$$

Using (37) and (12), α can be selected sufficiently small so that $h_j \cdot b''_{r_1} \leq 0$ for all $j = 1, \ldots, r_1 - 1, r_1 + 1, \ldots, n$. That is, $b''_{r_1} \in \mathcal{B} \cap \mathcal{C}_{r_1}$. Moreover, with $\beta \neq 0$,

$$\{b_1, \ldots, b_{r_1-1}, b''_{r_1}\}$$

is a linearly independent set. This contradicts (5a), where b_{r_1} depends only on $\{b_1, \ldots, b_{r_1-1}\}$. □

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