Time-varying Affine Feedback for Reach Control on Simplices

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Abstract

We study the reach control problem for affine systems on simplices, and the focus is on cases when it is known that the problem is not solvable by continuous state feedback. In previous work we used the reach control indices to construct a (discontinuous) piecewise affine control that solves the problem. Here we investigate the extent to which time-varying feedbacks can be used. A simple time-varying affine feedback is proposed and shown to solve the problem.

1 Introduction

This paper studies the *reach control problem* (RCP) on simplices. The problem is to make the closed-loop trajectories of an affine system leave a simplex through a prespecified facet in finite time. The problem was introduced in [7] and was further developed in [8], [9], [11], [3], [4], [5], [6]. See [1] for a complementary approach. Recently it was shown that under a certain choice of triangulation, affine feedback and continuous state feedback are equivalent from the point of view of solvability of RCP on simplices [3]. In [4,6] reach control indices were proposed to expose how affine or continuous state feedbacks may fail - such feedbacks induce closed-loop equilibria in sub-simplices that are inherently starved of sufficient inputs. Fortunately, the reach control indices also give insight on how to overcome the problem of insufficient inputs. In [5,6] we presented a subdivision procedure that triangulates the simplex into sub-simplices with sub-reach control problems that are solvable by affine feedback. The final outcome was that if the reach control problem is solvable by open-loop controls, then it is solvable by (discontinuous) piecewise affine feedback.

The objective of this paper is to explore whether other types of controls can be used to solve the problem, in the case when it is not solvable by continuous state feedback. We are especially interested in controls that are not discontinuous in order to circumvent issues of chattering due to measurement errors, and a natural choice is time-varying feedback. Here we present a method inspired by the information that is provided by the reach control indices. In particular, the indices indicate in which sub-simplices equilibria appear when affine feedback is used. This information is used to construct a compensator that dynamically shifts the set of equilibria so that trajectories can effectively "roll around" equilibria in order to exit the simplex.

Notation. The notation $h \cdot y$ denotes the dot product of two vectors $h, y \in \mathbb{R}^n$. The notation $\mathrm{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation $\mathrm{sp}\{y_1, y_2, \ldots\}$ denotes the span of vectors $y_i \in \mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector.

2 Reach Control Problem

Consider an n-dimensional simplex $\mathcal{S}$ with vertex set $V := \{v_0, v_1, \ldots, v_n\}$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ (the facet is indexed by the vertex it does not contain). Let h_i be the unit normal vector to the facet $\mathcal{F}_i$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet* of $\mathcal{S}$. Define the index set $I := \{1, \ldots, n\}$. Define the closed, convex cones $\mathcal{C}_i := \{\ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in I \setminus \{i\}\ \}$, $i = 0, \ldots, n$. We write $\mathrm{cone}(\mathcal{S}) := \mathcal{C}_0$ since $\mathcal{C}_0$ is the tangent cone to $\mathcal{S}$ at v_0. We consider the affine control system

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{S}, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\mathrm{rank}(B) = m$. Let $\phi_u(t, x_0)$ denote the trajectory of (1) starting at x_0 under some control law u.

Problem 1 (Reach Control Problem (RCP)) *Consider system (1) defined on $\mathcal{S}$. Find a feedback control $u(x)$ such that for every $x_0 \in \mathcal{S}$ there exist $T \geq 0$*

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and $\gamma > 0$ such that: (i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$; (ii) $\phi_u(T, x_0) \in \mathcal{F}_0$; and (iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \gamma)$.

In the sequel we use the shorthand notation $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ to denote that (i)-(iii) of Problem 1 hold under some control law. The following conditions guarantee that closed-loop trajectories cannot leave $\mathcal{S}$ through a non-exit facet [8]. We say the *invariance conditions are solvable* if for each $v_i \in V$ there exists $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a \in \mathcal{C}_i$. Equivalently,

$$h_j \cdot (Av_i + Bu_i + a) \leq 0\,, \;\; i \in \{0, \ldots, n\}, \;\; j \in I \backslash \{i\}\,. \quad (2)$$

Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Define $\mathcal{O} := \{\; x \in \mathbb{R}^n \;:\; Ax + a \in \mathcal{B}\}$ and $\mathcal{G} := \mathcal{S} \cap \mathcal{O}$. Notice that closed-loop equilibria can only appear in $\mathcal{G}$. The primary conclusion of [3] was that if the state space is triangulated so that $\mathcal{G}$ is either the empty set or a face of $\mathcal{S}$, then RCP is solvable by affine feedback if and only if it is solvable by continuous state feedback. The goal of this paper is to solve RCP in cases where continuous state feedback cannot be used.

3 Reach Control Indices

The results of [3] tell us that RCP is not solvable by continuous state feedback under the following assumptions.

Assumption 2 *Simplex $\mathcal{S}$ and system (1) satisfy the following conditions:*

- *(D1)* $\mathcal{G} = \mathrm{co}\{v_1, \ldots, v_{\kappa+1}\}$, $0 \leq \kappa < n$.
- *(D2)* $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$.
- *(D3)* $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}$, $i \in I_{\mathcal{G}}$, *where* $I_{\mathcal{G}} := \{1, \ldots, \kappa + 1\}$.
- *(D4) The maximum number of linearly independent vectors in any set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ (with only one vector for each $\mathcal{B} \cap \mathcal{C}_i, i \in I_{\mathcal{G}}$) is $\widehat{m}$ with $1 \leq \widehat{m} \leq \kappa$.*

Since $\widehat{m} \leq \kappa$, there exists $p \geq 1$ such that $\widehat{m} + p = \kappa + 1$. Evidently there are not enough independent vectors in $\mathcal{B}$ to resolve all invariance conditions for vertices in $\mathcal{G}$. The reach control indices provide a means to bookkeep those vertices $v_i \in \mathcal{G}$ that share degrees of freedom in $\mathcal{B}$, and they capture the strong restrictions placed on $\mathcal{B}$ imposed by (2).

Theorem 3 ([4,6]) *Suppose Assumption 2 holds. Then there exist integers $r_1, \ldots, r_p \geq 2$ and decomposition of $\mathcal{B}$ such that w.l.o.g. (by reordering indices) for all $k = 1, \ldots, p$ and $i = m_k, \ldots, m_k + r_k - 1$,*

$$\mathcal{B} \cap \mathcal{C}_i \subset \mathcal{B}_k := \mathrm{sp}\{b_{m_k}, \ldots, b_{m_k + r_k - 1}\}\,, \quad (3)$$

where $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $m_1 := 1$, and $m_k := r_1 + \cdots + r_{k-1} + 1$, $k = 2, \ldots, p$. Moreover, for each $k = 1, \ldots, p$, $\{b_{m_k}, \ldots, b_{m_k + r_k - 2}\}$ are linearly independent and

$$b_{m_k + r_k - 1} = c_{m_k} b_{m_k} + \cdots + c_{m_k + r_k - 2} b_{m_k + r_k - 2}, \quad (4)$$

with $c_i < 0$, $i = m_k, \ldots, m_k + r_k - 2$.

We observe that due to (4) the lists (3) have the property that any vector in a list on the right is dependent on all the other vectors in its list. Also, if any vector is removed from a list, the remaining vectors are linearly independent. In particular, the kth list contains $r_k - 1$ linearly independent vectors in $\mathcal{B}$, so $\dim(\mathcal{B}_k) = r_k - 1$. The integers $\{r_1, \ldots, r_p\}$ are called the *reach control indices* of system (1) with respect to simplex $\mathcal{S}$. Based on Theorem 3, we require an additional assumption that was proved to be necessary for solvability of a slightly stronger version of RCP [6].

Assumption 4 *Simplex $\mathcal{S}$ and system (1) satisfy Assumption 2 and the following condition.*

(D5) $\mathcal{B}_k \not\subset \mathcal{H}_0 := \{y \in \mathbb{R}^n \mid h_0 \cdot y = 0\}\,, \quad k = 1, \ldots, p.$

Finally, we summarize the algebraic consequences of the reach control indices. Proofs are provided in the Appendix. Let $H_{p,q} := [h_p \cdots h_q]$ and $Y_{p,q} := [b_p \cdots b_q]$, where the b_i's come from (3). Let $M_{p,q} := H_{p,q}^T Y_{p,q}$. We say a matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \leq 0$ for all $i \neq j$. A $\mathscr{Z}$-matrix M is a nonsingular $\mathscr{M}$-matrix if every real eigenvalue of M is positive [2]. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in I_{\mathcal{G}}$, each $M_{p,q}$ is a $\mathscr{Z}$-matrix. Define $r := r_1 + \cdots + r_p$ and for $k = 1, \ldots, p$ define $I_{\mathcal{G}_k} := \{m_k, \ldots, m_k + r_k - 1\}$ and $\mathcal{G}_k := \mathrm{co}\{v_{m_k}, \ldots, v_{m_k + r_k - 1}\}$.

Lemma 5 ([4,6]) *Suppose Assumption 4 holds. Then for each $k = 1, \ldots, p$, $h_j \cdot b_i = 0$ for $i \in I_{\mathcal{G}_k}$, $j \in I \setminus I_{\mathcal{G}_k}$.*

Lemma 6 *Suppose Assumption 4 holds. Then for each $k = 1, \ldots, p$,*

- *(i) Each principal submatrix of $M_{m_k, m_k + r_k - 1}$ is a nonsingular $\mathscr{M}$-matrix.*
- *(ii) Matrix $M_{m_k, m_k + r_k - 1} \in \mathbb{R}^{r_k \times r_k}$ is irreducible.*
- *(iii) Matrix $M_{m_k, m_k + r_k - 1}$ is a singular $\mathscr{M}$-matrix.*

Lemma 7 ([4,6]) *Suppose Assumption 4 holds. For each $k = 1, \ldots, p$, there does not exist $\beta \in \mathcal{B}$ such that $\{b_{m_k}, \ldots, b_{m_k + r_k - 2}, \beta\}$ are linearly independent and*

$$h_j \cdot \beta \leq 0\,, \qquad j \in I \setminus I_{\mathcal{G}_k}\,. \quad (5)$$

4 A Flow-like Condition

We begin our exploration of time-varying feedback to solve RCP in cases when it is not solvable by continuous state feedback. The development is divided into two parts. First we establish that a flow-like condition holds on $\mathcal{S}$ which has desirable properties relative to the subsimplices $\mathcal{G}_k$, $k = 1, \ldots, p$. By a *flow-like condition* we mean a condition of the form: $\xi^* \cdot (Ax + Bu(x) + a) \geq 0$, $x \in \mathcal{S}$, where $0 \neq \xi^* \in \mathbb{R}^n$. Second, we propose a time-varying compensator whose role in essence is to dynamically shift the set of equilibria generated by affine feedback in a direction opposite to the direction indicated by the flow-like condition. In this section the flow-like condition is developed.

Define $\widehat{B} = \left[\, b_1 \cdots \ b_{m_1+r_1-2} \cdots \ b_{m_p} \cdots \ b_{m_p+r_p-2}\right] \in \mathbb{R}^{n\times(r-p)}$ and let $\widehat{\mathcal{B}} = \mathrm{Im}(\widehat{B})$. Note that the columns of $\widehat{B}$ are ordered according to Theorem 3 and that vectors $b_{m_k+r_k-1}$, $k = 1, \ldots, p$, do not appear in the columns of $\widehat{B}$. Also, velocity vectors associated with v_i, $i = r+1, \ldots, \kappa+1$ are not yet defined, so they do not appear. By Theorem 3, $\dim(\mathcal{B}_k) = r_k - 1$ and $\mathcal{B}_1, \ldots, \mathcal{B}_p$ form a family of independent subspaces, so $\mathrm{rank}(\widehat{B}) = r - p$. The next result gives a specially selected vector used to strongly separate at least two vertices in each $\mathcal{G}_k$, $k = 1, \ldots, p$.

Lemma 8 *Suppose Assumption 4 holds. For each* $k \in \{1, \ldots, p\}$ *there exists* $\beta_k \in \mathrm{Ker}(\widehat{B}^T)$ *such that*

(i) $\beta_k = d_{m_k} h_{m_k} + \cdots + d_{m_k+r_k-1} h_{m_k+r_k-1}$, $d_i < 0$.
(ii) $\beta_k \cdot (v_i - v_0) = 0, \quad i \in I \setminus I_{\mathcal{G}_k}$.
(iii) $\beta_k \cdot (v_i - v_j) = 0, \quad i, j \in I_{\mathcal{G}} \setminus I_{\mathcal{G}_k}$.

PROOF. Let $k \in \{1, \ldots, p\}$. By Lemma 6(ii) and (iii), M_{m_k, m_k+r_k-1} is a singular, irreducible $\mathscr{M}$-matrix; therefore, so is $M^T_{m_k, m_k+r_k-1}$. By Theorem 6.4.16(2) of [2], there exists $d \prec 0$ such that $M^T_{m_k, m_k+r_k-1} d = 0$. Define $\beta_k := H_{m_k, m_k+r_k-1} d$. This gives the form (i). Next we show $\beta_k \in \mathrm{Ker}(\widehat{B}^T)$. First, we have $M^T_{m_k, m_k+r_k-1} d = Y^T_{m_k, m_k+r_k-1} H_{m_k, m_k+r_k-1} d = Y^T_{m_k, m_k+r_k-1} \beta_k = 0$. That is, $\beta_k \cdot b_i = 0$, $i = m_k, \ldots, m_k + r_k - 1$. Also from Lemma 5, $\beta_k \cdot b_i = 0$, $i = 1, \ldots, m_{k-1} + r_{k-1} - 1, m_{k+1}, \ldots, r$. We conclude $\beta \in \mathrm{Ker}(\widehat{B}^T)$. The statements (ii) and (iii) follow from Lemma 4.4 of [3].

Lemma 9 *Suppose Assumption 4 holds. Then* $\mathrm{Ker}(\widehat{B}^T) = \mathrm{sp}\{\beta_1, \ldots, \beta_p, h_{r+1}, \ldots, h_n\}$.

PROOF. By construction $\beta_k \in \mathrm{sp}\{h_{m_k}, \ldots, h_{m_k+r_k-1}\}$, so by Lemma 4.4 of [3], $\{\beta_1, \ldots, \beta_p, h_{r+1}, \ldots, h_n\}$ are linearly independent. From Lemma 5, $\widehat{\mathcal{B}} \perp \mathrm{sp}\{h_{r+1}, \ldots, h_n\}$. Thus, $h_{r+1}, \ldots, h_n \in \mathrm{Ker}(\widehat{B}^T)$. Also from Lemma 8, $\beta_1, \ldots, \beta_p \in \mathrm{Ker}(\widehat{B}^T)$. Now $\mathrm{rank}(\widehat{B}^T) = r - p$, so $\dim(\mathrm{Ker}(\widehat{B}^T)) = n - r + p$. Thus, $\{\beta_1, \ldots, \beta_p, h_{r+1}, \ldots, h_n\}$ is a basis of $\mathrm{Ker}(\widehat{B}^T)$.

The next result shows that for each $k = 1, \ldots, p$, vector β_k can be used to strongly separate at least two vertices in $\mathcal{G}_k$.

Lemma 10 *Suppose Assumption 4 holds. Consider* $\beta_1, \ldots, \beta_p$ *from Lemma 8. For each* $k \in \{1, \ldots, p\}$*, there exist* $i_k, j_k \in I_{\mathcal{G}_k}$ *such that* $\beta_k \cdot (v_{i_k} - v_{j_k}) \neq 0$.

PROOF. Let $k \in \{1, \ldots, p\}$ and suppose by way of contradiction that for every $\beta \in \mathrm{Ker}(\widehat{B}^T)$ and $i, j \in I_{\mathcal{G}_k}$, $\beta \cdot (v_i - v_j) = 0$. This implies $(v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k}) \in \widehat{\mathcal{B}}$. Suppose w.l.o.g. that $v_{m_k+1} - v_{m_k} = b' + b''$, where $b' \in \mathcal{B}_k$ and $0 \neq b'' \in \mathrm{sp}\{b_{m_1}, \ldots, b_{m_{k-1}+r_{k-1}-1}, b_{m_{k+1}}, \ldots, b_r\}$. Then by Lemma 4.4 of [3] and by Lemma 5, $0 = h_j \cdot (v_{m_k+1} - v_{m_k}) = h_j \cdot b' + h_j \cdot b'' = h_j \cdot b''$, $j \in I \setminus I_{\mathcal{G}_k}$. By construction $\{b_{m_k}, \ldots, b_{m_k+r_k-2}, b''\}$ are linearly independent. This contradicts Lemma 7. Thus, $b'' = 0$. This argument can be repeated for each $v_{m_k+i} - v_{m_k}$, $i = 1, \ldots, r_k - 1$ to get $(v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k}) \in \mathcal{B}_k$. By Lemma 4.4 of [3], $\{(v_{m_k+1} - v_{m_k}), \ldots, (v_{m_k+r_k-1} - v_{m_k})\}$ is a basis for $\mathcal{B}_k$, so $\mathcal{B}_k \subset \mathcal{H}_0$. This contradicts Assumption (D5). We deduce that there exist $i_k, j_k \in I_{\mathcal{G}_k}$ and $\beta \in \mathrm{Ker}(\widehat{B}^T)$ such that $\beta \cdot (v_{i_k} - v_{j_k}) \neq 0$. By Lemma 9, $\beta \in \mathrm{Ker}(\widehat{B}^T)$ can be expressed as $\beta = \alpha_1 \beta_1 + \cdots + \alpha_p \beta_p + \alpha_{r+1} h_{r+1} + \cdots + \alpha_n h_n$, $\alpha_i \in \mathbb{R}$. By Lemma 8(iii) and Lemma 4.4 of [3] $0 \neq \beta \cdot (v_{i_k} - v_{j_k}) = \alpha_k \beta_k \cdot (v_{i_k} - v_{j_k})$ implying that $\beta_k \cdot (v_{i_k} - v_{j_k}) \neq 0$.

In light of Lemma 10, we assume without loss of generality (by reordering the indices within each group $I_{\mathcal{G}_k}$) that for $k = 1, \ldots, p$, $v_{m_k} \in \arg\max_{i \in I_{\mathcal{G}_k}} \beta_k \cdot v_i$ and $v_{m_k+r_k-1} \in \arg\min_{i \in I_{\mathcal{G}_k}} \beta_k \cdot v_i$. We also define the sets $\mathcal{E}^0 := \mathrm{co}\{v_{m_1}, v_{m_2}, \ldots, v_{m_p}\}$ and $\mathcal{E}^\infty := \mathrm{co}\{v_{m_1+r_1-1}, v_{m_2+r_2-1}, \ldots, v_{m_p+r_p-1}\}$. In the next result we pick a single hyperplane that strongly separates the two sets $\mathcal{E}^0$ and $\mathcal{E}^\infty$.

Lemma 11 *Suppose Assumption 4 holds. Consider* $\beta_1, \ldots, \beta_p \in \mathrm{Ker}(\widehat{B}^T)$ *from Lemma 8. Define*

$$\xi^1 := \xi^1_1 \beta_1 + \cdots + \xi^1_p \beta_p \,, \quad \xi^1_i \in \mathbb{R} \,, \tag{6}$$

and $\mathcal{H} := \{x \in \mathbb{R}^n \mid \xi^1 \cdot (x - v_0) = 1\}$. *There exist* $\xi^1_1, \ldots, \xi^1_p > 0$ *such that* $\mathcal{H}$ *strongly separates* $\mathcal{E}^0$ *and* $\mathcal{E}^\infty$.

PROOF. Let $x \in \mathcal{E}^0$ and $y \in \mathcal{E}^\infty$. Then there exist $\alpha_i \geq 0$, $\sum_{i=1}^p \alpha_i = 1$, and $\gamma_i \geq 0$, $\sum_{i=1}^p \gamma_i = 1$ such that $x = \gamma_1 v_{m_1} + \cdots + \gamma_p v_{m_p}$ and $y = \alpha_1 v_{m_1+r_1-1} + \cdots + \alpha_p v_{m_p+r_p-1}$. For $k = 1, \ldots, p$ define $\Pi_k := \beta_k \cdot (v_{m_k} - v_0)$ and $\pi_k := \beta_k \cdot (v_{m_k+r_k-1} - v_0)$. By Lemma 8(i) we may write $\beta_k = d_{m_k} h_{m_k} + \cdots + d_{m_k+r_k-1} h_{m_k+r_k-1}$ with $d_i < 0$. By Lemma 4.4 in [3] we have that $\beta_k \cdot (v_i - v_0) = d_i h_i \cdot (v_i - v_0) > 0$, $i \in I_{\mathcal{G}_k}$. In particular, $\pi_k > 0$. By Lemma 10 we know that $\Pi_k \neq \pi_k$; thus, $0 < \pi_k < \Pi_k$, $k = 1, \ldots, p$. Select $\xi_k^1 \in (\frac{1}{\Pi_k}, \frac{1}{\pi_k}) \neq \emptyset$ for $k = 1, \ldots, p$. Then using Lemma 8(ii) $\xi^1 \cdot (x - v_0) = \gamma_1 \xi_1^1 \beta_1 \cdot (v_{m_1} - v_0) + \cdots + \gamma_p \xi_p^1 \beta_p \cdot (v_{m_p} - v_0) = \gamma_1 \xi_1^1 \Pi_1 + \cdots + \gamma_p \xi_p^1 \Pi_p \geq \min_k \{\xi_k^1 \Pi_k\} > 1$. Similarly, $\xi^1 \cdot (y - v_0) = \alpha_1 \xi_1^1 \pi_1 + \cdots + \alpha_p \xi_p^1 \pi_p \leq \max_k \{\xi_k^1 \pi_k\} < 1$. Thus, $\mathcal{H}$ strongly separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$.

The next result, based on a standard argument of convex analysis, shows that the open-loop system naturally drifts in some fixed direction for points away from $\mathcal{G}$.

Lemma 12 *Suppose Assumption 4 holds. Let* $\mathcal{P} := \mathrm{co}\{v_0, v_{\kappa+2}, \ldots, v_n\}$. *There exists* $\xi^2 \in \mathrm{Ker}(B^T)$ *such that*

$$\xi^2 \cdot (Ax + a) > 0 \,, \qquad x \in \mathcal{P} \,. \tag{7}$$

The following is the main result on a flow-like condition on $\mathcal{S}$.

Theorem 13 *Suppose Assumption 4 holds. Let* $u(x,t)$ *be a time-varying affine feedback such that for all* $t \geq 0$

$$Av_i + Bu(v_i,t) + a \in \mathcal{C}_i \,, \quad i = 0, r+1, \ldots, n \tag{8a}$$

$$Av_i + Bu(v_i,t) + a \in \widehat{\mathcal{B}} \,, \quad i = 1, \ldots, r \,. \tag{8b}$$

Then there exists $0 \neq \xi^* \in \mathrm{Ker}(\widehat{B}^T)$ *such that for* $t \geq 0$

$$\xi^* \cdot (Ax + Bu(x,t) + a) \geq 0 \,, \qquad x \in \mathcal{S} \,, \tag{9}$$

and such that $\mathcal{H}^*$ *strongly separates* $\mathcal{E}^0$ *and* $\mathcal{E}^\infty$ *where*

$$\mathcal{H}^* := \{x \in \mathbb{R}^n \mid \xi^* \cdot (x - v_0) = 1\} \,. \tag{10}$$

PROOF. Let $t \geq 0$, and consider ξ^1 from by Lemma 11 and ξ^2 from Lemma 12. Define

$$\xi^* := (1-\lambda)\xi^1 + \lambda \xi^2 \,, \qquad \lambda \in (0,1). \tag{11}$$

Since $\beta_k = H_{m_k, m_k+r_k-1} d$ with $d \prec 0$ and by (8a) we have for each $k = 1, \ldots, p$, $\beta_k \cdot (Av_i + Bu(v_i,t) + a) \geq 0$, $i = 0, r+1, \ldots, n$. Therefore from (6)

$$\xi^1 \cdot (Av_i + Bu(v_i,t) + a) \geq 0 \,, \quad i = 0, r+1, \ldots, n \,. \tag{12}$$

Using (7), that $\xi^2 \in \mathrm{Ker}(B^T)$, and that $r \leq \kappa + 1$, for $i = 0, \kappa+2, \ldots, n$, $\xi^* \cdot (Av_i + Bu(v_i,t) + a) = (1-\lambda)\xi^1 \cdot (Av_i + Bu(v_i,t) + a) + \lambda \xi^2 \cdot (Av_i + Bu(v_i,t) + a) > 0$. Similarly, for $i = r+1, \ldots, \kappa+1$, $\xi^* \cdot (Av_i + Bu(v_i,t) + a) = (1-\lambda)\xi^1 \cdot (Av_i + Bu(v_i,t) + a) \geq 0$, where we use (12), $\xi^2 \in \mathrm{Ker}(B^T)$, and $Av_i + a \in \mathcal{B}$, $i = r + 1, \ldots, \kappa+1$. Finally, since $\xi^1 \in \mathrm{Ker}(\widehat{B}^T)$, $\xi^2 \in \mathrm{Ker}(B^T)$, and $\mathrm{Ker}(B^T) \subset \mathrm{Ker}(\widehat{B}^T)$, then $\xi^* \in \mathrm{Ker}(\widehat{B}^T)$. Thus, by (8b), $\xi^* \cdot (Av_i + Bu(v_i,t) + a) = 0$, $i = 1, \ldots, r$. By convexity of $Ax + Bu(x,t) + a$ in x, $\xi^* \cdot (Av_i + Bu(x,t) + a) \geq 0$ for all $t \geq 0$, $x \in \mathcal{S}$, and $\lambda \in (0,1)$, which proves (9).

Let $x \in \mathcal{E}^0$ and $y \in \mathcal{E}^\infty$, and let γ_i and α_i be as in the proof of Lemma 11. Then $\xi^* \cdot (x - v_0) = (1-\lambda)\xi^1 \cdot (x - v_0) + \lambda \xi^2 \cdot (x - v_0)$, $\xi^* \cdot (y - v_0) = (1-\lambda)\xi^1 \cdot (y - v_0) + \lambda \xi^2 \cdot (y - v_0)$, and from the proof of Lemma 11, $\xi^1 \cdot (x - v_0) > 1$ and $\xi^1 \cdot (y - v_0) < 1$. Since functions $\xi^1 \cdot (x - v_0)$ and $\xi^2 \cdot (x - v_0)$ are continuous, they achieve a minimum and maximum on each compact set $\mathcal{E}^0$ and $\mathcal{E}^\infty$. This means we can select $\lambda \in (0,1)$ close enough to 0 such that

$$\xi^* \cdot (y - v_0) < 1 < \xi^* \cdot (x - v_0) \,, \quad x \in \mathcal{E}^0, y \in \mathcal{E}^\infty \,. \tag{13}$$

With this choice of λ, $\mathcal{H}^*$ strongly separates $\mathcal{E}^0$ and $\mathcal{E}^\infty$.

5 Time-varying Compensator

The time-varying compensator will be constructed so as to exploit the flow-like condition (9) and the separation property of $\mathcal{H}^*$ in (10). First, we define two affine feedbacks $u^0(x)$ and $u^\infty(x)$ that place equilibria at $\mathcal{E}^0$ and $\mathcal{E}^\infty$, respectively. Then we define a compensator $u(x,\alpha)$ with additional state $\alpha \in \mathbb{R}$. This compensator simply interpolates between $u^0(x)$ and $u^\infty(x)$ as α varies from 0 to 1. By construction when $\alpha = 0$, all closed-loop equilibria are in $\mathcal{E}^0$. When $\alpha = 1$, they are in $\mathcal{E}^\infty$. Thus, as α varies from 0 to 1, the set of closed-loop equilibria crosses from one side of $\mathcal{H}^*$ to the other in a direction with decreasing ξ^* component. Informally, we can say that trajectories flow downstream according to (9) while equilibria flow upstream, so that no trajectory can be "stuck" at an equilibrium. Ultimately, this enables all trajectories to exit $\mathcal{S}$, as shown in Theorem 16.

Suppose the invariance conditions for $\mathcal{S}$ are solvable; thus, there exist inputs $u_0^0, \ldots, u_n^0 \in \mathbb{R}^m$ such that (2) hold. Let $y_i^0 := Av_i + Bu_i^0 + a$, for $i = 0, \ldots, n$. We choose $u_1^0, \ldots, u_{\kappa+1}^0 \in \mathbb{R}^m$ such that

$$y_i^0 = 0 \,, \qquad i \in \{m_1, m_2, \ldots, m_p\} \tag{14a}$$

$$y_i^0 = b_i \,, \qquad i \in I_{\mathcal{G}} \setminus \{m_1, m_2, \ldots, m_p\} \,, \tag{14b}$$

where $b_i \in \widehat{\mathcal{B}} \cap \mathcal{C}_i$, $i = 1, \ldots, r$, are provided by Theorem 3; and $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i = r+1, \ldots, \kappa+1$, are selected so that $\widehat{m}$ independent directions in $\mathcal{B}$ are associated with $\mathcal{G}$, as per (D4). Finally, construct the associated affine feedback $u^0(x) = K^0 x + g^0$, and let $\phi^0(t, x_0)$ denote trajectories of the closed-loop system. Note that the closed-loop system has equilibria at $v_{m_1}, \ldots, v_{m_p}$.

Next we define a symmetrical controller $u^\infty(x)$ which is identical to $u^0(x)$ except that it places equilibria at $v_{m_k+r_k-1}$, $k=1,\ldots,p$. Let $y_i^\infty := Av_i + Bu_i^\infty + a$, for $i=0,\ldots,n$. First set $u_i^\infty = u_i^0$, $i=0,r+1,\ldots,n$. Then we choose $u_1^\infty,\ldots,u_r^\infty \in \mathbb{R}^m$ such that

$$y_i^\infty = 0\,, \qquad i \in \{m_1+r_1-1, m_2+r_2-1, \ldots, m_p+r_p-1\} \tag{15a}$$

$$y_i^\infty = b_i\,, i \in \{1,\ldots,r\} \setminus \{m_1+r_1-1, m_2+r_2-1, \ldots, m_p+r_p-1\}\,, \tag{15b}$$

where again $b_i \in \widehat{\mathcal{B}} \cap \mathcal{C}_i$, $i=1,\ldots,r$, are provided by Theorem 3. Finally, construct the associated affine feedback $u^\infty(x) = K^\infty x + g^\infty$, and let $\phi^\infty(t,x_0)$ denote trajectories of the closed-loop system. Note that this closed-loop system has equilibria at $v_{m_k+r_k-1}$, $k=1,\ldots,p$. The next result argues that, away from equilibria, trajectories do exit $\mathcal{S}$.

Lemma 14 *There exist* $\xi^0, \xi^\infty \in \mathbb{R}^n$ *such that*

$$\xi^0 \cdot (Ax + Bu^0(x) + a) > 0\,, \quad x \in \mathcal{S} \setminus \mathcal{E}^0\,, \tag{16}$$

$$\xi^\infty \cdot (Ax + Bu^\infty(x) + a) > 0\,, \quad x \in \mathcal{S} \setminus \mathcal{E}^\infty\,. \tag{17}$$

PROOF. We consider only (16), since the proof for (17) is analogous. First, we claim $Ax + Bu^0(x) + a \neq 0$ for all $x \in \mathcal{S} \setminus \mathcal{E}^0$. Suppose not. Then there exists $\overline{x} \in \mathcal{S} \setminus \mathcal{E}^0$ such that $A\overline{x} + Bu^0(\overline{x}) + a = 0$. Since necessarily $\overline{x} \in \mathcal{G}$, there exist $\lambda_1,\ldots,\lambda_{\kappa+1} \geq 0$ with $\sum \lambda_i = 1$ and not all λ_i, $i \in I_{\mathcal{G}} \setminus \{m_1,\ldots,m_p\}$ equal to zero such that $\overline{x} = \sum_{i=1}^{\kappa+1} \lambda_i v_i$. By convexity of $y^0(x) := Ax + Bu^0(x) + a$ and the fact that $y^0(v_i) = 0$ for $i = m_1,\ldots,m_p$, we have $0 = y^0(\overline{x}) = \sum_{i=1}^{\kappa+1} \lambda_i y^0(v_i) = \sum_{i \in I_{\mathcal{G}} \setminus \{m_1,\ldots,m_p\}} \lambda_i b_i$, with not all λ_i's equal to zero. The set $\{b_i \mid i \in I_{\mathcal{G}} \setminus \{m_1,\ldots,m_p\}\}$ consists of $\kappa+1-p = \widehat{m}$ vectors in $\mathcal{B}$. The first $r-p$ vectors are selected from the r vectors in (3), except that one vector has been removed from each group $\{b_{m_k},\ldots,b_{m_k+r_k-1}\}$, $k=1,\ldots,p$. The remaining $r-p$ vectors are linearly independent according to Theorem 3. The last $\widehat{m}-(r-p)$ vectors in the set $\{b_i \mid i \in I_{\mathcal{G}} \setminus \{m_1,\ldots,m_p\}\}$ are selected to fulfill (D4). In sum, $\{b_i \mid i \in I_{\mathcal{G}} \setminus \{m_1,\ldots,m_p\}\}$ are linearly independent. Thus, we reach a contradiction. Now let $\mathcal{P} := \mathbf{0}$ and $\mathcal{S}' := \mathrm{co}\{v_i \mid i \in \{0,\ldots,n\} \setminus \{m_1,\ldots,m_p\}\}$. Since $y^0(x)$ is affine, $\mathcal{P}' := \{y^0(x) \mid x \in \mathcal{S}'\}$ is compact and convex. By the argument above $\mathcal{P} \cap \mathcal{P}' = \emptyset$, so by Corollary 11.4.2 of [10], there exists $\xi^0 \in \mathbb{R}^n$ such that $\xi^0 \cdot (Ax + Bu^0(x) + a) > 0$, $x \in \mathcal{S}'$. Let $x \in \mathcal{S} \setminus \mathcal{E}^0$. That is, there exist $\lambda_0,\ldots,\lambda_n \geq 0$ with $\sum \lambda_i = 1$ and not all λ_i, $i \in \{0,\ldots,n\} \setminus \{m_1,\ldots,m_p\}$ equal to zero such that $x = \sum_{i=0}^n \lambda_i v_i$. By convexity of $y^0(x)$ we have $\xi^0 \cdot y^0(x) = \sum_{i=0}^n \lambda_i \xi^0 \cdot y^0(v_i)$. Since not all λ_i, $i \in \{0,\ldots,n\} \setminus \{m_1,\ldots,m_p\}$ are equal to zero, $\xi^0 \cdot y^0(v_i) > 0$ for $v_i \in \mathcal{S}'$, and $\xi^0 \cdot y^0(v_i) = 0$ for $v_i \in \mathcal{E}^0$, we get $\xi^0 \cdot y^0(x) > 0$, as desired.

Now we extend the state x by an additional state $\alpha \in \mathbb{R}$ with dynamics $\dot{\alpha} = -c\alpha + c$ with $\alpha(0) = 0$ and $c > 0$ a to-be-determined constant. Construct the extended state vector $x_e := (x, \alpha)$ and define a multiaffine feedback

$$u(x,\alpha) := (1-\alpha)u^0(x) + \alpha u^\infty(x)\,. \tag{18}$$

Clearly the role of $u(x,\alpha)$ is to interpolate from $u^0(x)$ to $u^\infty(x)$ as α varies from 0 to 1. Define the closed-loop system $y(x,\alpha) := Ax + Bu(x,\alpha) + a$.

Remark 15 *The function* $\alpha(t) = 1 - \mathrm{e}^{-ct}$ *has been chosen for its simplicity. Any monotone function increasing from* 0 *to* 1 *with sufficiently small derivative may also be used.*

Theorem 16 *Suppose Assumption 4 holds and suppose the invariance conditions* (2) *for* $\mathcal{S}$ *are solvable. There exists* $c > 0$ *sufficiently small such that* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *using* $u(x,\alpha)$ *defined in* (18).

PROOF. To construct $u(x,\alpha)$ in (18), we define $u^0(x)$ and $u^\infty(x)$. For $i \in I_{\mathcal{G}}$, assign $u^0(v_i)$ according to (14a)-(14b). For $i \in \{1,\ldots,r\}$ assign $u^\infty(v_i)$ according to (15a)-(15b). For $i \in \{r+1,\ldots,\kappa+1\}$ assign $u^\infty(v_i) = u^0(v_i)$. For $i \in \{0,\kappa+2,\ldots,n\}$, assign $u^0(v_i) = u^\infty(v_i)$ so that $Av_i + Bu^0(v_i) + a \in \mathcal{C}_i$. Let $u^0(x)$ and $u^\infty(x)$ be the associated affine feedbacks. By construction $u^0(x)$ and $u^\infty(x)$ satisfy (8a)-(8b) of Theorem 13. Now consider $\mathcal{H}^*$ given by (10). Define the compact, convex sets $\mathcal{P}^- := \{x \in \mathcal{S} \mid \xi^* \cdot (x - v_0) \leq 1\}$ and $\mathcal{P}^+ := \{x \in \mathcal{S} \mid \xi^* \cdot (x - v_0) \geq 1\}$. From (13), $\mathcal{E}^0 \subset \mathcal{P}^+$, $\mathcal{E}^\infty \subset \mathcal{P}^-$, $\mathcal{P}^- \subset \mathcal{S} \setminus \mathcal{E}^0$, and $\mathcal{P}^+ \subset \mathcal{S} \setminus \mathcal{E}^\infty$. First we discuss the behavior of trajectories using $u^0(x)$ and $u^\infty(x)$. By Lemma 14, a flow condition holds on $\mathcal{P}^-$ using $u^0(x)$. Since $\mathcal{P}^-$ is compact, by a standard argument all trajectories $\phi^0(t,x_0)$ exit $\mathcal{P}^-$ in finite time. Since the invariance conditions hold using $u^0(x)$, trajectories only exit $\mathcal{P}^-$ via $\mathcal{F}_0$ or $\mathcal{H}^*$. Similarly, by Lemma 14, a flow condition holds on $\mathcal{P}^+$ using $u^\infty(x)$. Since $\mathcal{P}^+$ is compact, all trajectories $\phi^\infty(t,x_0)$ exit $\mathcal{P}^+$ in finite time. Since the invariance conditions hold using $u^\infty(x)$, trajectories only exit $\mathcal{P}^+$ via $\mathcal{F}_0$ or $\mathcal{H}^*$. Because $u^\infty(x)$ satisfies (9), trajectories $\phi^\infty(t,x_0)$ cannot exit through $\mathcal{H}^*$. Hence all trajectories $\phi^\infty(t,x_0)$ starting in $\mathcal{P}^+$ exit $\mathcal{S}$ in finite time.

Now we consider the controller $u(x,\alpha)$ with associated trajectories $\phi(t,x_0)$. Abusing notation, it can be rewritten as a time-varying affine feedback

$$u(x,t) = \mathrm{e}^{-ct}u^0(x) + (1-\mathrm{e}^{-ct})u^\infty(x)\,.$$

It is easily verified using convexity that $u(x,t)$ satisfies (8a), and by construction it satisfies (8b); therefore (9) holds. For $c > 0$ sufficiently small, $u(x,t)$ is sufficiently close to $u^0(x)$ for a sufficiently long time interval $[0,\tau_1]$ so that (16) holds on $\mathcal{P}^-$ and all trajectories $\phi(t,x_0)$

initialized in $\mathcal{P}^-$ either exit $\mathcal{S}$ or enter $\mathcal{P}^+$ in a finite time $\tau < \tau_1$. Trajectories initialized in $\mathcal{P}^+$ either exit $\mathcal{S}$ or they remain in $\mathcal{P}^+$ (since they cannot cross over to $\mathcal{P}^-$ by (9)). There exists $\tau_2 > \tau_1$ when all trajectories remaining in $\mathcal{S}$ are in $\mathcal{P}^+$ and $u(x,t)$ is sufficiently close to $u^\infty(x)$ such that the flow condition (17) takes effect on $\mathcal{P}^+$. Thus, all trajectories must exit $\mathcal{P}^+$, and they do so through $\mathcal{F}_0$ and not $\mathcal{H}^*$, again, because of (9). Thus, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using feedback $u(x,t)$ (equivalently $u(x,\alpha)$).

References

[1] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*. vol. 35, pp. 407–428, March 1999.

[2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, New York. 1979.

[3] M.E. Broucke. Reach control on simplices by continuous state feedback. *SIAM Journal on Control and Optimization*. vol. 48, issue 5, pp. 3482-3500, February 2010.

[4] M.E. Broucke. On the reach control indices of affine systems on simplices. *8th IFAC Symposium on Nonlinear Control Systems* (NOLCOS '10). August 2010.

[5] M.E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *American Control Conference* (ACC '11). June 2011.

[6] M.E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *IEEE Transactions on Automatic Control*. Under Revision August 2011.

[7] L.C.G.J.M. Habets and J.H. van Schuppen. Control of piecewise-linear hybrid systems on simplices and rectangles, in: M.D. Di Benedetto and A.L. Sangiovanni-Vincentelli (Eds.) *Hybrid Systems: Computation and Control, Lecture Notes in Computer Science*. Springer Verlag, vol. 2034, pp. 261–274, 2001.

[8] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.

[9] L.C.G.J.M. Habets, P.J. Collins, and J.H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Trans. Automatic Control*. no. 51, pp. 938–948, 2006.

[10] R.T Rockafellar. *Convex Analysis*. Princeton University,Princeton, New Jersey 1970.

[11] B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. vol. 42, no. 11, pp. 1913–1918, November 2006.

A Appendix

PROOF. [Proof of Lemma 6] (i) We consider only $k = 1$ and the submatrix M_{1,r_1-1}. The other cases are analogous. First, we know M_{1,r_1-1} is a $\mathscr{Z}$-matrix because $h_j \cdot b_i \le 0$, $j \neq i$, so the off-diagonal entries are nonpositive. Second, we show $M_{1,r_1-1} = H^T_{1,r_1-1} Y_{1,r_1-1}$ is nonsingular. Suppose there exists $c \in \mathbb{R}^{r_1-1}$ such that $H^T_{1,r_1-1} Y_{1,r_1-1} c = 0$. Let $y := Y_{1,r_1-1} c$. Then $h_j \cdot y = 0$, $j = 1, \ldots, r_1 - 1$. Also by Lemma 5, $h_j \cdot y = 0$, $j = r_1 + 1, \ldots, n$. Thus, either $y \in \mathcal{B} \cap \text{cone}(\mathcal{S})$ or $-y \in \mathcal{B} \cap \text{cone}(\mathcal{S})$. By Assumption (D2), $y = 0$. However, $y = c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}$ and $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent, so $c = 0$. We conclude that M_{1,r_1-1} is nonsingular. Now we show M_{1,r_1-1} satisfies case (Q_{50}) of Theorem 6.2.3 of [2]. Suppose there exists $c \in \mathbb{R}^{r_1-1}$ with $c \neq 0$ and $c \succeq 0$ such that $M_{1,r_1-1} c \preceq 0$. Define the vector $\bar{y} = Y_{1,r_1-1} c \in \mathcal{B}$. Note that $\bar{y} \neq 0$ because $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent. Then $M_{1,r_1-1} c = H^T_{1,r_1-1} Y_{1,r_1-1} c = H^T_{1,r_1-1} \bar{y} \preceq 0$ implies $h_j \cdot \bar{y} \le 0$ for $j = 1, \ldots, r_1 - 1$. Also, since $c_i \ge 0$ and $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $h_j \cdot \bar{y} = \sum_{i=1}^{r_1-1} c_i (h_j \cdot b_i) \le 0$, $j = r_1, \ldots, n$. This implies $0 \neq \bar{y} \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, a contradiction. Therefore, M_{1,r_1-1} has the property that the only solution of the inequalities $c \succeq 0$ and $M_{1,r_1-1} c \preceq 0$ is $c = 0$. In sum, M_{1,r_1-1} is a nonsingular $\mathscr{Z}$-matrix satisfying Theorem 6.2.3, case (Q_{50}) of [2], so M_{1,r_1-1} is a nonsingular $\mathscr{M}$-matrix.

(ii) Suppose not. Then by the definition of reducibility there exists a permutation matrix P such that $P M_{m_k, m_k+r_k-1} P^T = \begin{bmatrix} M_1 & 0 \\ \star & M_2 \end{bmatrix}$ where $M_1 \in \mathbb{R}^{\rho \times \rho}$ and $M_2 \in \mathbb{R}^{(r_k-\rho)\times(r_k-\rho)}$ for some $1 \le \rho \le r_k - 1$. Without loss of generality suppose we have reordered the indices $\{m_k, \ldots, m_k + r_k - 1\}$ in accordance with the permutation matrix P. This means $H^T_{m_k, m_k+\rho-1} Y_{m_k+\rho, m_k+r_k-1} = 0$, or $h_j \cdot b_i = 0$, $i = m_k + \rho, \ldots, m_k + r_k - 1$, $j = m_k, \ldots, m_k + \rho - 1$. Combining with Lemma 5 we get $h_j \cdot b_i = 0$, $i = m_k + \rho, \ldots, m_k + r_k - 1$, $j \in I \setminus \{m_k + \rho, \ldots, m_k + r_k - 1\}$. Consider $M_{m_k+\rho, m_k+r_k-1}$. Since every principal submatrix of a nonsingular $\mathscr{M}$-matrix is a nonsingular $\mathscr{M}$-matrix, by Lemma 6(i) it is a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3 case (I_{28}) of [2], there exists $c \in \mathbb{R}^{r_k-\rho}$, $c \neq 0$, such that $c \preceq 0$ and $M_{m_k+\rho, m_k+r_k-1} c \prec 0$. Let $y := Y_{m_k+\rho, m_k+r_k-1} c$. Note that $y \neq 0$ since $\{b_{m_k+\rho}, \ldots, b_{m_k+r_k-1}\}$ are linearly independent for any $\rho \ge 1$ by Theorem 3. Then we have $M_{m_k+\rho, m_k+r_k-1} c = H^T_{m_k+\rho, m_k+r_k-1} Y_{m_k+\rho, m_k+r_k-1} c = H^T_{m_k+\rho, m_k+r_k-1} y \prec 0$. That is, $h_j \cdot y < 0$, $j = m_k + \rho, \ldots, m_k + r_k - 1$. Also from above, $h_j \cdot y = 0$, $j \in I \setminus \{m_k + \rho, \ldots, m_k + r_k - 1\}$. We conclude $0 \neq y \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, a contradiction.

(iii) By Lemma 4.4 of [3], $\text{rank}(H_{m_k, m_k+r_k-1}) = r_k$. Also we know $\text{rank}(Y_{m_k, m_k+r_k-1}) = r_k - 1$. Therefore, $\text{rank}(M_{m_k, m_k+r_k-1}) \le r_k - 1$, so it is clearly singular. Now we prove that M_{m_k, m_k+r_k-1} is an $\mathscr{M}$-matrix. By Lemma 6(i), each principal submatrix of M_{m_k, m_k+r_k-1} is a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3 case (D_{16}) of [2], every real eigenvalue of a nonsingular $\mathscr{M}$-matrix is positive. Therefore, every real eigenvalue of each principal submatrix of M_{m_k, m_k+r_k-1} is positive. By Theorem 6.4.6 case (A_2) of [2], M_{m_k, m_k+r_k-1} is an $\mathscr{M}$-matrix.