REACH CONTROL ON SIMPLICES BY PIECEWISE AFFINE FEEDBACK

MIREILLE E. BROUCKE* AND MARCUS GANNESS†

Abstract. We study the reach control problem for affine systems on simplices, and the focus is on cases when the problem is not solvable by continuous state feedback. We examine from a geometric viewpoint the structural properties of the system which make continuous state feedbacks fail. This structure is encoded by so-called *reach control indices*, which are defined and developed in the paper. Based on these indices, we propose a subdivision algorithm and associated piecewise affine feedback. The method is shown to solve the reach control problem in all remaining cases, assuming it is solvable by open-loop controls.

1. Introduction. This paper studies the *reach control problem* (RCP) on simplices[1]. The problem is for trajectories of an affine system defined on a simplex to reach a prespecified facet of the simplex in finite time. The overall concept of the problem and its setting were introduced in [15] and further developed in [16, 17, 26, 7]. The significance of the problem stems from its capturing the essential features of reachability problems for control systems: the presence of state constraints and the notion of trajectories reaching a goal in a guided and finite-time manner. The problem fits within a larger family of reachability problems; namely, to reach a target set $\mathcal{X}_f$ with state constraint in a set $\mathcal{X}$, denoted as $\mathcal{X} \xrightarrow{\mathcal{X}} \mathcal{X}_f$. In the present context, we assume that the state constraints give rise to a state space that is triangulable [18]; then the reachability specification is converted to a sequence of reachability problems on simplices of the triangulation. The reader is referred to [7, 15, 16, 17, 26, 21, 2] for further motivations including how the studied problem arises in fundamental problems concerning hybrid systems [14].

The present paper is a direct outgrowth of [7]. In [7] it was shown that under a special triangulation of the polytopic state space, namely Assumption 11, continuous state feedback and affine feedback are equivalent with respect to solvability of RCP. Also, [7] gave necessary and sufficient conditions for solvability of RCP by affine feedback in terms of the problem data, in contrast with [17, 26] where necessary and sufficient conditions for a *given* affine feedback to solve RCP were given. However, [7] left unresolved the question of what class of feedbacks suffices to solve RCP when continuous state feedbacks fail to do so. This paper fully addresses this question. First, we establish some necessary conditions for solvability of RCP by open-loop controls; they frame the search for a feedback class. Next, we elaborate ideas on fixed point theory

*Dept. of Electrical and Computer Engineering, University of Toronto. Email: broucke@control.utoronto.ca. Supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

†Dept. of Electrical and Computer Engineering, University of Toronto. Supported by NSERC.

[1]Preliminary versions of parts of this paper appeared in [8, 9].

and existence of equilibria using continuous state feedback in the context of RCP, specifically Proposition 7.2 and Theorem 7.3 of [7], to arrive at the *reach control indices*. These indices help to classify how a continuous state feedback fails through the appearance of equilibria in certain sub-simplices. A more detailed comparison with [7] can be found in Section 6.

RCP is one among several different research paths for analysis and synthesis of piecewise affine (PWA) feedback [4, 11, 27]. Recent progress on explicit Model Predictive Control (MPC) schemes has fueled the interest in PWA feedbacks [4], such feedbacks play a prominent role in linear switched systems [19], and PWA systems have significant applications in engineering and biology [28, 12, 23, 20]. A feature of our approach is that, rather than directly computing a controller numerically, we seek conditions for existence of controllers based on the problem data. This follows classical lines of thought which are well established in control theory. Another classical underpinning is to exploit system structure to understand the limits of a control system, again distinguishing our approach from numerical methods.

2. Contributions. In [7] it was shown that, under a suitable triangulation of the state space, affine feedback and continuous state feedback are equivalent from the point of view of solvability of the reach control problem. The approach is based, fundamentally, on fixed point theory. The latter allows to deduce that continuous state feedbacks always generate closed-loop equilibria in the simplex when affine feedbacks do. The current paper departs from these findings, and using a geometric approach, we explore the system structure that gives rise to equilibria. This structure is encoded in the so-called *reach control indices*. The first goal of this paper is to elucidate these indices. The second goal is to use the indices to obtain a subdivision of the simplex and an associated piecewise affine feedback to solve RCP in those cases when the problem is not solvable by continuous state feedback. It is shown that RCP is solvable by piecewise affine feedback if it is solvable by open-loop controls. This finding gives strong evidence to the relevance of the class of piecewise affine feedbacks in solving reachability problems.

The main ideas of the paper can be understood informally. Consider a 2D simplex $\mathcal{S} = \text{co}\{v_0, v_1, v_2\}$ the convex hull of vertices v_0, v_1, and v_2, with 1D facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$, as in Figure 2.1(a). Let cone($\mathcal{S}$) be the cone with apex at v_0 determined by $\mathcal{S}$. Consider a single-input control system $\dot{x} = Ax + bu + a$ defined on $\mathcal{S}$. The *reach control problem* is to find a state feedback $u = f(x)$ such that all closed-loop trajectories initialized in $\mathcal{S}$ leave $\mathcal{S}$ in finite time through the *exit facet* $\mathcal{F}_0$. The procedure to solve this control problem by continuous state feedback is to select control values u_i at the vertices v_i such that the velocity vectors $Av_i + bu_i + a$ lie in the tangent cone to cone($\mathcal{S}$) at v_i; otherwise trajectories may leave $\mathcal{S}$ through $\mathcal{F}_1$ or $\mathcal{F}_2$, which is disallowed. The controller $u = f(x)$ is formed as a continuous interpolation of the control values at the vertices. Label the vertex velocity vectors as $y_0 = Av_0 + bu_0 + a$,

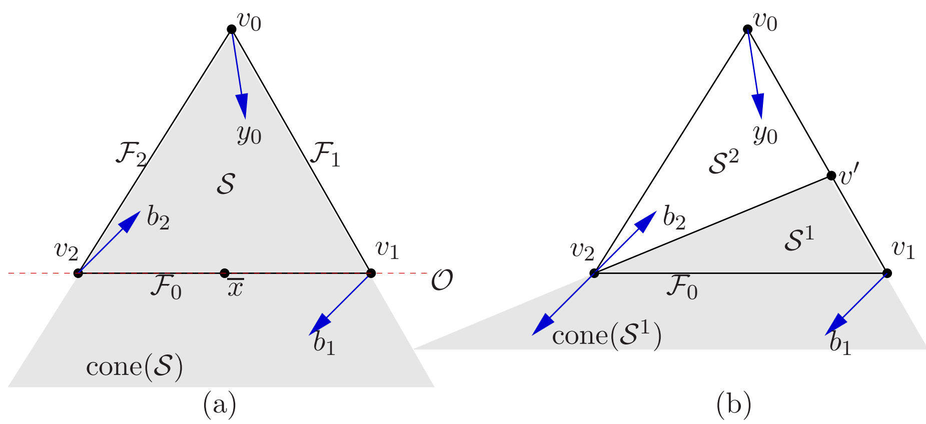


FIG. 2.1. *Main idea in a 2D example.*

and $b_i = Av_i + bu_i + a$, $i = 1, 2$, as in the figure. Let $\mathcal{O}$ be the set where $Ax + a \in \text{Im}(b)$, and suppose it is a line through v_1 and v_2. Clearly closed loop equilibria can only appear on the set $\mathcal{O}_\mathcal{S} := \mathcal{S} \cap \mathcal{O}$. Now it is obvious that this control problem cannot be solved by any continuous state feedback. For at v_1, b_1 has to point down but at v_2, b_2 has to point up. If we continuously interpolate along $\mathcal{F}_0$ from v_1 to v_2, the continuous vector field, always in $\text{Im}(b)$ along $\mathcal{F}_0$, must pass through zero (by the Intermediate Value Theorem) at some $\overline{x}$ along $\mathcal{F}_0$. The defect is that there are two vertices v_1 and v_2 that "share" the only control direction available, b.

Suppose we now allow discontinuous feedback. Place a point v' along the edge from v_0 to v_1 and define a new simplex $\mathcal{S}^1 = \text{co}\{v', v_1, v_2\}$. See Figure 2.1(b). Notice that as we slide v' from v_0 to v_1 the cone cone($\mathcal{S}^1$) with apex at v' widens at v_2 enough that $-b_2$ lies in the tangent cone to cone($\mathcal{S}^1$) at v_2. Notice also that v_1 is unaffected by sliding v'. Pick such a v'. Then one can construct an affine feedback $u = K^1 x + g^1$ on $\mathcal{S}^1$ that assigns a non-zero velocity vector at every point on $\mathcal{F}_0$, so there is no closed loop equilibrium in $\mathcal{S}^1$. By [17, 26], RCP is solved on $\mathcal{S}^1$. For the remaining simplex $\mathcal{S}^2$ it is also possible to devise an affine controller so there is no equilibrium in $\mathcal{S}^2$. This is because equilibria can only appear in $\mathcal{S}^2$ at $v_2 \in \mathcal{O}$. But at v_2 we can select the velocity vector $b_2 \neq 0$. Again RCP can be solved on $\mathcal{S}^2$ by affine feedback. Combining the two affine feedbacks, we get a discontinuous piecewise affine feedback that solves RCP on $\mathcal{S}$. Note that a discontinuity is introduced because we use two different control values at v_2.

The contribution of the paper is to make mathematically rigorous the informal ideas described above. The most significant outcome is Theorem 14 on equivalence of (discontinuous) piecewise affine feedback and open-loop controls for solving RCP. The main technical difficulty arises in dealing with multi-input systems. For this we bring in two tools. First we introduce the reach control indices to group together vertices in $\mathcal{O}_\mathcal{S}$ that share control inputs. These indices are similar in spirit to the controllability indices to group together states that share control inputs [10]. As with the controllability indices, the reach control indices require a special ordering of a set of linearly independent vectors; however, other technical details are different. The second tool is $\mathscr{M}$-matrices which help to concisely represent the constraints on the vector field at vertices of $\mathcal{O}_\mathcal{S}$. The reader is referred to Chapter 6 of [5] for relevant background.

The paper is organized as follows. In Section 3 we present notation and background results from linear algebra. In Section 4 we review the reach control problem. In Section 5 we give necessary conditions for solvability by open-loop controls. Using the necessary conditions as the foundation, in Section 5 we present the main result of the paper on equivalence of piecewise affine feedback and open-loop controls for solving RCP. The main result relies on a subdivision procedure that uses so-called reach control indices. These indices are developed in Section 7. In Section 8, the subdivision procedure and an associated piecewise affine feedback are given to solve RCP when continuous state feedback does not. Examples are presented in Section 9.

3. Background. In this section we present the notation of the paper and some background results of linear algebra. The proofs for this section can be found in the Appendix. For $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). Notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. Let χ be a finite set of elements. The notation $|\chi|$ denotes the cardinality of χ. The notation $\mathscr{B}$ denotes the open unit ball centered at the origin, and $\overline{\mathscr{B}}$ denotes its closure. The notation $\mathrm{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$, and $\mathrm{sp}\{y_1, y_2, \ldots\}$ denotes the span of vectors $y_i \in \mathbb{R}^n$. The notation (v_i, v_j) denotes the open segment in $\mathbb{R}^n$ between $v_i, v_j \in \mathbb{R}^n$. Finally, $T_\mathcal{S}(x)$ denotes the Bouligand tangent cone to set $\mathcal{S}$ at a point x [13].

A matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \leq 0$ for all $i \neq j$. A matrix M is *monotone* if $Mc \preceq 0$ implies $c \preceq 0$. A $\mathscr{Z}$-matrix M is a nonsingular $\mathscr{M}$-matrix if it is monotone [5].

The following two results will be used in Section 7 to construct the reach control indices.

Lemma 1. *Let $\{w_1, \ldots, w_r \mid w_i \in \mathbb{R}^n\}$ be a set of linearly independent vectors, and let $\mathcal{C}$ be a cone satisfying $\mathcal{C} \neq \mathbf{0}$ and $\mathcal{C} \subset \mathrm{sp}\{w_1, \ldots, w_r\}$. There exists a unique non-empty subset $\chi \subset \{w_1, \ldots, w_r\}$ of minimum cardinality such that $\mathcal{C} \subset \mathrm{sp}\ \ \chi$.*

The following lemma establishes that one can always find a vector in a cone $\mathcal{C}$ that depends on all the vectors in $\{w_1, \ldots, w_r\}$.

Lemma 2 ([8]). *Let $\{w_1, \ldots, w_r \mid w_i \in \mathbb{R}^n\}$ be a set of linearly independent vectors, and let $\mathcal{C}$ be a cone satisfying $\mathcal{C} \neq \mathbf{0}$ and $\mathcal{C} \subset \mathrm{sp}\{w_1, \ldots, w_r\}$. Suppose that for each $i \in \{1, \ldots, r\}$, $\mathcal{C} \not\subset \mathrm{sp}\{w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_r\}$. Then there* *exists* $\overline{y} \in \mathcal{C}$ *such that*

$$\overline{y} = c_1 w_1 + \cdots + c_r w_r \,, \qquad c_i \neq 0, \quad i = 1, \ldots, r \,. \tag{3.1}$$

Throughout the paper we make use of simplices, polytopes, and their faces and facets. An *n-dimensional simplex* $\mathcal{S}$ is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. A *face* of $\mathcal{S}$ is any sub-simplex of $\mathcal{S}$ which makes up its boundary. A *facet* of $\mathcal{S}$ is an $(n-1)$-dimensional face of $\mathcal{S}$. An *n-dimensional polytope* $\mathcal{P}$ is the convex hull of a finite set of points in $\mathbb{R}^n$ which contains $(n+1)$ affinely independent points. In Section 8 we make use of triangulations of a polytope $\mathcal{P}$.

Definition 3 ([18]). *A triangulation* $\mathbb{T}$ *of an* n*-dimensional polytope* $\mathcal{P}$ *is a finite collection of* n*-dimensional simplices* $\mathcal{S}_1, \ldots, \mathcal{S}_L$ *such that (i)* $\mathcal{P} = \bigcup_{i=1}^{L} \mathcal{S}_i$*; (ii) for all* $i, j \in \{1, \ldots, L\}$ *with* $i \neq j$*, the intersection* $\mathcal{S}_i \cap \mathcal{S}_j$ *is either empty or a common face of* $\mathcal{S}_i$ *and* $\mathcal{S}_j$*.*

4. Problem Statement. Consider an n-dimensional simplex $\mathcal{S}$, the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. Let its vertex set be $V := \{v_0, \ldots, v_n\}$ and its facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$. The facet will be indexed by the vertex it does not contain. Let $h_j \in \mathbb{R}^n$, $j = 0, \ldots, n$ be the unit normal vector to each facet $\mathcal{F}_j$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet* of $\mathcal{S}$. Define the index set $I := \{1, \ldots, n\}$. For $x \in \mathcal{S}$ define the closed, convex cone

$$\mathcal{C}(x) := \{ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \; j \in I \;\; s.t. \;\; x \in \mathcal{F}_j \} \,.$$

We'll write $\text{cone}(\mathcal{S}) := \mathcal{C}(v_0)$ because $\mathcal{C}(v_0)$ is the tangent cone to $\mathcal{S}$ at v_0. We consider the affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a \,, \tag{4.1}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\text{rank}(B) = m$. Let $\mathcal{B} = \text{Im}(B)$, the image of B. We take as open-loop controls for (4.1) any measurable function $\mu : [0, \infty) \to \mathbb{R}^m$ that is bounded on compact intervals. In the sequel, solutions of (4.1) under either a feedback control $u = f(x)$ or an open-loop control $\mu(t)$ will be interpreted in the sense of Caratheodory. We use the notation $\phi_u(t, x_0)$ to denote the trajectory of (4.1) starting at x_0 under a feedback $u = f(x)$. Similiarly, $\phi_\mu(t, x_0)$ will denote the trajectory of (4.1) starting at x_0 under an open-loop control $\mu(t)$. Finally, define $\mathcal{O} := \{ \; x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B} \}$ and $\mathcal{O}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{O}$. Note that closed-loop equilibria of (4.1) can only appear in $\mathcal{O}$.

Example 4. *Consider Figure 4.1 where we illustrate the notation in a 2D example. We have a full-dimensional simplex in* $\mathbb{R}^2$ *given by* $\mathcal{S} = \text{co}\{v_0, v_1, v_2\}$ *with vertex set* $V = \{v_0, v_1, v_2\}$ *and facets* $\mathcal{F}_0$,$\mathcal{F}_1$, *and* $\mathcal{F}_2$*. Each facet* $\mathcal{F}_j$ *has an outward normal vector* h_j*. The only vertex not in facet* $\mathcal{F}_j$ *is vertex* v_j*.* $\mathcal{F}_0$ *is the exit facet. If we assume that* $v_0 = 0$*, then the subspace* $\mathcal{B}$ *is shown*

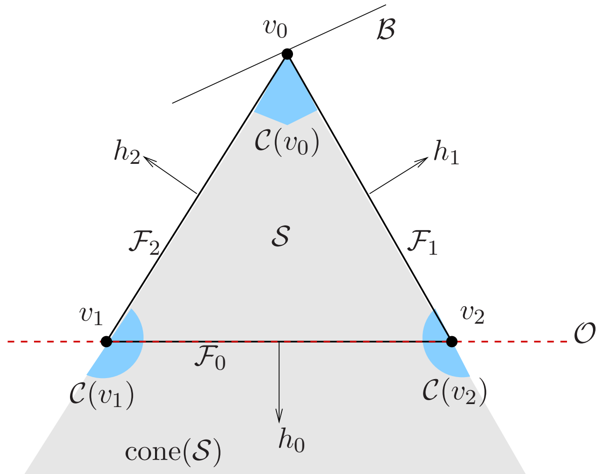


FIG. 4.1. *Notation for reach control problem.*

passing through v_0*. The set* $\mathcal{O}$ *is an affine space shown passing through* $\mathcal{F}_0$*. Notice in this case* $\mathcal{O}_{\mathcal{S}} = \mathcal{S} \cap \mathcal{O} = \text{co}\{v_1, v_2\}$*. The cone* $\text{cone}(\mathcal{S})$ *is the cone with apex at* v_0 *determined by* $\mathcal{S}$*. It is indicated in the figure as the shaded area. The cones* $\mathcal{C}(v_i)$*,* $i = 0, 1, 2$ *are depicted as darker shaded cones attached at each vertex. Of course, the apex of each* $\mathcal{C}(v_i)$ *is at the origin, but we depict it as being attached at the corresponding vertex* v_i *since it will be used to describe allowable directions for the vector field at the vertices. Notice that* $\mathcal{C}(v_0)$ *is the tangent cone to* $\mathcal{S}$ *at* v_0*. Instead the cones* $\mathcal{C}(v_1)$ *and* $\mathcal{C}(v_2)$ *are not tangent cones to* $\mathcal{S}$ *at* v_1 *and* v_2*, respectively; however, they are tangent cones to* $\text{cone}(\mathcal{S})$ *if* $v_0 = 0$*.*

We are interested in formulating a problem to make the closed-loop trajectories of (4.1) exit $\mathcal{S}$ through the exit facet $\mathcal{F}_0$ only. For this, we require conditions that disallow trajectories to exit from any other facet $\mathcal{F}_i$, $i \in I$. We say the invariance conditions are solvable at vertex $v_i \in V$ if there exists $u_i \in \mathbb{R}^m$ such that

$$Av_i + Bu_i + a \in \mathcal{C}(v_i)\,. \tag{4.2}$$

We say the invariance conditions are solvable if (4.2) is solvable at each $v_i \in V$. The conditions (4.2) are called *invariance conditions*. They are used to construct affine feedbacks such that trajectories of the closed-loop system cannot exit from the facets $\mathcal{F}_i$, $i \in I$ [16]. For general state feedbacks, stronger conditions (also called invariance conditions) are needed. We say a state feedback $u = f(x)$ satisfies the invariance conditions if for all $x \in \mathcal{S}$,

$$Ax + Bf(x) + a \in \mathcal{C}(x)\,. \tag{4.3}$$

EXAMPLE 5. *Consider Figure 4.2. Attached at each vertex is a velocity vector* $y_i := Av_i + Bu_i + a$*,* $i \in \{0\} \cup I$*. The invariance conditions* (4.2) *require that* $y_i \in \mathcal{C}(v_i)$*, as illustrated. Notice that velocity vectors at* $v_i \in \mathcal{F}_0$ *may or*

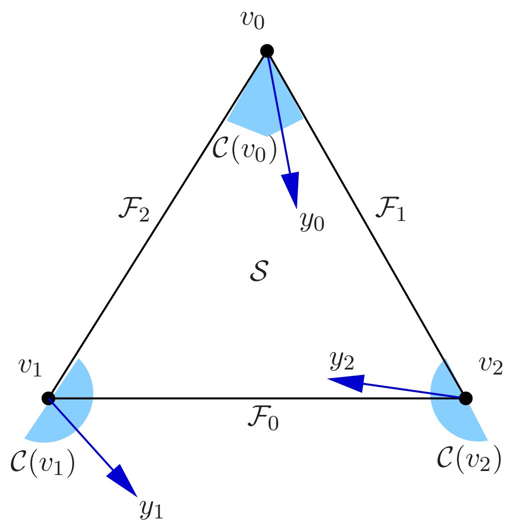


FIG. 4.2. *The invariance conditions state that* $y_i := Av_i + Bu_i + a \in \mathcal{C}(v_i)$ *for* $i = 0, \ldots, n$.

may not point out of $\mathcal{S}$. *If the control is an affine feedback* $u = Kx + g$ *such that* $u_i = Kv_i + g$, *then since* $x \mapsto (A + BK)x + Bg + a$ *is an affine function,* $\mathcal{F}_i$ *is a convex set, and the cones* $\mathcal{C}(x)$ *are convex,* (4.3) *holds at every* $x \in \mathcal{F}_i$, $i \in I$. *If the input is a continuous state feedback* $u = f(x)$, *then invariance conditions for every* $x \in \mathcal{F}_i$, $i \in I$, *must be explicitly stated, since convexity is not guaranteed; hence* (4.3).

PROBLEM 1 (Reach Control Problem (RCP)). *Consider system (4.1) defined on* $\mathcal{S}$. *Find a state feedback* $u = f(x)$ *such that:*

(i) For every $x \in \mathcal{S}$ *there exist* $T \geq 0$ *and* $\gamma > 0$ *such that* $\phi_u(t, x) \in \mathcal{S}$ *for all* $t \in [0, T]$, $\phi_u(T, x) \in \mathcal{F}_0$, *and* $\phi_u(t, x) \notin \mathcal{S}$ *for all* $t \in (T, T + \gamma)$.
(ii) There exists $\varepsilon > 0$ *such that for every* $x \in \mathcal{S}$, $\quad \|Ax + Bf(x) + a\| > \varepsilon$.
(iii) Feedback $u = f(x)$ *satisfies the invariance conditions* (4.3).

We remark that no regularity assumptions are placed on f at this point. The goal of the paper is to discover a class of feedbacks that have sufficient regularity to ensure closed-loop solutions in the sense of Caratheodory. In the sequel we will use the shorthand notation $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ to denote that (i)-(iii) of Problem 1 hold under some control law. Condition (i) is the same condition that appears in the standard formulation of RCP [17, 26]. It states that all closed-loop trajectories must exit $\mathcal{S}$ through $\mathcal{F}_0$ in finite time without first exiting from another facet. Condition (ii) and (iii) are new, and they are introduced to deal with pathologies that can only happen when using discontinuous feedbacks. It can be shown that if continuous state feedback is used, then condition (i) implies conditions (ii) and (iii) [16]. Therefore, results on affine feedbacks [17, 26] and continuous state feedbacks [7] remain valid.

EXAMPLE 6. *In this example we illustrate the need for condition (ii). Figure 4.3(a) illustrates a 2D simplex* $\mathcal{S} = \mathrm{co}\{v_0, v_1, v_2\}$. *Let* $f_1(x)$ *be a continu-*

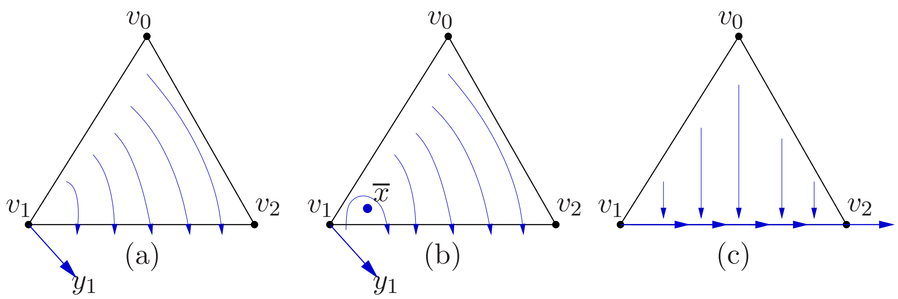


FIG. 4.3. *Illustration of pathologies that arise using discontinuous feedback to solve RCP.*

ous vector field on $\mathcal{S}$ such that $f_1(v_i) \in \mathcal{C}(v_i)$, $i \in \{0, \ldots, n\}$ and $f_1(v_1) = 0$. Second, let $y_1 \in \mathcal{C}(v_1)$ be a non-zero vector. Then we define a discontinuous vector field $f(x)$ on $\mathcal{S}$ given by $f(x) = f_1(x)$ for $x \in \mathcal{S} \setminus \{v_1\}$ and $f(v_1) = y_1$. The notable feature of $f(x)$ is that it is arbitrarily close to zero on $\mathcal{S}$; nevertheless, it can be proved that $f(x)$ satisfies the requirements of RCP. If $f(x)$ is the closed-loop vector field arising from a discontinuous feedback and the system parameters (A, B, a) are slightly perturbed, then there can appear an equilibrium $\overline{x}$ of the perturbed system in the interior of $\mathcal{S}$, as shown in Figure 4.3(b). Thus, RCP is not solved for the perturbed system. Condition (ii) disallows such non-robust behavior specifically arising from the appearance of unwanted equilibria (it does not address non-robustness that may arise from chattering due to discontinuous feedback). Any solution of RCP based on continuous feedback will never exhibit such non-robust behavior.

EXAMPLE 7. *Next consider Figure 4.3(c) which represents a second pathological solution to RCP using discontinuous feedback. Here trajectories reach $\mathcal{F}_0$ in finite time, and then they slide along $\mathcal{F}_0$ out of the simplex along a direction at v_2 that violates v_2's invariance conditions. In order to circumvent this behavior, it is sufficient to disallow feedbacks that violate the invariance conditions* (4.3)*, particularly on $\mathcal{F}_0$. This is the purpose of condition (iii).*

5. Necessary Conditions.

In this section we present two necessary conditions for solvability of RCP using open-loop controls. First, we define what is meant by a solution of RCP by open-loop controls.

DEFINITION 8. *Consider system (4.1) defined on $\mathcal{S}$. We say $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$* by open-loop controls *if there exists a map $T : \mathcal{S} \to \mathbb{R}^+$ and a set of open-loop controls $\{\mu_x \mid x \in \mathcal{S}\}$ such that:*

(i) For every $x \in \mathcal{S}$ there exists $\gamma > 0$ such that $\phi_{\mu_x}(t, x) \in \mathcal{S}$ for all $t \in [0, T(x)]$, $\phi_{\mu_x}(T(x), x) \in \mathcal{F}_0$, and $\phi_{\mu_x}(t, x) \notin \mathcal{S}$ for all $t \in (T(x), T(x) + \gamma)$.

(ii) There exists $\varepsilon > 0$ such that for every $x \in \mathcal{S}$ and $t \in [0, T(x)]$, $\|A\phi_{\mu_x}(t, x) + B\mu_x(t) + a\| > \varepsilon$.

(iii) For every $x \in \mathcal{S}$ and $t \in [0, T(x)]$, $(A\phi_{\mu_x}(t, x) + B\mu_x(t) + a) \in \mathcal{C}((\phi_{\mu_x}(t, x)))$.

The first result of the section is that solvability of the invariance conditions (4.2) is necessary for solvability of RCP by open-loop controls in the sense of condition (i) only. This extends the analogous result in [16] on the necessity of the invariance conditions for solvability of RCP (in the sense of condition (i) only) for continuous state feedbacks. Proofs are in the Appendix.

THEOREM 9. *If* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by open-loop controls in the sense of condition (i) only, then the invariance conditions* (4.2) *are solvable.*

The second result says that if RCP is solvable by open-loop controls, then it is possible to assign non-zero velocity vectors satisfying (4.2) at vertices $v_i \in V \cap \mathcal{O}_\mathcal{S}$. This is an immediate consequence of condition (ii). We know that $Av_i + a \in \mathcal{B}$ for vertices $v_i \in \mathcal{O}_\mathcal{S}$. Theorem 9 says that if RCP is solvable by open-loop controls (in the sense of condition (i)), then $\mathcal{B} \cap \mathcal{C}(v_i) \neq \emptyset$, for $v_i \in V \cap \mathcal{O}_\mathcal{S}$. The next result says that, moreover, the zero vector cannot be the only element of $\mathcal{B} \cap \mathcal{C}(v_i)$, $v_i \in V \cap \mathcal{O}_\mathcal{S}$.

THEOREM 10. *If* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by open-loop controls, then* $\mathcal{B} \cap \mathcal{C}(v_i) \neq \mathbf{0}$*,* $v_i \in V \cap \mathcal{O}_\mathcal{S}$*.*

6. Main Result. In this section we state the main result of the paper. The next two sections will be devoted to proving the main step of the result. The necessary and sufficient conditions for a given affine feedback to solve RCP are: (a) the invariance conditions (4.2) hold; and (b) the closed-loop system has no equilibrium in $\mathcal{S}$ [17, 26]. We wish to study the extent to which affine feedbacks can solve RCP. For this it is useful to exploit information about $\mathcal{O}$ to determine whether and how equilibria appear in $\mathcal{S}$. For example, if it is known that $\mathcal{O}_\mathcal{S} = \emptyset$, then any affine feedback satisfying the invariance conditions will solve RCP. If $\mathcal{O}_\mathcal{S} = \text{co}\{v_1, \dots, v_{\kappa+1}\}$, then one can carefully choose control values at these vertices to avoid equilibria in $\mathcal{S}$. This observation motivates the following assumption.

ASSUMPTION 11. *Simplex* $\mathcal{S}$ *and system* (4.1) *satisfy the following condition: if* $\mathcal{O}_\mathcal{S} \neq \emptyset$*, then* $\mathcal{O}_\mathcal{S}$ *is a* κ*-dimensional face of* $\mathcal{S}$*, where* $0 \leq \kappa \leq n$*.*

Under Assumption 11, the following cases when affine feedbacks solve RCP have been identified.

- Suppose $\mathcal{O}_\mathcal{S} = \emptyset$. If RCP is solvable by open-loop controls, then it is solvable by affine feedback (Theorem 6.1 of [7]).
- Suppose $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$. If RCP is solvable by open-loop controls, then it is solvable by affine feedback (Theorem 6.2 of [7]).
- Suppose $v_0 \in \mathcal{O}$. If RCP is solvable by open-loop controls, then it is solvable by affine feedback (Remark 7.1 of [7]).
- Suppose $\mathcal{O}_\mathcal{S} = \text{co}\{v_1, \dots, v_{\kappa+1}\}$ and there exists a linearly independent set $\{b_1, \dots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$. If RCP is solvable by open-loop controls, then it is solvable by affine feedback (Theorem 6.7 of [7]).

These results show that when $\mathcal{O}_\mathcal{S}$ is non-trivial (neither the empty set nor containing v_0), there are two strategies to avoid closed-loop equilibria in $\mathcal{O}_\mathcal{S}$. The first strategy is to select a single control direction $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, and then assign an affine feedback that inserts a large b component in the velocity vector at each vertex of $\mathcal{S}$. The second strategy is to assign a linearly independent set of control directions in $\mathcal{B}$ at the vertices of $\mathcal{O}_{\mathcal{S}}$. In both cases a convexity argument shows there can be no closed-loop equilibria in $\mathcal{S}$. Having identified those cases under Assumption 11 when affine feedback is known to solve RCP, we can now summarize all remaining cases for which the class of feedbacks to solve RCP is still unknown. Observe that Assumption (A4) below is no loss of generality due to Theorem 10.

ASSUMPTION 12. *Simplex $\mathcal{S}$ and system* (4.1) *satisfy the following conditions.*

(A1) $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{v_1, \ldots, v_{\kappa+1}\}$, *with* $0 \le \kappa < n$.

(A2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$.

(A3) The maximum number of linearly independent vectors in any set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ *(with only one vector for each* $\mathcal{B} \cap \mathcal{C}(v_i)$*) is* $\widehat{m}$ *with* $0 \le \widehat{m} < \kappa + 1$.

(A4) $\mathcal{B} \cap \mathcal{C}(v_i) \neq \mathbf{0}$, $i = 1, \ldots, \kappa + 1$.

EXAMPLE 13. *Consider Figure 4.1. We have* $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{v_1, v_2\}$, *which satisfies (A1). Notice (A1) is a strengthening of Assumption 11 - it imposes that* $v_0 \notin \mathcal{O}_{\mathcal{S}}$*; otherwise RCP is not solvable [7]. (A2) is also illustrated in Figure 4.1. At* v_0, $\mathcal{B}$ *has no vectors in common with* $\mathrm{cone}(\mathcal{S})$ *except the zero vector. Next, we see that (A3) is satisfied with* $\widehat{m} = m = 1$*. In particular,* $b_1 \in \mathcal{B} \cap \mathcal{C}(v_1)$ *and* $b_2 \in \mathcal{B} \cap \mathcal{C}(v_2)$ *are linearly dependent. Note also that (A3) specifies that* $\widehat{m} < \kappa + 1$*. If* $\widehat{m} = \kappa + 1$*, then RCP is solvable by affine feedback [7]. Finally (A4) is taken from Theorem 10. It says that at each vertex in* $\mathcal{O}_{\mathcal{S}}$*, there exists a non-zero* $b_i \in \mathcal{B}$ *satisfying the invariance conditions of* v_i *for* $i = 1, 2$.

It has been shown in Theorem 8.1 of [7] that under Assumption 12, RCP is not solvable by continuous state feedback. The main result of this paper, stated next, is that piecewise affine feedbacks are a sufficiently rich class to solve RCP when it is solvable by open-loop controls. The proof shows by a process of elimination that either RCP is solvable by affine feedback [17, 26, 7] or it is solvable by (discontinuous) PWA feedback via a Subdivision Algorithm to be presented in Section 8. The main result on synthesis of the PWA feedback appears in Theorem 33 of Section 8.

THEOREM 14. *Suppose Assumption 11 holds. Then the following are equivalent:*

1. $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by piecewise affine feedback.*
2. $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by open-loop controls.*

Proof. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1) Suppose $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls. By Theorem 9, the invariance conditions are solvable. Let $\mathcal{O}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{O}$. If $\mathcal{O}_{\mathcal{S}} = \emptyset$, then by Theorem 6.1 of [7], $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose instead $\mathcal{O}_{\mathcal{S}} \neq \emptyset$. If $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq \mathbf{0}$, then by Theorem 6.2 of [7], $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

Suppose instead $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. From Theorem 10, $v_0 \notin \mathcal{O}_{\mathcal{S}}$, so by reordering indices, $\mathcal{O}_{\mathcal{S}} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$, where $0 \leq \kappa < n$. Let $\{b_1, \ldots, b_{\widehat{m}} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ be a maximal linearly independent set as in (A3). If $\kappa < \widehat{m}$, then by Theorem 6.7 of [7], $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose instead $\kappa \geq \widehat{m}$. By Theorem 10, $\mathcal{B} \cap \mathcal{C}(v_i) \neq \mathbf{0}$ for $i \in \{1, \ldots, \kappa+1\}$. Then Assumption 12 holds and by Theorem 33, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by piecewise affine feedback. $\square$

The remainder of the paper is devoted to proving Theorem 33, which provides the piecewise affine feedback to solve RCP under Assumption 12. This result depends on so-called reach control indices, which will be developed in the next section. In preparation for their development, we now give some intuition and motivation for these indices.

Suppose Assumption 12 holds and consider $\mathcal{O}_{\mathcal{S}}$. Because each v_i, $i = 1, \ldots, \kappa+1$ belongs to $\mathcal{O}$, it is possible to select an affine feedback so that every $x \in \mathcal{O}_{\mathcal{S}}$ is a closed-loop equilibrium. One can say that this is the *maximal set* of closed-loop equilibria possible in $\mathcal{S}$. But is there a *minimal set*? That is, is it possible to select an affine feedback to achieve a smallest closed-loop equilibrium set in $\mathcal{O}_{\mathcal{S}}$? The reach control indices serve to quantify this smallest closed-loop equilibrium set.

The indices are motivated by the observation that evidently, when $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$, equilibria can only be avoided by assigning linearly independent control directions in $\mathcal{B}$ at the vertices of $\mathcal{O}_{\mathcal{S}}$. Suppose there are insufficient independent control directions in $\mathcal{B}$ available to the resolve the invariance conditions at the vertices in $\mathcal{O}_{\mathcal{S}}$. Then certain vertices in $\mathcal{O}_{\mathcal{S}}$ must share control directions. The reach control indices, denoted $r_1, r_2, \ldots$ effectively partition the set of vertices of $\mathcal{O}_{\mathcal{S}}$ into cosets of cardinality r_i where the r_i vertices in a coset have available to them only $r_i - 1$ independent control directions in $\mathcal{B}$ to resolve all their invariance conditions. This shortage of one independent control direction then implies that using any affine feedback, an equilibrium will arise in the sub-simplex formed by the convex hull of the vertices in the coset.

This information about the location of equilibria under affine feedback makes it possible to contrive a subdivision algorithm, presented in Section 8, whose goal is to remove one vertex from each coset in order to remove the linear dependence on control directions. This subdivision is orchestrated so that in each sub-simplex of the obtained triangulation of $\mathcal{S}$, a condition resembling (A3) is restored.

7. Reach Control Indices. In this section we develop the reach control indices which will be used to obtain the main result Theorem 14 on equivalence of open-loop controls and piecewise affine feedback for solving RCP. The reach control indices are defined in the situation corresponding to Assumption 12 when it is known that RCP is not solvable by continuous state feedback but it is still solvable by open-loop controls.

In the development that follows we will frequently have use of the following operation on the indices I. Let $\tilde{I} = \{i_1, \ldots, i_n\}$ be a permutation of I with the property that $\{i_1, \ldots, i_{\kappa+1}\}$ is a permutation of $\{1, \ldots, \kappa+1\}$. We say we *reorder indices* if we define $\tilde{v}_j := v_{i_j}$, $j \in I$. Second, define $\tilde{h}_j$ and $\tilde{\mathcal{F}}_j$, $j \in I$, according to our convention that $\tilde{\mathcal{F}}_j$ is the facet that does not contain $\tilde{v}_j$, and $\tilde{h}_j$ is its outward normal vector. If we have a selection $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ then define $\tilde{b}_j := b_{i_j}$, $j \in \{1, \ldots, \kappa+1\}$. Finally, we drop the tilde's so that all data v_i, h_i, $\mathcal{F}_i$, b_i are relabeled with the new indices. This notion can be extended to reorder subsets of indices of the form $\{k_1, \ldots, k_2\}$ where $1 \leq k_1 \leq k_2 \leq n$. In that case the remaining indices $\{1, \ldots, k_1-1, k_2+1, \ldots, n\}$ are left the same.

EXAMPLE 15. *Consider the simplex* $\mathcal{S} := \mathrm{co}\{v_0, v_1, v_2, v_3\} \subset \mathbb{R}^3$ *with* $v_0 = (0,0,0)$, $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, *and* $v_3 = (0,0,1)$. *We have* $h_1 = (-1,0,0)$, $h_2 = (0,-1,0)$, *and* $h_3 = (0,0,-1)$. *Suppose it is required to perform two consecutive reorderings of indices. The first reordering is applied to the indices* $\{1,2,3\}$ *based on a permutation* $\{2,1,3\}$. *The second reordering is applied to the subset of indices* $\{2,3\}$ *(of the newly reordered indices) based on a permutation* $\{3,2\}$. *After reordering indices according to the first permutation, the new assignment of vertices is* $v_1 = (0,1,0)$, $v_2 = (1,0,0)$, *and* $v_3 = (0,0,1)$ *and the new assignment of normal vectors is* $h_1 = (0,-1,0)$, $h_2 = (-1,0,0)$, *and* $h_3 = (0,0,-1)$. *After reordering indices according to the second permutation, the new assignment of vertices is* $v_1 = (0,1,0)$, $v_2 = (0,0,1)$, *and* $v_3 = (1,0,0)$ *and the new assignment of normal vectors is* $h_1 = (0,-1,0)$, $h_2 = (0,0,-1)$, *and* $h_3 = (-1,0,0)$. *Similar operations would be performed on the* $\mathcal{F}_i$*'s and* b_i*'s.*

Now consider assumption (A3). Select any $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $i = 1, \ldots, \kappa+1$, and write the list $\{b_1, \ldots, b_{\kappa+1}\}$. Clearly there exists a list with a maximum number $\widehat{m}$ of linearly independent vectors. W.l.o.g., we reorder indices $\{1, \ldots, \kappa+1\}$ (leaving the indices $0, \kappa+2, \ldots, n$ the same) so that $\{b_1, \ldots, b_{\widehat{m}}\}$ are linearly independent. Notice by the maximality of $\{b_1, \ldots, b_{\widehat{m}}\}$ that for each $i = \widehat{m}+1, \ldots, \kappa+1$ and for each $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $b_i \in \mathrm{sp}\{b_1, \ldots, b_{\widehat{m}}\}$. Now consider the cone $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+1})$. By (A4), $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+1}) \neq \mathbf{0}$. By Lemma 1 there exists a unique, non-empty subset χ of $\{b_1, \ldots, b_{\widehat{m}}\}$ such that: (i) $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+1}) \subset \mathrm{sp}\ \chi$; (ii) χ has the minimum cardinality among all subsets of $\{b_1, \ldots, b_{\widehat{m}}\}$ with property (i). In particular, there exists $2 \leq r_1 \leq \widehat{m}+1$ such that w.l.o.g. (reordering indices $1, \ldots, \widehat{m}$ and leaving the indices $0, \widehat{m}+1, \ldots, n$ the same), $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+1}) \subset \mathrm{sp}\{b_1, \ldots, b_{r_1-1}\}$, where $r_1 - 1$ is the minimum cardinality of any subset of $\{b_1, \ldots, b_{\widehat{m}}\}$ whose span contains the cone $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+1})$.

The indices involved in the construction above (after the reorderings) are $\{1, \ldots, r_1 - 1, \widehat{m} + 1\}$. So that these indices are consecutive, we reorder indices again according to the permutation $\tilde{I} = \{1, \ldots, r_1 - 1, \widehat{m} + 1, r_1 + 1, \ldots, \widehat{m}, r_1, \widehat{m} + 2, \ldots, n\}$ of I. That is, we effectively swap the indices $\widehat{m} + 1$ and r_1, so we can write

$$\mathcal{B} \cap \mathcal{C}(v_{r_1}) \subset \mathrm{sp}\{b_1, \ldots, b_{r_1-1}\}\,. \tag{7.1}$$

By Lemma 2 there exists $b_{r_1} \in \mathcal{B} \cap \mathcal{C}(v_{r_1})$ such that

$$b_{r_1} = c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}\,, \qquad c_i \neq 0, \quad i = 1, \ldots, r_1 - 1\,. \tag{7.2}$$

So far we have only used (A3) and some general facts of linear algebra comprising Lemmas 1, 34, and 2. In particular, we have not explicitly used the condition (A2) that $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \mathbf{0}$. We will now see that it determines the nature of the dependency between b_{r_1} and $b_1, \ldots, b_{r_1-1}$.

Lemma 16. *Suppose Assumption 12 and* (7.1)-(7.2) *hold. Then the coefficients in* (7.2) *satisfy* $c_i < 0$, $i = 1, \ldots, r_1 - 1$.

Proof. Suppose w.l.o.g. (by reordering indices $\{1, \ldots, r_1 - 1\}$), there exists $1 \le \rho < r_1 - 1$ such that $c_i > 0$ for $i = 1, \ldots, \rho$ and $c_i < 0$ for $i = \rho+1, \ldots, r_1-1$. Consider the vector $\beta := b_{r_1} - c_{\rho+1} b_{\rho+1} - \cdots - c_{r_1-1} b_{r_1-1} = c_1 b_1 + \cdots + c_\rho b_\rho$. Notice that $\beta \neq 0$ since $\{b_1, \ldots, b_\rho\}$ are linearly independent. Since $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $i \in \{1, \ldots, r_1\}$, we have $h_j \cdot \beta = h_j \cdot \big(b_{r_1} - c_{\rho+1} b_{\rho+1} - \cdots - c_{r_1-1} b_{r_1-1}\big) \le 0$, $j = 1, \ldots, \rho, r_1 + 1, \ldots, n$. Also $h_j \cdot \beta = h_j \cdot \big(c_1 b_1 + \cdots + c_\rho b_\rho\big) \le 0$, $j = \rho + 1, \ldots, n$. In sum, $h_j \cdot \beta \le 0$, $i \in I$; that is, $\beta \in \mathcal{B} \cap \operatorname{cone}(\mathcal{S})$. By Assumption (A2), $\beta = 0$, a contradiction. ∎

Remark 17.

1. *A notable feature of Lemma 16 is that any* b_i, $i = 1, \ldots, r_1$, *can be expressed as a negative linear combination of the remaining vectors* $\{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{r_1}\}$. *This means we may reorder indices within the set* $\{1, \ldots, r_1\}$ *with impunity in the sense that the formula* (7.2) *will still hold with strictly negative coefficients. Such a reordering will be immediately invoked below to generate the second index* r_2 *and it will also be invoked in Lemma 27 of the next section.*
2. *We remark on an implication of Lemma 16 on affine feedbacks. Suppose we assign an affine feedback* $u = Kx + g$ *on* $\mathcal{S}$ *such that* $(A + BK)v_i + Bg + a = b_i$, $i = 1, \ldots, r_1$. *Lemma 16 implies that* $0 \in \operatorname{co}\{b_1, \ldots, b_{r_1}\}$. *This means an equilibrium of the closed-loop system* $\dot{x} = (A + BK)x + Bg + a$ *lies in the relative interior of the simplex* $\operatorname{co}\{v_1, \ldots, v_{r_1}\}$. *Notice it cannot lie on a face of* $\operatorname{co}\{v_1, \ldots, v_{r_1}\}$ *since any* $\{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{r_1}\}$ *are linearly independent.*

We review the construction so far. Starting from the initial assumption (A3), adding a new vector b_{r_1} via Lemma 2, and applying successive index reorderings, we have generated a list of vectors associated with vertices in $\mathcal{O}_{\mathcal{S}}$ of the form

$$\{b_1, \ldots, b_{r_1-1}, b_{r_1}, b_{r_1+1}, \ldots, b_{\widehat{m}+1}\} \tag{7.3}$$

where $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $i \in \{1, \ldots, \widehat{m} + 1\}$. This list has the following properties:

(a) Any b_i, $i = 1, \ldots, r_1$, is a strictly negative linear combination of $\{b_1, \ldots, b_{i-1},\ b_{i+1}, \ldots, b_{r_1}\}$.

(b) Any set of the form $\{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{r_1}\}$, $i = 1, \ldots, r_1$, consists of $r_1 - 1$ linearly independent vectors.

(c) Due to the previous property, $\{b_2, \dots, b_{\widehat{m}+1}\}$ consists of a maximal number $\widehat{m}$ of linearly independent vectors. (This set is different than the one initially proposed in (A3) because of the introduction of b_{r_1} and the index reorderings).

(d) Vectors in $\mathcal{B}$ associated with vertices $\{v_{\widehat{m}+2}, \dots, v_{\kappa+1}\}$ of $\mathcal{O}_{\mathcal{S}}$ have not yet been defined. However, by the maximality of $\widehat{m}$ we know that for any $i \in \{\widehat{m}+2, \dots, \kappa+1\}$ and $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, b_i is linearly dependent on $\{b_2, \dots, b_{\widehat{m}+1}\}$.

The fact that any $r_1 - 1$ vectors in the set $\{b_1, \dots, b_{r_1}\}$ are linearly independent and each vector is a strictly negative linearly combination of the others puts strong restrictions on $\mathcal{B}$. Indeed for the vectors $\{b_1, \dots, b_{r_1}\}$ to meet these properties and to lie in their respective cones, they have a special geometric relationship with $\mathcal{S}$ which is captured in the next result.

LEMMA 18. *Suppose Assumption 12 and* (7.1)-(7.2) *hold. Then*

$$h_j \cdot b_i = 0\,, \qquad i \in \{1, \dots, r_1\}\,, \qquad j \in I \setminus \{1, \dots, r_1\}\,. \tag{7.4}$$

Proof. Let b_{r_1} be as in (7.2). Since $b_{r_1} \in \mathcal{B} \cap \mathcal{C}(v_{r_1})$, $h_j \cdot b_{r_1} = h_j \cdot \big(c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}\big) \leq 0$, $j \in I \setminus \{1, \dots, r_1\}$. Since $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $h_j \cdot b_i \leq 0$ for $i \in \{1, \dots, r_1 - 1\}$ and $j \in I \setminus \{1, \dots, r_1\}$. Also, by Lemma 16, $c_i < 0$. Thus, every term in the sum $c_1 h_j \cdot b_1 + \cdots + c_{r_1-1} h_j \cdot b_{r_1-1}$ is non-negative. The result immediately follows. ∎

Whereas Lemmas 1, 34, and 2 are standard facts of linear algebra, and Lemma 16 is an easy consequence of the condition $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \mathbf{0}$, the constraints (7.4) are the most significant property to emerge about $\{b_1, \dots, b_{r_1}\}$. They place strong geometric constraints on $\mathcal{B}$ enabling us to find a decomposition of $\mathcal{B}$ relative to the simplex.

EXAMPLE 19. *Lemmas 16 and 18 are illustrated for a 3D example in Figure 7.1. We have $\mathcal{S} = \operatorname{co}\{v_0, \dots, v_3\}$, $\mathcal{O}_{\mathcal{S}} = \mathcal{S} \cap \mathcal{O} = \operatorname{co}\{v_1, v_2\}$, and with $v_0 = 0$ we see that $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \mathbf{0}$. Also, $\widehat{m} = m = 1$. Vector b_i shown attached at v_i lies in the cone $\mathcal{B} \cap \mathcal{C}(v_i)$, $i = 1, 2$. Now we observe that $b_2 = -c_1 b_1$, $c_1 > 0$, to satisfy $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$. This is the content of Lemma 16. Second, we observe from the figure that the only way $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $i = 1, 2$, can hold simultaneously is if $h_3 \cdot b_i = 0$, $i = 1, 2$. That is, b_1 and b_2 lie in the 2D plane containing $\mathcal{F}_3$. This is the content of Lemma 18.*

Next we consider the cone $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+2})$. Proceeding as above, there exists a subset of $\{b_2, \dots, b_{\widehat{m}+1}\}$ with minimum cardinality $r_2 - 1$ whose span contains $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+2})$. If we independently reorder each index set $\{2, \dots, r_1\}$ and $\{r_1 + 1, \dots, \widehat{m}+1\}$ so that the indices $\{2, \dots, r_1\}$ are only permuted with each other (this is allowed by Remark 17(1)), we have $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+2}) \subset \operatorname{sp}\{b_\rho, \dots, b_{\rho+r_2-2}\}$, for some $2 \leq \rho \leq r_1 + 1$ and $\rho \leq \rho + r_2 - 2 \leq \widehat{m} + 1$. (Lemma 22 below will demonstrate that the reordering of $\{2, \dots, r_1\}$ is actually not performed).

Now Lemmas 2 and 16 can be adapted for $\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+2})$ since we have exactly the same situation as for $\mathcal{B} \cap \mathcal{C}(v_{r_1})$, only the indices are different.

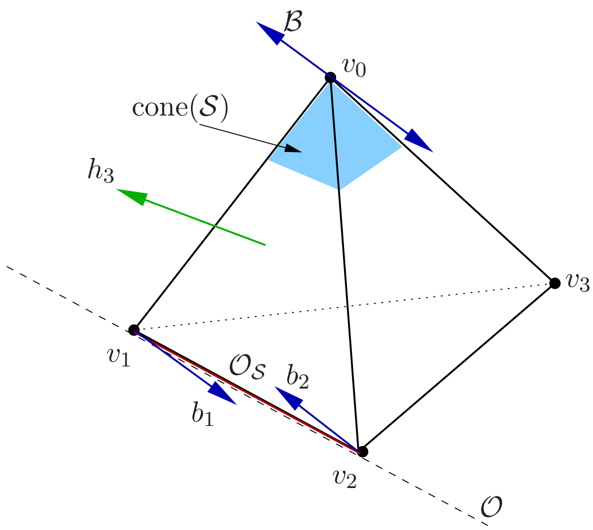


FIG. 7.1. *Illustration for Lemma 18.*

Thus, we get

$$\mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+2}) \subset \mathrm{sp}\{b_\rho, \ldots, b_{\rho+r_2-2}\} \tag{7.5}$$

$$(\exists b_{\widehat{m}+2} \in \mathcal{B} \cap \mathcal{C}(v_{\widehat{m}+2})) \quad b_{\widehat{m}+2} = c_\rho b_\rho + \cdots + c_{\rho+r_2-2} b_{\rho+r_2-2}\,, \quad c_i < 0\,. \tag{7.6}$$

We can similarly invoke Lemma 18 to obtain

$$h_j \cdot b_i = 0\,, \quad i = \rho, \ldots, \rho + r_2 - 2, \widehat{m} + 2\,, j \in I \setminus \{\rho, \ldots, \rho + r_2 - 2, \widehat{m} + 2\}\,. \tag{7.7}$$

We have now generated a list of vectors associated with vertices in $\mathcal{O}_\mathcal{S}$ of the form

$$\{b_1, b_2, \ldots, b_\rho, \ldots, b_{r_1+1}, \ldots, b_{\rho+r_2-2}, \ldots, b_{\widehat{m}+1}, b_{\widehat{m}+2}\} \tag{7.8}$$

where $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$. This list has the following properties:

(a) Vectors $\{b_2, \ldots, b_{\widehat{m}+1}\}$ are linearly independent since reordering the indices $\{2, \ldots, r_1\}$ and $\{r_1 + 1, \ldots, \widehat{m} + 1\}$ does not affect linear independence.

(b) The properties of $\{b_1, \ldots, b_{r_1}\}$ are the same as before. Namely, any b_i, $i = 1, \ldots, r_1$, is a strictly negative linear combination of $\{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{r_1}\}$, and any set $\{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{r_1}\}$, $i = 1, \ldots, r_1$, consists of $r_1 - 1$ linearly independent vectors. This is because these properties are invariant to permutations of the indices $\{1, \ldots, r_1\}$ as noted in Remark 17(1).

(c) Any b_i, $i = \rho, \ldots, \rho + r_2 - 2, \widehat{m} + 2$ is a strictly negative linear combination of $\{b_\rho, \ldots, b_{\rho+r_2-2}, b_{\widehat{m}+2}\} \setminus \{b_i\}$.

(d) Any set of the form $\{b_\rho, \ldots, b_{\rho+r_2-2}, b_{\widehat{m}+2}\} \setminus \{b_i\}$, $i = \rho, \ldots, \rho + r_2 - 2, \widehat{m} + 2$, consists of $r_2 - 1$ linearly independent vectors.

At this point we know $\rho \leq r_1 + 1$. Next we show that actually $\rho = r_1 + 1$. This means that the lists $\{b_1, \ldots, b_{r_1}\}$ and $\{b_\rho, \ldots, b_{\rho+r_2-2}\}$ have no vectors in common. The ensuing proof is facilitated by $\mathscr{M}$-matrices. Let $1 \leq \alpha \leq \beta \leq \kappa + 1$, $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, and define $H_{\alpha,\beta} := [h_\alpha \cdots h_\beta]$, $Y_{\alpha,\beta} := [b_\alpha \cdots b_\beta]$, and $M_{\alpha,\beta} := H^T_{\alpha,\beta} Y_{\alpha,\beta}$.

Lemma 20. *Suppose $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$. Let $1 \leq \alpha \leq \beta \leq \kappa + 1$ and suppose $\{b_\alpha, \ldots, b_\beta \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ are linearly independent and satisfy*

$$h_j \cdot b_i = 0\,, \qquad i \in \{\alpha, \ldots, \beta\}\,, \qquad j \in I \setminus \{\alpha, \ldots, \beta\}\,. \tag{7.9}$$

Then $M_{\alpha,\beta}$ is a nonsingular $\mathscr{M}$-matrix.

Proof. Consider $M_{\alpha,\beta} = H^T_{\alpha,\beta} Y_{\alpha,\beta}$. First, we know $M_{\alpha,\beta}$ is a $\mathscr{Z}$-matrix because $b_i \in \mathcal{C}(v_i)$ implies $h_j \cdot b_i \leq 0$, $j \neq i$, so the off-diagonal entries of $M_{\alpha,\beta}$ are non-positive. Second, we show $M_{\alpha,\beta}$ is monotone. Let $c = (c_\alpha, \ldots, c_\beta)$ be such that $M_{\alpha,\beta} c \preceq 0$. Define $y := Y_{\alpha,\beta} c$. Then $h_j \cdot y \leq 0$, $j = \alpha, \ldots, \beta$. By (7.9), $h_j \cdot y = 0$, $j = 1, \ldots, \alpha - 1, \beta + 1, \ldots, n$. Thus, $y \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S})$. Since $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$, $y = 0$. However, by assumption $\{b_\alpha, \ldots, b_\beta\}$ are linearly independent, so $c = 0$. Thus, $M_{\alpha,\beta}$ is monotone. Finally, by Theorem 6.2.3, case (N_{39}), of [5], $M_{\alpha,\beta}$ is a nonsingular $\mathscr{M}$-matrix. □

Remark 21. *A similar result to Lemma 20 first appeared in [7]; a step of the proof was clarified in [1]. Here we present a simpler argument based on monotonicity.*

Lemma 22. *Suppose Assumption 12 holds. Consider the list* (7.8) *where b_{r_1} and $b_{\widehat{m}+2}$ are given by* (7.2) *and* (7.6)*, respectively. Suppose $\{b_2, \ldots, b_{\widehat{m}+1}\}$ are linearly independent and* (7.4) *and* (7.7) *hold. Then $\rho = r_1 + 1$.*

Proof. Suppose by way of contradiction that $\rho < r_1 + 1$. Let $\sigma = \min\{r_1, \rho + r_2 - 2\}$. Combining (7.4) with (7.7) we obtain

$$h_j \cdot b_i = 0\,, \quad i = \rho, \ldots, \sigma\,, j = 1, \ldots, \rho - 1, \sigma + 1, \ldots, n\,. \tag{7.10}$$

Consider $M_{\rho,\sigma} = H^T_{\rho,\sigma} Y_{\rho,\sigma}$. By (A2), (7.10), and the linear independence of $\{b_\rho, \ldots, b_\sigma\}$ (since $\rho \geq 2$ and $\sigma \leq \widehat{m} + 1$), we can apply Lemma 20 to conclude $M_{\rho,\sigma}$ is a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3 (case I_{28}) of [5] there exists $c = (c_\rho, \ldots, c_\sigma)$ such that $c \preceq 0$ and $M_{\rho,\sigma} c \prec 0$. Define $y := Y_{\rho,\sigma} c \neq 0$. The statement $H^T_{\rho,\sigma} y = M_{\rho,\sigma} c \prec 0$ is equivalent to

$$h_j \cdot y < 0, \qquad j = \rho, \ldots, \sigma\,. \tag{7.11}$$

By (7.10),

$$h_j \cdot y = h_j \cdot (c_\rho b_\rho + \cdots + c_\sigma b_\sigma) = 0\,, \qquad j = 1, \ldots, \rho - 1, \sigma + 1, \ldots, n\,. \tag{7.12}$$

In sum, (7.11)-(7.12) imply $y \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S})$. By Assumption (A2), $y = 0$, a contradiction. □

Lemma 22 tells us that $\rho + r_2 - 1 = r_1 + r_2$ so variable ρ will now be dropped. We can rewrite the list (7.8) as

$$\{b_1, b_2, \ldots, b_{r_1}, b_{r_1+1}, \ldots, b_{r_1+r_2-1}, \ldots, b_{\widehat{m}+1}, b_{\widehat{m}+2}\} \tag{7.13}$$

and each $b_i \in \{b_{r_1+1}, \ldots, b_{r_1+r_2-1}, b_{\widehat{m}+2}\}$ is a strictly negative linear combination of the others. The indices involved in the construction of r_2 (after the reorderings) are $\{r_1 + 1, \ldots, r_1 + r_2 - 1, \widehat{m} + 2\}$. So that these indices are consecutive, we reorder indices again according to the permutation $\tilde{I} = \{1, \ldots, r_1 + r_2 - 1, \widehat{m} + 2, r_1 + r_2 + 1, \ldots, \widehat{m} + 1, r_1 + r_2, \widehat{m} + 3, \ldots, n\}$ of I. That is, we effectively swap the indices $\widehat{m} + 2$ and $r_1 + r_2$. Notice that indices $\{1, \ldots, r_1\}$ are not changed because $r_2 \geq 2$ implies $r_1 + r_2 > r_1 + 1$. Therefore the swapped indices $r_1 + r_2$ and $\widehat{m} + 2$ belong to the index set $I \setminus \{1, \ldots, r_1\}$.

Two indices r_1 and r_2 have been put in place. By iterating on Lemmas 18, 22, and our index swap, we can further decompose $\mathcal{B}$ relative to the cones $\mathcal{B} \cap \mathcal{C}(v_i)$ associated with $\mathcal{O}_{\mathcal{S}}$. Let

$$p := \kappa + 1 - \widehat{m}\,, \quad r := r_1 + \cdots + r_p\,.$$

The procedure generates a specially ordered list of the form

$$\{b_1, \ldots, b_{r_1}, b_{r_1+1}, \ldots, b_{r_1+r_2}, \ldots, b_{r_1+\cdots+r_{p-1}+1}, \ldots, b_r, b_{r+1}, \ldots, b_{\kappa+1}\}\,. \tag{7.14}$$

For each $k = 1, \ldots, p$, $b_{r_1+\cdots+r_k}$ is a strictly negative linear combination of the previous $r_k - 1$ vectors in the list.

Theorem 23. *Suppose Assumption 12 holds. There exist integers $r_1, \ldots, r_p \geq 2$ such that w.l.o.g. (by reordering indices)*

$$\mathcal{B} \cap \mathcal{C}(v_i) \subset \operatorname{sp}\{b_{m_1}, \ldots, b_{m_1+r_1-1}\}\,, \qquad i = m_1, \ldots, m_1 + r_1 - 1\,, \tag{7.15a}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathcal{B} \cap \mathcal{C}(v_i) \subset \operatorname{sp}\{b_{m_p}, \ldots, b_{m_p+r_p-1}\}\,, \qquad i = m_p, \ldots, m_p + r_p - 1\,, \tag{7.15b}$$

where $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$ for $i \in \{1, \ldots, r\}$, $m_1 := 1$, and

$$m_k := r_1 + \cdots + r_{k-1} + 1\,, \qquad k = 2, \ldots, p\,. \tag{7.16}$$

Moreover, for each $k = 1, \ldots, p$, $\{b_{m_k}, \ldots, b_{m_k+r_k-2}\}$ are linearly independent and

$$b_{m_k+r_k-1} = c_{m_k} b_{m_k} + \cdots + c_{m_k+r_k-2} b_{m_k+r_k-2}\,,\; c_i < 0\,, i = m_k, \ldots, m_k + r_k - 2\,. \tag{7.17}$$

Proof. The vectors $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$ in (7.15a)-(7.15b) are provided by (7.14), including those from Lemma 2. Lemma 16 gives (7.17). It remains only to prove (7.15a)-(7.15b). We consider only (7.15a). Consider any $i \in \{1, \ldots, r_1\}$

and any $\beta_i \in \mathcal{B} \cap \mathcal{C}(v_i)$ such that $\beta_i = \alpha_1 b_1 + \cdots + \alpha_{r_1} b_{r_1} + \beta$, where $\alpha_i \in \mathbb{R}$ and $\beta \in \mathcal{B}$. W.l.o.g. we may assume β is independent of $\{b_1, \ldots, b_{r_1}\}$. From (4.2) and Lemma 18, $h_j \cdot \beta_i = h_j \cdot (c_1 b_1 + \cdots + c_{r_1} b_{r_1} + \beta) = h_j \cdot \beta \leq 0$, for $j = r_1 + 1, \ldots, n$. By the proof of Proposition 7.2 in [7], $\beta = 0$. Hence, for any $i \in \{1, \ldots, r_1\}$ and $\beta_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $\beta_i \in \mathrm{sp}\{b_1, \ldots, b_{r_1}\}$, as desired. ◻

REMARK 24. *A number of relationships between the integers κ, $\widehat{m}$, p, and r are implied by our construction. The following $\sum_{1=1}^{p} (r_i - 1) = r - p$ vectors are linearly independent:*

$$\{b_1, \ldots, b_{r_1 - 1}, b_{r_1 + 1}, \ldots, b_{r_1 + r_2 - 1}, \ldots, b_{r_1 + \cdots + r_{p-1} + 1}, \ldots, b_{r-1}\}\,.$$

Therefore, $\widehat{m} \geq r - p$. Also, $r_i \geq 2$, $i = 1, \ldots, p$, so $r \geq 2p$. Combining these two inequalities we have $\widehat{m} \geq p = \kappa + 1 - \widehat{m}$. We conclude that

$$\widehat{m} \geq \frac{\kappa + 1}{2}\,.$$

This condition is interpreted to say that RCP is only solvable if there are sufficient inputs.

The integers $\{r_1, \ldots, r_p\}$ are called the *reach control indices* of system (4.1) with respect to simplex $\mathcal{S}$.

8. Piecewise Affine Feedback. The reach control indices catalog the degeneracies (caused by insufficient inputs) that lead to the appearance of equilibria in $\mathcal{S}$ whenever $p \geq 1$ and continuous state feedback is applied. Thus, any control method that overcomes the limits of continuous state feedback must confront this degeneracy and will necessarily draw upon the degrees of freedom in $\mathcal{B}$ provided to $\mathcal{O}_\mathcal{S}$ which are inscribed by the indices. In this section we investigate the extent to which piecewise affine feedback can solve RCP, in cases when continuous state feedback cannot. We construct a triangulation [18] of the simplex $\mathcal{S}$ such that RCP is solvable for each simplex of the triangulation. The next result shows that because of condition (iii) of Problem 1 a situation like the one in Figure 4.3(c) cannot happen. Correspondingly one recovers a third necessary condition for solvability of RCP by open-loop controls - in essence saying that $\mathcal{B}$ cannot be parallel to $\mathcal{F}_0$.

LEMMA 25. *Suppose Assumption 12 and* (7.15)-(7.17) *hold. If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls, then* $\mathrm{sp}\{b_{m_k}, \ldots, b_{m_k + r_k - 2}\} \not\subset \mathcal{H}_0 := \{y \in \mathbb{R}^n \mid h_0 \cdot y = 0\}$ *for each $k = 1, \ldots, p$.*

Proof. W.l.o.g. we consider only $k = 1$. Define $\widehat{\mathcal{F}}_0 := \mathrm{co}\{v_1, \ldots, v_{r_1}\} \subset \mathcal{F}_0$. Let $\{\mu_x\}$ be open-loop controls satisfying (i)-(iii) of Definition 8. Let $x \in \mathcal{S}$ and consider any $t \in [0, T(x)]$ such that $\phi_{\mu_x}(t, x) \in \widehat{\mathcal{F}}_0$. First, by condition (iii) of Definition 8, $h_l \cdot (A\phi_{\mu_x}(t, x) + B\mu_x(t) + a) \leq 0$ for $l \in I$, $\phi_{\mu_x}(t, x) \in \mathcal{F}_l$. Second, let $A\phi_{\mu_x}(t, x) + B\mu_x(t) + a = \alpha_1 b_1 + \cdots + \alpha_{r_1} b_{r_1} + \beta$, where $\alpha_i \in \mathbb{R}$ and $\beta \in \mathcal{B}$. By the same argument as in Theorem 23, $\beta = 0$. Then by Lemma 18, $h_j \cdot (A\phi_{\mu_x}(t, x) + B\mu_x(t) + a) = 0$ for $j = r_1 + 1, \ldots, n$. Suppose by way of contradiction that $\mathrm{sp}\{b_1, \ldots, b_{r_1}\} \subset \mathcal{H}_0$.

Then $h_0 \cdot (A\phi_{\mu_x}(t,x) + B\mu_x(t) + a) = 0$. On the other hand, for $z \in \widehat{\mathcal{F}}_0$, $T_{\widehat{\mathcal{F}}_0}(z) = \{y \in \mathbb{R}^n \mid h_j \cdot y = 0, h_l \cdot y \leq 0, j = 0, r_1+1, \ldots, n, l \in I \ s.t.\ z \in \mathcal{F}_l\}$. We conclude that for all $x \in \mathcal{S}$ and $t \in [0, T(x)]$, if $\phi_{\mu_x}(t,x) \in \widehat{\mathcal{F}}_0$, then $A\phi_{\mu_x}(t,x) + B\mu_x(t) + a \in T_{\widehat{\mathcal{F}}_0}(x)$. Using uniqueness of solutions, we obtain $\widehat{\mathcal{F}}_0$ is a positively invariant set, a contradiction. ◻

DEFINITION 26. *Given system* (4.1) *and a state feedback* $u = f(x)$, *we say* $f(x)$ *is a* piecewise affine feedback *if there exists a triangulation* $\mathbb{T}$ *of* $\mathcal{S}$ *such that for each* n*-dimensional* $\mathcal{S}^j \in \mathbb{T}$, *there exist* $K^j \in \mathbb{R}^{m \times n}$ *and* $g^j \in \mathbb{R}^m$ *such that* $f(x) = K^j x + g^j$, $x \in \mathcal{S}^j$.

This definition of piecewise affine feedback allows for discontinuities at the boundaries of simplices; moreover, the feedback is a multi-valued function, distinct from the usual notion in algebraic topology where piecewise affine functions are single-valued and continuous [22]. Resolving what control value to use at points lying in more than one simplex is treated as a problem of implementation. The artifact of a *discrete supervisory controller* [24] will be introduced to convert the multi-valued function to a single-valued feedback.

We now explain informally an inductive procedure for subdividing $\mathcal{S}$ in order that RCP can be solved by piecewise affine feedback. First, in Lemma 27 we show that because of Lemma 25, each simplex $\mathrm{co}\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$, $k = 1, \ldots, p$, has a vertex v_i (among $\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$) with $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$ pointing out of $\mathcal{S}$. By convention, we reorder indices so this vertex is the first one in each list $\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$. We make a subdivision of $\mathcal{S}$ by placing a new vertex v' along the edge (v_0, v_{m_k}). In particular, at the first iteration we would have $v' \in (v_0, v_1)$, and we form two simplices $\mathcal{S}^1$ and $\mathcal{S}'$ as in Figure 8.1. Lemma 29 shows that because $b_{m_k} \in \mathcal{B} \cap \mathcal{C}(v_{m_k})$ points out of $\mathcal{S}$ at v_{m_k} and because the invariance conditions for $\mathcal{S}$ are solvable at v_0, a convexity argument (precisely, (8.3)) gives that v' can be placed along (v_0, v_{m_k}) so that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1) \neq \mathbf{0}$. Finally one applies Theorem 6.2 of [7] to obtain that RCP is solved for $\mathcal{S}^1$. Essentially $\mathcal{S}^1$ can be removed from further consideration, and the induction step is repeated with $\mathcal{S}$ replaced by the remainder $\mathcal{S}'$. See Figure 8.2. To guarantee that the induction is sound, one must show that $\mathcal{S}'$ inherits the relevant properties of $\mathcal{S}$, especially the property of Lemma 25. This is done in Lemma 30. Lemmas 27-30 demonstrate the first step of a triangulation algorithm that partitions $\mathcal{S}$ into a set of $p+1$ simplices. Each step of the algorithm will correspond to one reach control index. The algorithm is therefore guaranteed to terminate with a finite partition.

LEMMA 27. *Suppose Assumption 12 and* (7.15a)-(7.17) *hold. Then w.l.o.g. (by reordering indices* $\{m_k, \ldots, m_k + r_k - 1\}$*),* $h_0 \cdot b_{m_k} > 0$, $k = 1, \ldots, p$.

Proof. We prove the result only for $k = 1$. If for some $i \in \{1, \ldots, r_1\}$, $h_0 \cdot b_i > 0$, then the proof is finished. Instead suppose that for all $i \in \{1, \ldots, r_1\}$, $h_0 \cdot b_i \leq 0$. Using Lemma 25 and by reordering the indices $1, \ldots, r_1$, assume $h_0 \cdot b_{r_1} < 0$. By (7.17), $b_1 = \frac{1}{c_1}(b_{r_1} - c_2 b_2 - \cdots - c_{r_1-1} b_{r_1-1})$ with $c_i < 0$.

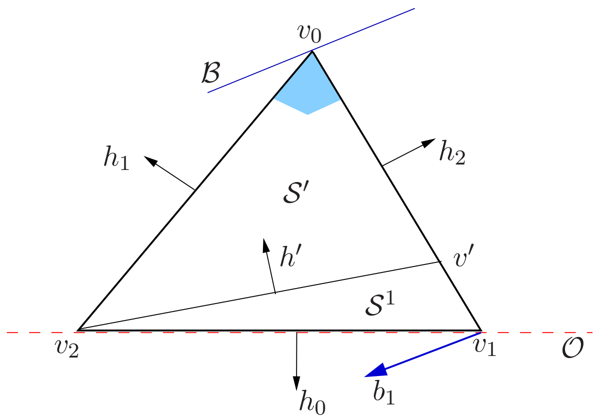


FIG. 8.1. *Subdivision into two simplices $\mathcal{S}'$ and $\mathcal{S}^1$.*

Thus we obtain

$$h_0 \cdot b_1 = h_0 \cdot \frac{1}{c_1} (b_{r_1} - c_2 b_2 - \cdots - c_{r_1 - 1} b_{r_1 - 1}) \geq \frac{1}{c_1} h_0 \cdot b_{r_1} > 0 \,.$$

□

EXAMPLE 28. *Lemmas 25 and 27 are illustrated in Figure 8.1 for a 2D example. We have $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{v_1, v_2\}$, $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$, and $\mathcal{B} \cap \mathcal{C}(v_i) \neq \mathbf{0}$, $i = 1, 2$, as required by Assumption 12. We observe that $\mathcal{B}$ is not parallel to $\mathcal{F}_0$. Otherwise, the only way for trajectories to exit $\mathcal{F}_0$ would be by violating the invariance conditions at v_1 or v_2 as depicted in Figure 4.3(c). Therefore, $\mathcal{B}$ cannot be parallel to $\mathcal{F}_0$. This is the essence of Lemma 25. Next, since $\mathcal{B}$ is not parallel to $\mathcal{F}_0$ there is $b_1 \in \mathcal{B} \cap \mathcal{C}(v_1)$ that points out of $\mathcal{S}$. This is the content of Lemma 27.*

Following Lemma 27, suppose that b_1 satisfies $h_0 \cdot b_1 > 0$. We consider any point v' in the open segment (v_0, v_1). That is, let $\lambda \in (0, 1)$ and define

$$v' = \lambda v_1 + (1 - \lambda) v_0 \,. \tag{8.1}$$

Now define the following simplices in $\mathcal{S}$:

$$\begin{aligned} \mathcal{S}' &= \mathrm{co}\{v_0, v', v_2, \ldots, v_n\} \\ \mathcal{S}^1 &= \mathrm{co}\{v', v_1, v_2, \ldots, v_n\} \,. \end{aligned}$$

Also define the new exit facet for $\mathcal{S}'$ by $\mathcal{F}_0' := \mathrm{co}\{v', v_2, \ldots, v_n\}$. See Figure 8.1. Suppose that $h_0 = -\gamma_1 h_1 - \ldots - \gamma_n h_n$ with $\gamma_i > 0$, and let $\lambda \in (0, 1)$. Then the normal vector to $\mathcal{F}_0'$ pointing out of $\mathcal{S}^1$ is

$$h' = \gamma_1 h_1 + \lambda \sum_{j=2}^{n} \gamma_j h_j = \gamma_1 (1 - \lambda) h_1 - \lambda h_0 \,. \tag{8.2}$$

For the next result we observe that $\mathrm{cone}(\mathcal{S}^1) = \{y \in \mathbb{R}^n \mid h' \cdot y \leq 0\,, h_j \cdot y \leq 0, \quad j \in \{2, \cdots, n\}\}$.

LEMMA 29. *Suppose Assumption 12 and* (7.15a)-(7.17) *hold. There exists* $v' \in (v_0, v_1)$*, such that* $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1) \neq \mathbf{0}$*. Moreover,* $b_1 \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1)$ *with* $h' \cdot b_1 < 0$*. If the invariance conditions for* $\mathcal{S}$ *are solvable, then* $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ *by affine feedback.*

Proof. We show there is an interval of values for λ such that $0 \neq b_1 \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1)$, where we assume the index ordering of Lemma 27 so that $h_0 \cdot b_1 > 0$. First, since $b_1 \in \mathcal{B} \cap \mathcal{C}(v_1)$ we know $h_j \cdot b_1 \leq 0$ for $j \in \{2, \ldots, n\}$. We must only show that there exists $\lambda \in (0, 1)$ such that $h' \cdot b_1 < 0$. Using (8.2) we have

$$h' \cdot b_1 = \gamma_1 (1 - \lambda) h_1 \cdot b_1 - \lambda h_0 \cdot b_1 \,. \tag{8.3}$$

Since $h_0 \cdot b_1 > 0$ (by Lemma 27), it is clear from (8.3) that we can select $\lambda = \lambda'$ sufficiently close to 1 such that $h' \cdot b_1 < 0$. Setting $v' = \lambda' v_1 + (1 - \lambda') v_0$, we get $b_1 \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1)$.

Next, we show that $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback. By assumption, $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1) \neq \mathbf{0}$. We show that the invariance conditions are solvable for $\mathcal{S}^1$. First, consider the vertex v'. Since the invariance conditions for $\mathcal{S}$ are solvable, there exist control inputs $u_0, u_1 \in \mathbb{R}^m$ such that the invariance conditions for $\mathcal{S}$ at v_0 and v_1 are satisfied, i.e. $y_0 := A v_0 + B u_0 + a \in \mathrm{cone}(\mathcal{S})$ and $y_1 := A v_1 + B u_1 + a \in \mathcal{B} \cap \mathcal{C}(v_1)$. In particular, $h_j \cdot y_i \leq 0$ for $i = 0, 1$ and $j = 2, \ldots, n$. Also from above, $h_j \cdot b_1 \leq 0$ for $j = 2, \ldots, n$. Let w_1 be such that $b_1 = B w_1$. Set $\epsilon_1 > 0$ and let $u' := \lambda u_1 + (1 - \lambda) u_0 + \epsilon_1 w_1$. Then $y' := A v' + B u' + a = \lambda y_1 + (1 - \lambda) y_0 + \epsilon_1 b_1$. Thus, $h_j \cdot y' \leq 0$ for $j = 2, \ldots, n$ and for $\epsilon_1 > 0$ sufficiently large, $h' \cdot y' < 0$. That is, the invariance conditions for $\mathcal{S}^1$ are solvable at v'.

Next consider v_1. Since the invariance conditions for $\mathcal{S}^1$ at v_1 are identical to those for $\mathcal{S}$ at v_1, and since the latter are by assumption solvable, the former are also solvable. Finally, consider vertices v_i, $i = 2, \ldots, n$. There exist control inputs $u_i \in \mathbb{R}^m$ such that $y_i := A v_i + B u_i + a$ satisfy $h_j \cdot y_i \leq 0$ for $j = 2, \ldots, i-1, i+1, \ldots, n$. As above let w_1 be such that $b_1 = B w_1$. Set $\epsilon_1 > 0$ and let $u_i' := u_i + \epsilon_1 w_1$. Then $y_i' = A v_i + B u_i' + a = y_i + \epsilon_1 b_1$. Thus, $h_j \cdot y_i' \leq 0$ for $j = 2, \ldots, i-1, i+1, \ldots, n$ and for $\epsilon_1 > 0$ sufficiently large, $h' \cdot y_i' < 0$. That is, the invariance conditions for $\mathcal{S}^1$ are solvable at v_i. In sum, we can apply Theorem 6.2 of [7] to obtain that $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback. ∎

LEMMA 30. *Suppose Assumption 12 and* (7.15a)-(7.17) *hold. Let* v' *be as in Lemma 29. If the invariance conditions for* $\mathcal{S}$ *are solvable then*

(i) The invariance conditions for $\mathcal{S}'$ *are solvable.*

(ii) $(-h') \cdot b_{m_k} > 0\,, \quad k = 1, \ldots, p.$

Proof. First we prove (i). By assumption the invariance conditions for $\mathcal{S}$ are solvable, and since the invariance conditions for $\mathcal{S}'$ are identical (the only facet that changed for $\mathcal{S}'$ is $\mathcal{F}_0$, which plays no role in invariance conditions),

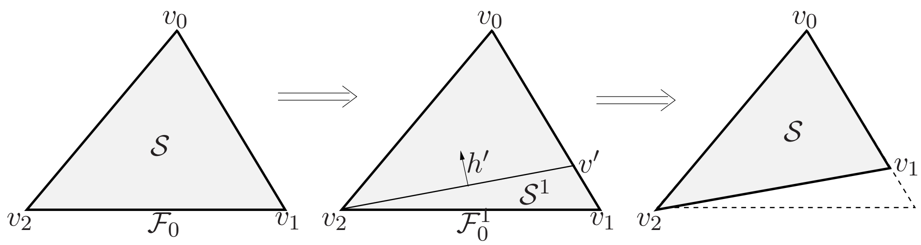


Fig. 8.2. *Notation for the subdivision algorithm.*

they are also solvable for $\mathcal{S}'$. Next we prove (ii). First we have $(-h') \cdot b_{m_1} > 0$ by Lemma 29. Second, since $b_{m_k} \in \mathcal{B} \cap \mathcal{C}(v_{m_k})$, we have $h_1 \cdot b_{m_k} \leq 0$, for $k = 2, \ldots, p$. Also by Lemma 27, $h_0 \cdot b_{m_k} > 0$, for $k = 2, \ldots, p$. Thus using (8.2), $(-h') \cdot b_{m_k} = -\gamma_1(1-\lambda)h_1 \cdot b_{m_k} + \lambda h_0 \cdot b_{m_k} > 0$, $k = 2, \ldots, p$. □

We have demonstrated the first step of a triangulation procedure that partitions $\mathcal{S}$ into simplices on each of which a reach control problem is solvable. Now we present a triangulation algorithm that iterates on the presented subdivision method. It consists of p iterations, one for each set $\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$, $k = 1, \ldots, p$. The notation $\mathcal{S}^k := \text{co}\{v', v_1, \ldots, v_n\}$ is understood to mean that all $n+1$ vertices of $\mathcal{S}^k$ are assigned simultaneously in the order presented. The vertices of $\mathcal{S}^k$ are later identified as $\{v_0^k, \ldots, v_n^k\}$. The algorithm generates simplices $\mathcal{S}^1, \ldots, \mathcal{S}^{p+1}$ starting from the given simplex $\mathcal{S}$. At the kth iteration, the current declaration of $\mathcal{S}$ is split into a lower simplex $\mathcal{S}^k$ and an upper simplex. The lower simplex is then "thrown away" and the remainder - the upper simplex - is declared to be $\mathcal{S}$ with vertices called $\{v_0, \ldots, v_n\}$ (overloading the vertices of the previous $\mathcal{S}$). See Figure 8.2. In this way each iterate mimics the first subdivision developed in the discussion above.

Subdivision Algorithm:

1. Set $k = 1$.
2. Select $v' \in (v_0, v_{m_k})$ such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^k) \neq \mathbf{0}$, where $\mathcal{S}^k := \text{co}\{v', v_1, \ldots, v_n\}$.
3. Set $\mathcal{S} := \text{co}\{v_0, v_1, \ldots, v_{m_k-1}, v', v_{m_k+1}, \ldots, v_n\}$.
4. If $k < p$, set $k := k+1$ and go to step 2.
5. Set $\mathcal{S}^{p+1} := \mathcal{S}$.

Example 31. *Consider the output of the subdivision algorithm for an example with $p = 3$:*

- $\mathcal{S}^1 := \{v_0^1, v_{m_1}, \ldots, v_n\}$ *where* $v_0^1 \in (v_0, v_{m_1})$.
- $\mathcal{S}^2 := \{v_0^2, v_0^1, v_{m_1+1}, \ldots, v_n\}$ *where* $v_0^2 \in (v_0, v_{m_2})$.
- $\mathcal{S}^3 := \{v_0^3, v_0^1, v_{m_1+1}, \ldots, v_0^2, v_{m_2+1}, \ldots, v_n\}$ *where* $v_0^3 \in (v_0, v_{m_3})$.
- $\mathcal{S}^4 := \{v_0, v_0^1, v_{m_1+1}, \ldots, v_0^2, v_{m_2+1}, \ldots, v_0^3, v_{m_3+1}, \ldots, v_n\}$.

From this example we observe several features:

- For each $k = 1, \ldots, p$, we have $v_0^k \in \mathcal{S}^k \cap \cdots \cap \mathcal{S}^{p+1}$ and $v_{m_k} \in \mathcal{S}^1 \cap \cdots \cap \mathcal{S}^k$.
- Simplex $\mathcal{S}^{p+1}$ has all the same vertices as $\mathcal{S}$ except that the vertices $v_{m_1}, \ldots, v_{m_p}$ have been replaced by new vertices $v_0^1, \ldots, v_0^p$, respec-

tively. In particular, v_0 is a vertex of both $\mathcal{S}$ and $\mathcal{S}^{p+1}$. This means that the only difference between $\mathcal{S}$ and $\mathcal{S}^{p+1}$ is the original exit facet $\mathcal{F}_0 = \text{co}\{v_1, \dots, v_n\}$ of $\mathcal{S}$ has been modified for $\mathcal{S}^{p+1}$ to be $\mathcal{F}_0^{p+1} = \{v_0^1, v_{m_1+1}, \dots, v_{m_1+r_1-1}, \dots, v_0^p, v_{m_p+1}, \dots, v_{m_p+r_p-1}, v_{r+1}, \dots, v_n\}$.

- Because of Assumption (A1) and the fact that $v_0^k \in (v_0, v_{m_k})$, we have $v_0^1, \dots, v_0^p \notin \mathcal{O}$.
- Because of the previous two properties, $\mathcal{O}_{\mathcal{S}}^{p+1} := \mathcal{S}^{p+1} \cap \mathcal{O}$ has dropped in dimension to $\kappa - p = \widehat{m} - 1$ because p vertices originally in $\mathcal{O}$ have been removed from $\mathcal{S}^{p+1}$.

Let $\mathcal{F}_0^k = \text{co}\{v_1^k, \dots, v_n^k\}$ denote the exit facet of $\mathcal{S}^k = \text{co}\{v_0^k, \dots, v_n^k\}$. The triangulation generated by the algorithm has the property that $\mathcal{S}^k \cap \mathcal{S}^{k-1} = \mathcal{F}_0^k$, $k = 2, \dots, p+1$, and closed-loop trajectories follow paths through simplices with decreasing indices. Thus, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ is achieved by implementing affine controllers that achieve $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$ for $k = 1, \dots, p+1$. In order to guarantee that switching occurs in the proper sequence (with decreasing simplex indices), a *discrete supervisor* should accompany the implementation of the piecewise affine feedback. The supervisor enforces the following rule:

(DS) At a point $x \in \mathcal{S}$ belonging to more than one simplex $\mathcal{S}^j$, the controller for the simplex with the highest index is used.

REMARK 32. *The rule (DS) is imposed to guarantee that condition (iii) of Problem 1 is met. For example, in the center figure of Figure 8.2, the controller for the upper simplex would be applied at the points along the segment from v_2 to v'. In particular, at v_2 that controller would satisfy invariance conditions at v_2 for the simplex $\mathcal{S}$, whereas the controller for simplex $\mathcal{S}^1$ violates the invariance conditions of $\mathcal{S}$ at v_2.*

THEOREM 33. *Suppose Assumption 12 and* (7.15a)-(7.17) *hold. If the invariance conditions for $\mathcal{S}$ are solvable, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by piecewise affine feedback.*

Proof. Form the triangulation $\{\mathcal{S}^1, \dots, \mathcal{S}^{p+1}\}$ of $\mathcal{S}$ based on the Subdivision Algorithm. We show by induction that $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$ by affine feedback for $k = 1, \dots, p$ (momentarily ignoring the rule (DS)). For the initial step, by assumption the invariance conditions for $\mathcal{S}$ are solvable and by Lemma 27, $h_0 \cdot b_{m_k} > 0$ for $k = 1, \dots, p$. Thus, by Lemma 29, $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback. Now assume that at the jth step the invariance conditions are solvable for (the current) $\mathcal{S}$ and $h_0 \cdot b_{m_k} > 0$ for $k = 1, \dots, p$. Then by Lemma 29, $\mathcal{S}^j \xrightarrow{\mathcal{S}^j} \mathcal{F}_0^j$ by affine feedback. Now consider the $(j+1)$th step. By the algorithm $\mathcal{S} := \text{co}\{v_0, v_1, \dots, v_{m_j-1}, v', v_{m_j+1}, \dots, v_n\}$ and $h_0 = -h'$, where v' and h' are provided by the jth step. By Lemma 30, the invariance conditions are solvable for $\mathcal{S}$ and $h_0 \cdot b_{m_k} > 0$ for $k = 1, \dots, p$. Then by Lemma 29, $\mathcal{S}^{j+1} \xrightarrow{\mathcal{S}^{j+1}} \mathcal{F}_0^{j+1}$ by affine feedback.

Next consider $\mathcal{S}^{p+1}$. We observe that $\mathcal{S}^{p+1}$ and $\mathcal{S}$ share the same invariance conditions since they only differ in their exit facets, so the invariance conditions for $\mathcal{S}^{p+1}$ are solvable. Now let $\mathcal{O}_{\mathcal{S}}^{p+1} := \mathcal{S}^{p+1} \cap \mathcal{O}$. Then by the algorithm, $\mathcal{O}_{\mathcal{S}}^{p+1} = \text{co}\{v_2, \ldots, v_{m_2-1}, v_{m_2+1}, \ldots, v_{m_p-1}, v_{m_p+1}, \ldots, v_{\kappa+1}\}$. We can see that the algorithm has removed the p vertices $v_{m_1}, v_{m_2}, \ldots, v_{m_p}$ from $\mathcal{O}_{\mathcal{S}}$. There remain $\widehat{m}$ linearly independent vectors in $\mathcal{B}$ associated with $\mathcal{O}_{\mathcal{S}}^{p+1}$ (an $(\widehat{m}-1)$-dimensional simplex) given by $\{b_2, \ldots, b_{m_2-1}, b_{m_2+1}, \ldots, b_{m_p-1}, b_{m_p+1}, \ldots, b_{\kappa+1}\}$. Therefore, we can apply Theorem 6.7 of [7] to obtain $\mathcal{S}^{p+1} \xrightarrow{\mathcal{S}^{p+1}} \mathcal{F}_0^{p+1}$.

Next, we must prove that trajectories progress through simplices with decreasing indices only. Consider w.l.og. the boundary between $\mathcal{S}^1$ and $\mathcal{S}^2$ given by $\mathcal{F}_0^2 = \text{co}\{v', v_2, \ldots, v_n\}$, and let $u = K_1 x + g_1$ be the affine feedback obtained for $\mathcal{S}^1$. We show that for any $x_0 \in \mathcal{S}^1 \setminus \mathcal{F}_0^2$, closed-loop trajectories do not reach $\mathcal{F}_0^2$. This in turn means that trajectories never return to $\mathcal{S}^2$ from $\mathcal{S}^1$ after leaving $\mathcal{S}^2$. This can be deduced from the proof of Lemma 29 where it is shown that the controls $\{u', u_2, \ldots, u_n\}$ can be selected so that $h' \cdot (Av' + Bu' + a) < 0$ and $h' \cdot (Av_i + Bu_i + a) < 0$, $i = 2, \ldots, n$. Since $x \mapsto (A + BK_1)x + Bg_1 + a$ is an affine function, $\mathcal{F}_0^2$ is a convex set, and the cones $\mathcal{C}(x)$ are convex, $h' \cdot (Ax + B(K_1 x + g_1) + a) < 0$ for all $x \in \mathcal{F}_0^2$, from which the result easily follows.

Finally we verify conditions (ii) and (iii) of RCP. Condition (ii) follows immediately because there is a finite number of affine feedbacks each defined on a compact set $\mathcal{S}^k$ that does not contain an equilibrium. For (iii) we must verify that the piecewise affine feedback $u = f(x)$ resulting from (DS) satisfies (4.3). We show that it satisfies (4.2) and therefore also (4.3). First consider $\mathcal{S}^{p+1}$. Its exit facet is

$$\mathcal{F}_0^{p+1} = \{v_0^1, v_{m_1+1}, \ldots, v_{m_1+r_1-1}, \ldots, v_0^p, v_{m_p+1}, \ldots, v_{m_p+r_p-1}, v_{r+1}, \ldots, v_n\}.$$

The invariance conditions for $\mathcal{S}^{p+1}$ are identical to those for $\mathcal{S}$ and the controller for $\mathcal{S}^{p+1}$ takes precedence over controllers for simplices with lower index. This implies the invariance conditions for $\mathcal{S}$ hold at v_0 and all vertices of $\mathcal{F}_0^{p+1}$. The only vertices of $\mathcal{F}_0$ that are not in $\mathcal{F}_0^{p+1}$ are $v_{m_1}, v_{m_2}, \ldots, v_{m_p}$. For these vertices we have: $v_{m_1} \in \mathcal{S}^1$, $v_{m_2} \in \mathcal{S}^1 \cap \mathcal{S}^2$,...,$v_{m_p} \in \mathcal{S}^1 \cap \cdots \cap \mathcal{S}^p$. We use the affine controller for the simplex with the highest index. But the invariance conditions for $\mathcal{S}^k$ at v_{m_k} are precisely those for $\mathcal{S}$. We can see this because the invariance conditions for v_{m_k} do not include the normal vector $-h'$ given in (8.2). ∎

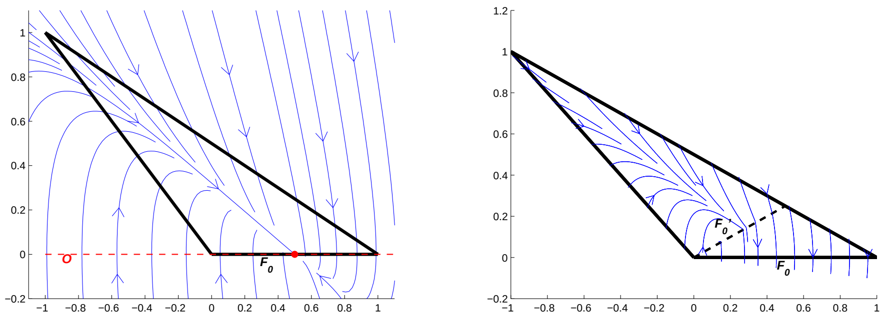
9. Examples.

9.1. Example 1.

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Safety constraints on both x_1 and x_2 determine a polyhedral state space within which the dynamics evolve. The polyhedral state space is triangulated according to Assumption 11. We focus on the reach control problem



(a) (b)

FIG. 9.1. *Closed-loop vector fields using (a) affine feedback and (b) piecewise affine feedback.*

for a specific simplex of the triangulation: consider the simplex $\mathcal{S}$ determined by vertices $v_0 = (-1, 1)$, $v_1 = (1, 0)$ and $v_2 = (0, 0)$. It can be verified that $\mathcal{O} = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$, $\mathcal{O}_{\mathcal{S}} = \text{co}\{v_1, v_2\}$, $\kappa = 1$, and $\widehat{m} = m = 1$. Also $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. By the results of [7], RCP is not solvable by continuous state feedback. For example, suppose we choose control values $u_0 = -\frac{3}{4}$, $u_1 = -1$, and $u_2 = 1$ to satisfy the invariance conditions (4.2). By the method in [16], this yields an affine feedback $u = \begin{bmatrix} -2 & -3.75 \end{bmatrix} x + 1$. Simulation of the closed-loop system is shown in Figure 9.1(a). We observe there exists a closed-loop equilibrium point on $\mathcal{O}_{\mathcal{S}}$. Now we show the problem is solvable by piecewise affine feedback.

Let $b_1 = (0, -1) \in \mathcal{B} \cap \mathcal{C}(v_1)$. Since $h_0 = (0, -1)$, we have $h_0 \cdot b_1 > 0$, verifying Lemma 27. Next, we choose $v' = (0.5, 0.25)$ along the simplex edge (v_0, v_1) such that from (8.2), $h' = (-0.25, 0.5)$. Then $h' \cdot b_1 < 0$ and $b_1 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^1)$, verifying Lemma 29. Let $\mathcal{S}^1 := \text{co}\{v', v_1, v_2\}$, $\mathcal{S}^2 := \text{co}\{v_0, v', v_2\}$, and $\mathcal{F}'_0 = \text{co}\{v', v_2\}$. To satisfy the invariance conditions for $\mathcal{S}^1$ we choose control inputs at the vertices to be $u' = -1$, $u_1 = -1$, and $u_{12} = -1$. Similarly, for $\mathcal{S}^2$ we choose $u_0 = -\frac{3}{4}$, $u' = -1$, and $u_{22} = 1$. The piecewise affine feedback is

$$u := \begin{cases} \begin{bmatrix} 0 & 0 \end{bmatrix} x - 1\,, & x \in \mathcal{S}^1 \\ \begin{bmatrix} -2.0833 & -3.833 \end{bmatrix} x + 1\,, & x \in \mathcal{S}^2\,. \end{cases}$$

By Theorem 6.2 of [7], $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ using u. Because $\mathcal{O}^2_{\mathcal{S}} := \mathcal{S}^2 \cap \mathcal{O} = \{v_2\}$, we have $\widehat{m}^2 = 1$ and $\kappa^2 = 0$ for $\mathcal{S}^2$. By Theorem 6.2 of [7], $\mathcal{S}^2 \xrightarrow{\mathcal{S}^2} \mathcal{F}'_0$ using u. The closed-loop vector field is shown in Figure 9.1(b), where it is clear that RCP is solved.

9.2. Example 2. Consider the simplex $\mathcal{S}$ in $\mathbb{R}^4$ defined by the vertices $v_0 = (0, 0, 0, 0)$, $v_1 = (1, 0, 0, 0)$, $v_2 = (0, 1, 0, 0)$, $v_3 = (0, 0, 1, 0)$, and $v_4 =$

$(0,0,0,1)$. Consider the system

$$\dot{x} = \begin{bmatrix} -3 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \\ -3 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & -2 \\ 0 & 1 \\ -2 & 0 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We compute $\mathcal{O} = \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 - 1 = 0\}$. Thus, $\mathcal{O}_{\mathcal{S}} = \mathcal{F}_0$, and we note that $\kappa = 3$, $\widehat{m} = m = 2$, and $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. By the results of [7], RCP is not solvable by continuous state feedback. Now we show it is solvable by piecewise affine feedback. First we examine the structure of $\mathcal{B}$ (note that indices are not reordered, as is the convention in our proofs). We find by inspect that $b_1 := (-2,1,0,0) \in \mathcal{B} \cap \mathcal{C}(v_1)$, $b_3 := (0,0,-2,1) \in \mathcal{B} \cap \mathcal{C}(v_3)$, and $\mathcal{B} = \text{sp}\{b_1, b_3\}$. In particular, $b_2 := -b_1 \in \mathcal{B} \cap \mathcal{C}(v_2)$ and $b_4 := -b_3 \in \mathcal{B} \cap \mathcal{C}(v_4)$. Thus, $r_1 = 2$ and $r_2 = 2$.

9.2.1. First subdivision. In the first iteration $\mathcal{S}$ is subdivided into simplices $\mathcal{S}^1$ and $\mathcal{S}'$. Since $b_2 \cdot h_0 > 0$, we choose $v' = (0, 0.75, 0, 0) \in (v_0, v_2)$ such that we obtain the condition $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq \mathbf{0}$. Hence $\mathcal{S}' = \text{conv}\{v_0, v_1, v', v_3, v_4\}$ and $\mathcal{S}^1 = \text{conv}\{v', v_1, v_2, v_3, v_4\}$. In order to satisfy the invariance conditions for $\mathcal{S}^1$ the control inputs at the vertices of $\mathcal{S}^1$ are chosen as $u' = (-1,-2)$, $u_{11} = (-1,-2)$, $u_{12} = (-1,-2)$, $u_{13} = (-1,-2)$, and $u_{14} = (1,0)$. This yields an affine feedback

$$u := \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \qquad x \in \mathcal{S}^1.$$

For $\mathcal{S}^1$ the invariance conditions are solvable and $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq \mathbf{0}$, so by Theorem 6.2 of [7], $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ using u. For $\mathcal{S}'$ we have $\mathcal{O}'_{\mathcal{S}} := \mathcal{S}' \cap \mathcal{O} = \text{co}\{v_1, v_3, v_4\}$. Since $\kappa' = 2$ and $m = 2$, RCP is not solvable by continuous state feedback on $\mathcal{S}'$, and further subdivision of $\mathcal{S}'$ is required.

9.2.2. Second subdivision. Consider the simplex $\mathcal{S}' = \text{co}\{v_0, v_1, v', v_3, v_4\}$, where $v' \in (v_0, v_2) = (0, 0.75, 0, 0)$ and the exit facet is $\mathcal{F}'_0 = \text{conv}\{v_1, v', v_3, v_4\}$. We subdivide $\mathcal{S}'$ into simplices $\mathcal{S}^3$ and $\mathcal{S}^2$ and use a piecewise affine feedback law to solve RCP on $\mathcal{S}'$. It is clear that $b_4 \cdot h'_0 > 0$ and therefore we can choose $v'' \in (v_0, v_4)$ such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^2) \neq \mathbf{0}$. One choice is $v'' := (0,0,0,0.8)$. Let $\mathcal{S}^3 = \text{co}\{v_0, v_1, v', v_3, v''\}$ and $\mathcal{S}^2 = \text{co}\{v'', v_1, v', v_3, v_4\}$. It can be verified that $b_4 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^2)$. To satisfy the invariance conditions for $\mathcal{S}^2$ we choose $u'' = (-4, 0.6)$, $u_{21} = (-5,-1)$, $u' = (-1,-2)$, $u_{23} = (-5,-1)$, and $u_{24} = (-3,1)$. To satisfy the invariance conditions for $\mathcal{S}^3$ we choose $u_0 = (0,0)$, $u_{31} = (-1,0)$, $u' = (-1,-2)$, $u_{33} = (0,-1)$, and $u'' = (-4, 0.6)$. This yields a piecewise affine feedback

$$u = \begin{cases} \begin{bmatrix} -1 & -1.33 & 0 & -5 \\ 0 & -2.66 & -1 & 0.75 \end{bmatrix} x, & x \in \mathcal{S}^3 \\ \begin{bmatrix} 3 & 9.33 & 3 & 5 \\ 0 & -1.33 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} -8 \\ -1 \end{bmatrix}, & x \in \mathcal{S}^2. \end{cases}$$

For $\mathcal{S}^2$ the invariance conditions are solvable and $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}^2) \neq \mathbf{0}$, so by Theorem 6.2 of [7], $\mathcal{S}^2 \xrightarrow{\mathcal{S}^2} \mathcal{F}_0'$ using u. For $\mathcal{S}^3$ we have $\mathcal{O}_{\mathcal{S}}{}^3 := \mathcal{S}^3 \cap \mathcal{O} = \mathrm{co}\{v_1, v_3\}$. Since $\kappa^3 = 1$ and $\widehat{m}^3 = 2$, by Theorem 6.7 of [7], $\mathcal{S}^3 \xrightarrow{\mathcal{S}^3} \mathcal{F}''$ using u. Indeed, $\{b_1, b_3 \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ is a linearly independent set associated with $\mathcal{O}_{\mathcal{S}}{}^3$.

10. Conclusion. The paper studies the reach control problem on simplices, and we investigate cases when the problem is not solvable by continuous state feedback. It is shown that the class of piecewise affine feedbacks is sufficient to solve the problem in all cases of interest; namely, those cases when the problem is solvable by open-loop controls.

Acknowledgements. The authors are grateful to the reviewers for valuable comments which helped us improve the readability of the paper.

REFERENCES

[1] G. Ashford and M. Broucke Reach control on simplices by time-varying affine feedback. *Automatica*. Vol. 49, issue 5, pp. 1365–1369, May 2013.

[2] C. Belta and L.C.G.J.M. Habets. Controlling a class of nonlinear systems on rectangles. *IEEE Trans. Autom. Control*. vol. 51, no. 11, pp. 17491759, Nov. 2006.

[3] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*. vol. 35, pp. 407–428, March 1999.

[4] A. Bemporad, M. Morari, V. Dua, E. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*. vol. 38, pp. 3-20, 2002.

[5] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York. 1979.

[6] K. Border. *Fixed Point Theorems with Applications to Economics and Game Theory.* Cambridge University Press, 1985.

[7] M.E. Broucke. Reach control on simplices by continuous state feedback. *SIAM Journal on Control and Optimization*. vol. 48, issue 5, pp. 3482-3500, February 2010.

[8] M.E. Broucke. On the reach control indices of affine systems on simplices. *8th IFAC Symposium on Nonlinear Control Systems*. August 2010.

[9] M.E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *American Control Conference*. pp. 2633-2638, June 2011.

[10] P. Brunovsky. A classification of linear controllable systems. *Kybernetika*. vol. 3, pp. 173-187, 1970.

[11] M.K. Camlibel, W.P.M.H. Heemels, and J.M. Schumacher. Algebraic necessary and sufficient conditions for the controllability of conewise linear systems. *IEEE Trans. on Automatic Control*. 53(3), pp. 762–774, 2008.

[12] R. Casey, H. De Jong, and J.L. Gouze. Piecewise linear models of genetic regulatory networks: Equilibria and their stability. Journal of Mathematical Biology, 52(1), pp. 27 56, 2006.

[13] F.H. Clarke, Y.S. Ledyaev, R.J. Stern, and P.R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer, 1998.

[14] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid Dynamical Systems. *IEEE Control Systems Magazine*. vol. 29, no. 2, pp. 28–93, April 2009.

[15] L.C.G.J.M. Habets and J.H. van Schuppen. Control of piecewise-linear hybrid systems on simplices and rectangles, in: M.D. Di Benedetto and A.L. Sangiovanni-Vincentelli (Eds.) *Hybrid Systems: Computation and Control, Lecture Notes in Computer Science*. Springer Verlag, vol. 2034, pp. 261–274, 2001.

[16] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.
[17] L.C.G.J.M. Habets, P.J. Collins, and J.H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Trans. Automatic Control*. no. 51, pp. 938–948, 2006.
[18] C. W. Lee. Subdivisions and triangulations of polytopes. *Handbook of Discrete and Computational Geometry*. CRC Press Series Discrete Math. Appl., pp. 271–290, 1997.
[19] D. Liberzon. *Switching in Systems and Control*. Boston, MA: Birkhauser, 2003.
[20] H.H. Lin, C.L. Beck, and M.J. Bloom. On the use of multivariable piecewise linear models for predicting human response to anesthesia. *IEEE Trans. on Biomedical Engineering*. 51(11), pp. 18761887, 2004.
[21] Z. Lin and M.E. Broucke. On a reachability problem for affine hypersurface systems on polytopes. *Automatica*. vol. 47, issue 4, pp. 769-775, April 2011.
[22] J.R. Munkres. *Elements of Algebraic Topology*. Perseus Books Publishing, Cambridge, Massachusetts, 1984.
[23] H. Oktem. A survey on piecewise linear models of regulatory dynamical systems. *Nonlinear Analysis: Theory Methods and Applications*. 63(3), pp. 336349, 2005.
[24] R.J. Ramadge and W.M. Wonham. Supervisory control of a class of discrete event processes. *SIAM J. Control and Optimization*. vol. 25, no. 1, pp. 206-230, 1987.
[25] R.T Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey 1970.
[26] B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. vol. 42, no. 11, pp. 1913–1918, November 2006.
[27] B. de Schutter and T.J.J. van den Boom. MPC for continuous piecewise-affine systems. *Systems and Control Letters*. 52(3-4), pp. 179192, 2004.
[28] F. Tahami and B. Molaei. Piecewise affine system modeling and control of PWM converters. Journal of Circuits Systems and Computers, 16(1), pp. 113 128, 2007.
[29] W.M. Wonham. *Linear Multivariable Control: a Geometric Approach*. 3rd Edition, Springer, 1985.

11. Appendix.

The proof of Lemma 1 relies on a fact about subspace intersection.

LEMMA 34. *Let $\{w_1, \ldots, w_r \mid w_i \in \mathbb{R}^n\}$ be a set of linearly independent vectors. Given integers q, p, and s such that $1 \le q \le p \le r$ and $q \le s \le r$, define $\mathcal{W}_1 = \mathrm{sp}\{w_1, \ldots, w_p\}$ and $\mathcal{W}_2 = \mathrm{sp}\{w_q, \ldots, w_s\}$. Then $\mathcal{W}_1 \cap \mathcal{W}_2 = \mathrm{sp}\{w_q, \ldots, w_p\}$.*

Proof. [Proof of Lemma 34] First, it is clear that $\mathrm{sp}\{w_q, \ldots, w_p\} \subset \mathcal{W}_1 \cap \mathcal{W}_2$. Now we show the converse. Let $0 \neq \beta \in \mathcal{W}_1 \cap \mathcal{W}_2$. Then there exist $c_1, \ldots, c_p \in \mathbb{R}$ (not all zero since $\{w_1, \ldots, w_p\}$ are linearly independent) and $d_q, \ldots, d_s \in \mathbb{R}$ (not all zero since $\{w_q, \ldots, w_s\}$ are linearly independent) such that

$$\beta = c_1 w_1 + \cdots + c_p w_p$$
$$\beta = d_q w_q + \cdots + d_s w_s \,.$$

Then

$$0 = c_1 w_1 + \cdots + c_{q-1} w_{q-1} + (c_q - d_q) w_q + \cdots + (c_p - d_p) w_p - d_{p+1} w_{p+1} - \cdots - d_s w_s \,.$$

Since $\{w_1, \ldots, w_s\}$ are linearly independent, we obtain $c_i = 0$, $i = 1, \ldots, q-1$, and $d_i = 0$, $i = p+1, \ldots, s$. Thus, $\beta \in \mathrm{sp}\{w_q, \ldots, w_p\}$. We conclude $\mathcal{W}_1 \cap \mathcal{W}_2 \subset$

sp$\{w_q,\dots,w_p\}$ as desired. □

Proof. [Proof of Lemma 1] By assumption $\mathcal{C} \subset \text{sp}\{w_1,\dots,w_r\}$ so trivially there exists a non-empty subset of $\{w_1,\dots,w_r\}$ whose span contains $\mathcal{C}$. It remains to show there exists a unique subset of minimum cardinality whose span contains $\mathcal{C}$. Let $\chi, \chi' \subset \{w_1,\dots,w_r\}$ such that $\mathcal{C} \subset \text{sp}\ \ \chi$ and $\mathcal{C} \subset \text{sp}\ \ \chi'$. Let $|\chi|$ denote the cardinality of χ. Moreover, suppose $p := |\chi| = |\chi'| \geq 1$ is the minimum cardinality of any subset of $\{w_1,\dots,w_r\}$ whose span contains $\mathcal{C} \neq \mathbf{0}$. We want to show $\chi = \chi'$. Suppose not. Define the subspaces $\mathcal{W} := \text{sp}\ \ \chi$ and $\mathcal{W}' := \text{sp}\ \ \chi'$. Clearly $\mathcal{W} \neq \mathcal{W}'$. Since $\mathcal{C} \subset \mathcal{W}$ and $\mathcal{C} \subset \mathcal{W}'$, then $\mathcal{C} \subset \mathcal{W} \cap \mathcal{W}'$, so $\mathcal{W} \cap \mathcal{W}' \neq \mathbf{0}$.

W.l.o.g. let $\chi = \{w_1,\dots,w_p\}$ and $\chi' = \{w_q,\dots,w_{q+p-1}\}$, where $1 \leq p, q+p-1 \leq r$. Then the following statements can be made about p and q:

$$\begin{aligned} \mathcal{W} \cap \mathcal{W}' \neq \mathbf{0} &\implies q \leq p \\ \mathcal{W} \neq \mathcal{W}' &\implies 1 < q\,. \end{aligned}$$

We conclude that

$$1 < q \leq p \leq r\,. \tag{11.1}$$

By Lemma 34, we get $\mathcal{C} \subset \mathcal{W} \cap \mathcal{W}' = \text{sp}\{w_q,\dots,w_p\}$, and by (11.1), $p - q + 1 < p$. This contradicts that p is the minimum cardinality of any subset of $\{w_1,\dots,w_r\}$ whose span contains $\mathcal{C}$. □

Proof. [Proof of Theorem 9] Let $x_0 \in \mathcal{S} \setminus \mathcal{F}_0$. By assumption there exists $\mu_{x_0}(t)$ and a time $T(x_0) > 0$ such that $\phi_{\mu_{x_0}}(t, x_0) \in \mathcal{S}$ for all $t \in [0, T(x_0)]$. Since $\mu_{x_0}(t)$ is an open-loop control, there exists $c \geq 0$ such that $\|\mu_{x_0}(t)\| \leq c$, for all $t \in [0, T(x_0)]$. Define $\mathcal{Y}(x) := \big\{Ax + Bw + a \mid w \in \mathbb{R}^m\big\}$ and $\mathcal{Y}_c(x) := \big\{Ax + Bw + a \mid w \in \mathbb{R}^m, \|w\| \leq c\big\}$. Now take a sequence $\{t_i \mid t_i \in (0, T(x_0)]\}$ with $t_i \to 0$. Since $\{y \in \mathcal{Y}_c(x) \mid x \in \mathcal{S}\}$ is bounded, there exists $M > 0$ such that $\|\phi_{\mu_{x_0}}(t_i, x_0) - x_0\| \leq M t_i$. Therefore $\left\{\frac{\phi_{\mu_{x_0}}(t_i,x_0)-x_0}{t_i}\right\}$ is a bounded sequence, and there exists a convergence subsequence (with indices relabeled) such that

$$\lim_{i\to\infty} \frac{\phi_{\mu_{x_0}}(t_i, x_0) - x_0}{t_i} =: v\,.$$

Since $\phi_{\mu_{x_0}}(t_i, x_0) \in \mathcal{S}$, by the definition of the Bouligand tangent cone, $v \in T_{\mathcal{S}}(x_0)$. On the other hand, we have

$$\frac{\phi_{\mu_{x_0}}(t_i, x_0) - x_0}{t_i} = \frac{1}{t_i}\int_0^{t_i} \big[A\phi_{\mu_{x_0}}(\tau, x_0) + B\mu_{x_0}(\tau) + a\big]d\tau\,. \tag{11.2}$$

Taking the limit, we get

$$v = Ax_0 + B\lim_{i\to\infty} \mu_{x_0}(t_i) + a \in \mathcal{Y}(x_0)\,.$$

Note that $\lim_{i\to\infty} \mu_{x_0}(t_i)$ exists by passing to a subsequence, if necessary, because μ_{x_0} is bounded on compact intervals. We conclude that $\mathcal{Y}(x_0) \cap T_{\mathcal{S}}(x_0) \neq$

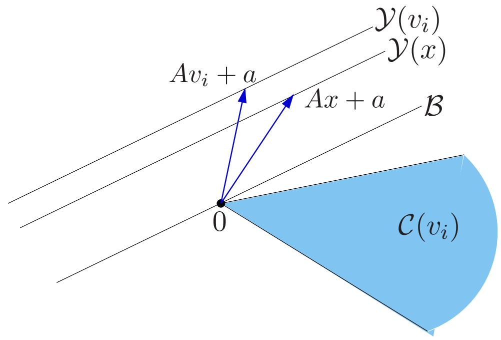


FIG. 11.1. *Illustration for the proof of Theorem 9.*

$\emptyset$, $x_0 \in \mathcal{S} \setminus \mathcal{F}_0$. Since $T_{\mathcal{S}}(v_0) = \text{cone}(\mathcal{S})$, and $T_{\mathcal{S}}(x) = \mathcal{C}(v_i)$ for $x \in (v_0, v_i)$, it follows that the invariance conditions are solvable at v_0 and along simplex edges $(v_0, v_i), i \in I$.

Now consider $v_i, i \in I$. If $v_i \in \mathcal{O}$, then the invariance conditions are solvable by selecting $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = 0$. Instead suppose $v_i \notin \mathcal{O}$. Suppose by way of contradiction that $\mathcal{Y}(v_i) \cap \mathcal{C}(v_i) = \emptyset$. Then $\mathcal{Y}(v_i)$ and $\mathcal{C}(v_i)$ are non-empty disjoint polyhedral convex sets in $\mathbb{R}^n$. By Corollary 19.3.3 of [25], they are strongly separated. That is, there exists $\epsilon > 0$ such that $\inf_{y \in \mathcal{Y}(v_i), z \in \mathcal{C}(v_i)} \|y - z\| > \epsilon$. By the upper semicontinuity of $x \mapsto \mathcal{Y}(x)$, there exists $\delta > 0$ such that if $\|x - v_i\| < \delta$, then $\mathcal{Y}(x) \subset \mathcal{Y}(v_i) + \frac{\epsilon}{2}\mathscr{B}$. Now taking any $x \in (v_0, v_i)$ with $\|x - v_i\| < \delta$, we get $\mathcal{Y}(x) \cap \mathcal{C}(v_i) = \emptyset$. See Figure 11.1. However, this is a contradiction with the result obtained above that for $x \in (v_0, v_i)$, $\mathcal{Y}(x) \cap T_{\mathcal{S}}(x) = \mathcal{Y}(x) \cap \mathcal{C}(v_i) \neq \emptyset$. ∎

Proof. [Proof of Theorem 10] Consider $v_i \in V \cap \mathcal{O}_{\mathcal{S}}$. Suppose $\mathcal{B} \cap \mathcal{C}(v_i) = \mathbf{0}$. Since $Av_i + a \in \mathcal{B}$, there exists $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = 0$. By (ii)-(iii) of Definition 8, there exists $\varepsilon > 0$ such that for all $x \in \mathcal{S} \setminus \mathcal{F}_0$, there exists $u_x \in \mathbb{R}^m$ such that $Ax + Bu_x + a \in T_{\mathcal{S}}(x)$ and $\|Ax + Bu_x + a\| > \varepsilon$. By continuity there exists $\delta > 0$ such that if $\|x - v_i\| < \delta$, then $\|Ax + Bu_i + a\| < \varepsilon/2$. Thus, for $x \in \mathcal{S} \setminus \mathcal{F}_0$ with $\|x - v_i\| < \delta$, we have $\|B(u_x - u_i)\| > \varepsilon/2$. Since $\mathcal{B} \cap \mathcal{C}(v_i) = \mathbf{0}$ and $\mathcal{C}(v_i)$ is a closed cone, there exists $\alpha > 0$ such that if $b \in \mathcal{B}$ satisfies $\|b\| > \varepsilon/2$, then $(b + \alpha\mathscr{B}) \cap \mathcal{C}(v_i) = \emptyset$. In particular, we can choose $x \in (v_0, v_i)$ sufficiently close to v_i such that $\|Ax + Bu_i + a\| < \min\{\alpha, \epsilon/2\}$. Then $Ax + Bu_x + a = (Ax + Bu_i + a) + B(u_x - u_i) \notin \mathcal{C}(v_i) = T_{\mathcal{S}}(x)$, a contradiction. ∎