Reach Control on Simplices by Piecewise Affine Feedback

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***Abstract*— We study the reach control problem for affine systems on simplices, and the focus is on cases when it is known that the problem is not solvable by continuous state feedback. Using the reach control indices for affine systems on simplices, we propose a subdivision algorithm and associated piecewise affine feedback. The main result is that if the reach control problem is solvable by open-loop controls, then it is solvable by piecewise affine feedback.**

I. Introduction

This paper studies the *reach control problem* on simplices. The problem is for an affine system defined on a simplex to reach a prespecified facet of the simplex in finite time. The problem has been developed in [3]–[10]. The significance of the problem stems from its capturing the essential features of reachability problems for control systems: the presence of state constraints and the notion of trajectories reaching a goal in a guided and finite-time manner. See [2] for motivations and an alternative approach.

In [3] it was shown that affine feedback and continuous state feedback are equivalent from the point of view of solvability of the reach control problem (RCP). In [4] we developed reach control indices which expose how affine or continuous state feedbacks may fail - such feedbacks induce closed-loop equilibria in sub-simplices that are inherently starved of sufficient inputs. Fortunately, the reach control indices also give insight on how to overcome the problem of insufficient inputs. We present here a subdivision procedure that triangulates the simplex into sub-simplices with sub-reach-control problems. The approach generalizes a subdivision method for hypersurface systems (having $n-1$ inputs) first presented in [8]. It enables a reassigment of controls to the effect that the shortage of inputs can be overcome. The final outcome is that if the reach control problem is solvable by open-loop controls, then it is solvable by piecewise affine feedback.

Notation. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\mathrm{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. Symbol $\mathbb{U}$ represents a control class such as open-loop, continuous state feedback, affine feedback, etc. For a vector $x \in \mathbb{R}^n$, the notation $x \prec 0$ ($x \preceq 0$) means $x_i < 0$ ($x_i \le 0$) for $1 \le i \le n$.

II. Background

Consider an n-dimensional simplex $\mathcal{S}$ with vertex set $V := \{v_0, v_1, \ldots, v_n\}$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ (the facet is indexed by the vertex not contained). Let h_i, $i = 0, \ldots, n$ be the unit normal vector to each facet $\mathcal{F}_i$ pointing outside of the simplex. Let $\mathcal{F}_0$ be the target set in $\mathcal{S}$. Define the index sets $I := \{1, \ldots, n\}$ and $I_i := I \setminus \{i\}$ (note $I_0 = I$). For $i \in I \cup \{0\}$, define the closed, convex cone

$$\mathcal{C}_i := \{ \ y \in \mathbb{R}^n \ : \ h_j \cdot y \le 0, \ j \in I_i \ \}.$$

We'll write $\mathrm{cone}(\mathcal{S}) := \mathcal{C}_0$, since $\mathcal{C}_0$ is the tangent cone to $\mathcal{S}$ at v_0. We consider the affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + a + Bu\,, \qquad x \in \mathcal{S}, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\mathrm{rank}(B) = m$. Let $\phi_u(t, x)$ denote the solution of (1) starting from x_0 under a control law u.

Definition 2.1: We say the *invariance conditions* are solvable if there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that $Av_i + a + Bu_i \in \mathcal{C}_i$. Equivalently,

$$h_j \cdot (Av_i + a + Bu_i) \le 0\,, \qquad i \in \{0, \ldots, n\}\,, j \in I_i\,. \tag{2}$$

Definition 2.2: We say a state feedback $u(x)$ satisfies the invariance conditions if for all $j \in I$ and $x \in \mathcal{F}_j$,

$$h_j \cdot (Ax + Bu(x) + a) \le 0\,. \tag{3}$$

Problem 2.1 (Reach Control Problem (RCP)): Consider system (1) defined on $\mathcal{S}$. Find a feedback control $u(x)$ such that:

(i) For every $x_0 \in \mathcal{S}$ there exist $T \ge 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T+\gamma)$.

(ii) There exists $\varepsilon > 0$ such that for all $x \in \mathcal{S}$, $\|Ax + a + Bu(x)\| > \varepsilon$.

(iii) Feedback $u(x)$ satisfies the invariance conditions (3) on $\mathcal{F}_0$.

Definition 2.3: A point $x_0 \in \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ with control class $\mathbb{U}$, denoted by $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a control u of class $\mathbb{U}$ such that properties (i)-(iii) of Problem 2.1 hold. We write $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by control class $\mathbb{U}$ if for every $x_0 \in \mathcal{S}$, $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$ with control of class $\mathbb{U}$.

Theorem 2.1: [7], [10] Given the system (1) and an affine feedback $u(x) = Kx + g$, where $K \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$, and $u_0 = u(v_0), \ldots, u_n = u(v_n)$, the closed-loop system satisfies $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ if and only if

(a) The invariance conditions (2) hold.

(b) There is no equilibrium in $\mathcal{S}$.

Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Define

$$\mathcal{O} := \{ \ x \in \mathbb{R}^n \ : \ Ax + a \in \mathcal{B} \}\,.$$

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Notice that the vector field $Ax + Bu + a$ on $\mathcal{O}$ can vanish for an appropriate choice of u, so $\mathcal{O}$ is the set of all possible equilibrium points of the system. Define

$$\mathcal{G} := \mathcal{S} \cap \mathcal{O} .$$

Associated with $\mathcal{G}$ is its vertex index set $I_{\mathcal{G}} := \{i \mid v_i \in V \cap \mathcal{G}\}$. We make an important assumption concerning the placement of $\mathcal{O}$ with respect to $\mathcal{S}$. The reader is referred to [3] for the motivation for and a method of triangulation of the state space that achieves it.

Assumption 2.1: Simplex $\mathcal{S}$ and system (1) satisfy the following condition: if $\mathcal{G} \neq \emptyset$, then $\mathcal{G}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \leq \kappa \leq n$.

Theorem 2.2 ([3]): Suppose $\mathcal{G} = \emptyset$. If the invariance conditions (2) are solvable, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

Theorem 2.3 ([3]): Suppose Assumption 2.1 holds and $\mathcal{G} \neq \emptyset$. If the invariance conditions (2) are solvable and $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

The primary conclusion of [3] is that RCP is solvable by affine feedback if and only if it is solvable by continuous state feedback. The goal of this paper is to solve RCP in cases where continuous state feedback cannot be used. We consider the following assumptions.

Assumption 2.2: Simplex $\mathcal{S}$ and system (1) satisfy the following conditions.

(A1) $\mathcal{G} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < n$.
(A2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$.
(A3) The maximum number of linearly independent vectors in any set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ (with only one vector for each $\mathcal{B} \cap \mathcal{C}_i$) is $\widehat{m}$ with $1 \leq \widehat{m} \leq \kappa + 1$.

Assumption (A1) rules out the application of Theorem 2.2, and it enforces that $v_0 \notin \mathcal{O}$. The latter requirement is because when $v_0 \in \mathcal{O}$ and (A2) holds, then RCP is not solvable. Assumption (A2) rules out the application of Theorem 2.3. Finally, (A3) introduces a new condition in terms of the variable $\widehat{m}$, which necessarily satisfies $\widehat{m} \leq \kappa + 1$. When $\kappa = \widehat{m} - 1$, an affine feedback solves RCP, as stated below. The remaining cases when $\kappa \geq \widehat{m}$ are the topic of this paper.

Theorem 2.4 ([3]): Suppose Assumption 2.2 holds. If the invariance conditions (2) are solvable and $\widehat{m} = \kappa + 1$, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

III. Necessary Conditions

In this section we summarize necessary conditions for solvability of RCP using open-loop controls. (Some proofs in the paper are supressed due to space constraints). We say that a function $u : [0, \infty) \to \mathbb{R}^m$ is an *open-loop control* if it is bounded on any compact interval and it is measurable. By Caratheodory's theorem solutions of (1) using open-loop controls exist and are unique.

Theorem 3.1: Suppose there exist open-loop controls such that condition (i) of RCP holds. Then the invariance conditions (2) are solvable.

Theorem 3.2: If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls, then

$$\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0} , \qquad i \in I_{\mathcal{G}} .$$

IV. Piecewise Affine Feedback

In this section we investigate the extent to which piecewise affine feedback can solve RCP, in cases when continuous state feedback cannot. We construct a triangulation of the simplex $\mathcal{S}$ comprised of sub-simplices such that a sub-RCP is solvable for each sub-simplex. Noteworthy is the way we exploit the reach control indices. According to [4], the reach control indices are defined under the following assumptions.

Assumption 4.1: Simplex $\mathcal{S}$ and system (1) satisfy the following conditions.

(P1) $\mathcal{G} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$, where $0 \leq \kappa < n$.
(P2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$.
(P3) $\widehat{\mathcal{B}} := \text{sp}\{b_1, \ldots, b_{\widehat{m}} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$, where $\widehat{m} < \kappa + 1$.
(P4) $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}\,, \quad i \in I_{\mathcal{G}}$.

Condition (P3) encodes the fact that there is a preferred basis that is maximal with respect to $\mathcal{G}$ in the sense of (A3). By condition (P1), we have $\kappa+1 = \widehat{m}+p$ for some $p \geq 1$. Using (P4), if we select any set $\{b_{\widehat{m}+1}, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ and we use $\{b_1, \ldots, b_{\widehat{m}}\}$ as in (P3), then p denotes the number of linearly dependent vectors in the set $\{b_1, \ldots, b_{\kappa+1}\}$.

Theorem 4.1 ([4]): Suppose Assumption 4.1 hold. Then there exist integers $r_1, \ldots, r_p \geq 0$ and a decomposition of $\mathcal{B}$ into p subsets such that w.l.o.g. (reordering indices)

$$\mathcal{B} \cap \mathcal{C}_i \subset \mathcal{B}_1 := \text{sp}\{b_{m_1}, \ldots, b_{m_1+r_1-1}\}\,, i = m_1, \ldots, m_1 + r_1 - 1\,, \tag{4}$$

$$\mathcal{B} \cap \mathcal{C}_i \subset \mathcal{B}_2 := \text{sp}\{b_{m_2}, \ldots, b_{m_2+r_2-1}\}\,, i = m_2, \ldots, m_2 + r_2 - 1 \tag{5}$$

$$\vdots$$

$$\mathcal{B} \cap \mathcal{C}_i \subset \mathcal{B}_p := \text{sp}\{b_{m_p}, b_{m_p+r_p-1}\}\,, i = m_p, \ldots, m_p + r_p - 1\,. \tag{6}$$

where $m_k := r_1 + \cdots + r_{k-1} + 1$ for $k = 1, \ldots, p$ and $r := r_1 + \cdots + r_p$.

The importance of the reach control indices stems from their ability to isolate closed-loop equilibria when using continuous state feedback. Define for $k = 1, \ldots, p$

$$\widehat{\mathcal{S}}_k := \text{co}\big\{v_{m_k}, \ldots, v_{m_k+r_k-1}\big\}\,.$$

In [4] it was shown that each $\widehat{\mathcal{S}}_k$ contains a closed-loop equilibrium when using continuous state feedback. We now show a control method that breaks up the dependencies in $\mathcal{B}$ to remove these equilibria.

Assumption 4.2: Simplex $\mathcal{S}$ and system (1) satisfy (P1)-(P4) and also the following conditions.

(P5) $\exists \ \{r_1, \ldots, r_p\}$ such that (4)-(6) hold.
(P6) $\mathcal{B}_k \not\subset \mathcal{H}_0 := \{y \in \mathbb{R}^n \mid h_0 \cdot y = 0\}\,, \quad k = 1, \ldots, p$.

Condition (P5) comes directly from Theorem 4.1 while the necessity of (P6) is stated below.

Lemma 4.1: If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls, then $\mathcal{B}_k \not\subset \mathcal{H}_0$ for each $k = 1, \ldots, p$.

Definition 4.1: Given a state feedback $u(x)$, we say u is a *piecewise affine feedback* if there exists a triangulation $\mathbb{T}$ of $\mathcal{S}$ such that for each (full-dimensional) $\mathcal{S}_i \in \mathbb{T}$, there exist $K_i \in \mathbb{R}^{m\times n}$ and $g_i \in \mathbb{R}^m$ such that $u(x) = K_i x + g_i$, $x \in \mathcal{S}_i$.

This definition of piecewise affine feedback allows for discontinuities at boundaries of simplices. A *discrete supervisory controller* will be introduced later to resolve which control value must be used at points lying in more than one simplex, thus ensuring the feedback is well-defined.

We now explain in general terms an inductive procedure for subdividing $\mathcal{S}$ in order that RCP can be solved by piecewise affine feedback. First, an immediate consequence of (P6) stated in Lemma 4.2 is that each subsimplex $\hat{\mathcal{S}}_k$, for $k = 1, \ldots, p$, has a vertex (among $\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$) with $b_i \in \mathcal{B} \cap \mathcal{C}_i$ pointing out of $\mathcal{S}$. By convention, we reorder indices so this vertex is the first one in each list $\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$. We make a subdivision of $\mathcal{S}$ by placing a new vertex v' along the edge (v_0, v_{m_k}). In particular, at the first iteration we would have $v' \in (v_0, v_1)$, and we form two sub-simplices $\mathcal{S}^1$ and $\mathcal{S}'$ as in Figure 1. Lemma 4.4 shows that because $b_{m_k} \in \mathcal{B} \cap \mathcal{C}_{m_k}$ points out of $\mathcal{S}$ at v_{m_k} and because the invariance conditions for $\mathcal{S}$ are solvable at v_0, a convexity argument gives that v' can be placed along (v_0, v_{m_k}) so that $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq \mathbf{0}$. Then in Lemma 4.5 one applies Theorem 2.3 to obtain that RCP is solved for $\mathcal{S}^1$. Essentially $\mathcal{S}^1$ can be removed from further consideration, and the induction step is repeated with $\mathcal{S}$ replaced by the remainder $\mathcal{S}'$. To guarantee that the induction is sound, one must show that $\mathcal{S}'$ inherits the relevant properties of $\mathcal{S}$, especially condition (P6). This is done in Lemma 4.6.

Lemma 4.2: Suppose Assumption 4.2 holds. Then w.l.o.g. (by reordering indices) $h_0 \cdot b_{m_k} > 0$ for $k = 1, \ldots, p$.

Proof: We prove the result for $k = 1$. If for any $j \in \{1, \ldots, r_1\}$, $h_0 \cdot b_j > 0$, then the proof is finished. Instead suppose that for all $i \in \{1, \ldots, r_1\}$, $h_0 \cdot b_i \leq 0$. Using (P6) and by reordering the indices $1, \ldots, r_1$, assume $h_0 \cdot b_{r_1} < 0$. By Lemma 18 of [4] there exists $b_1 \in \mathcal{B} \cap \mathcal{C}_1$ such that

$$b_1 = c_2 b_2 + \cdots + c_{r_1} b_{r_1}, \quad c_i \neq 0, \quad i = 2, \ldots, r_1,$$

and $\{b_2, \ldots, b_{r_1}\}$ are linearly independent. Let $c := (c_2, \ldots, c_{r_1})$. Define matrices $H := [h_2 \cdots h_{r_1}]$ and $Y := [b_2 \cdots b_{r_1}]$. Since $b_1 \in \mathcal{C}_1$, it satisfies the invariance conditions:

$$H^T b_1 = H^T Y c \preceq 0.$$

By (P2) and Lemma 6.4 of [3], $H^T Y$ is a non-singular $\mathscr{M}$-matrix. By Theorem 2.3 (case N_{39}) of [1], this implies $c \preceq 0$. Since $c_i \neq 0$, we have moreover $c \prec 0$. Thus we obtain

$$h_0 \cdot b_1 = h_0 \cdot (c_2 b_2 + \cdots + c_{r_1} b_{r_1}) \geq c_{r_1} h_0 \cdot b_{r_1} > 0.$$

■

Following Lemma 4.2, suppose that b_1 satisfies $h_0 \cdot b_1 > 0$. We consider any point v' in the open segment (v_0, v_1). That is, let $\lambda \in (0, 1)$ and define

$$v' = \lambda v_1 + (1 - \lambda) v_0. \tag{7}$$

Now define the following sub-simplices of $\mathcal{S}$:

$$\begin{aligned} \mathcal{S}' &= \text{co}\{v_0, v', v_2, \ldots, v_n\} \\ \mathcal{S}^1 &= \text{co}\{v', v_1, v_2, \ldots, v_n\}. \end{aligned}$$

Also define the new exit facet for $\mathcal{S}'$ by

$$\mathcal{F}_0' := \text{co}\{v', v_2, \ldots, v_n\}.$$

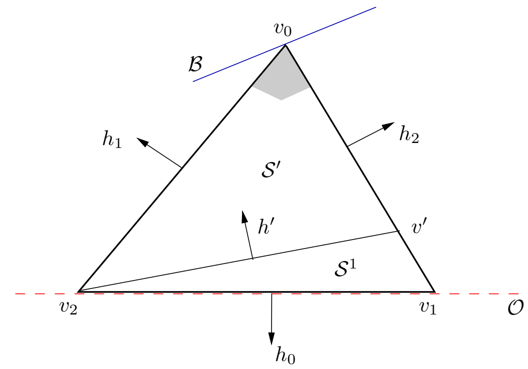

Fig. 1. Subdivision into two simplices $\mathcal{S}'$ and $\mathcal{S}^1$.

See Figure 1. The following lemma provides a formula for the normal vector h' of $\mathcal{F}_0'$.

Lemma 4.3: Suppose $h_0 = -\gamma_1 h_1 - \ldots - \gamma_n h_n$ with $\gamma_i > 0$. Then the normal vector to $\mathcal{F}_0'$ pointing out of $\mathcal{S}^1$ is

$$h' = \gamma_1 (1 - \lambda) h_1 - \lambda h_0. \tag{8}$$

Lemma 4.4: Suppose Assumption 4.2 holds. There exists $v' \in (v_0, v_1)$, such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq \mathbf{0}$. Moreover, one can choose $b' \in \mathcal{B} \cap \text{cone}(\mathcal{S}^1)$ such that $h' \cdot b' < 0$.

Proof: Observe that $\text{cone}(\mathcal{S}^1) = \{y \in \mathbb{R}^n \mid h' \cdot y \leq 0, h_j \cdot y \leq 0, \quad j \in \{2, \cdots, n\}\}$. We show there is an interval of values for λ such that $0 \neq b_1 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^1)$, where b_1 is provided by Lemma 4.2. First, since $b_1 \in \mathcal{B} \cap \mathcal{C}_1$ we know $h_j \cdot b_1 \leq 0$ for $j \in \{2, \ldots, n\}$. We must only show that there exists $\lambda \in (0, 1)$ such that $h' \cdot b_1 < 0$. From Lemma 4.3 we have

$$h' \cdot b_1 = \gamma_1 (1 - \lambda) h_1 \cdot b_1 - \lambda h_0 \cdot b_1. \tag{9}$$

Since $h_1 \cdot b_1 > 0$ (because $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$) and $h_0 \cdot b_1 > 0$ (by Lemma 4.2), it is clear from (9) that we can select $\lambda = \lambda'$ sufficiently close to 1 such that $h' \cdot b_1 < 0$. Setting $v' = \lambda' v_1 + (1 - \lambda') v_0$, we get $b_1 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^1)$. ■

Lemma 4.5: Suppose Assumption 4.2 holds. If the invariance conditions for $\mathcal{S}$ are solvable, then $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback.

Proof: By Lemma 4.4, we have $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq 0$. We show that the invariance conditions are solvable for $\mathcal{S}^1$. First, consider the vertex v'. By assumption there exist control inputs $u_0, u_1 \in \mathbb{R}^m$ such that the invariance conditions for $\mathcal{S}$ at v_0 and v_1 are satisfied, i.e.

$$\begin{aligned} y_0 &:= Av_0 + Bu_0 + a \in \text{cone}(\mathcal{S}) \\ y_1 &:= Av_1 + Bu_1 + a \in \mathcal{B} \cap \mathcal{C}_1. \end{aligned}$$

In particular, $h_j \cdot y_i \leq 0$ for $i = 0, 1$ and $j = 2, \ldots, n$. Now by Lemma 4.4, there exists $\lambda \in (0, 1)$ such that with $v' := \lambda v_1 + (1 - \lambda) v_0$, $h' \cdot b_1 < 0$ and $h_j \cdot b_1 \leq 0$ for $j = 2, \ldots, n$. Let w_1 be such that $b_1 = Bw_1$. Set $\epsilon_1 > 0$ and let $u' := \lambda u_1 + (1 - \lambda) u_0 + \epsilon_1 w_1$. Then

$$y' := Av' + Bu' + a = \lambda y_1 + (1 - \lambda) y_0 + \epsilon_1 b_1.$$

Thus, $h_j \cdot y' \leq 0$ for $j = 2, \ldots, n$ and for $\epsilon_1 > 0$ sufficiently large, $h' \cdot y' < 0$. That is, the invariance conditions for $\mathcal{S}^1$

are solvable at v'.

Next consider v_1. Since the invariance conditions for $\mathcal{S}^1$ at v_1 are identical to those for $\mathcal{S}$ at v_1, and since the latter are by assumption solvable, the former are also solvable. Finally, consider vertices v_i, $i = 2, \ldots, n$. There exist control inputs $u_i \in \mathbb{R}^m$ such that

$$y_i := Av_i + Bu_i + a$$

satisfy $h_j \cdot y_i \leq 0$ for $j = 2, \ldots, i-1, i+1, \ldots, n$. As above let w_1 be such that $b_1 = Bw_1$. Set $\epsilon_1 > 0$ and let $u_i' := u_i + \epsilon_1 w_1$. Then the closed-loop vector field for $\mathcal{S}^1$ at v_i is

$$y_i' = Av_i + Bu_i' + a = y_i + \epsilon_1 b_1 \,.$$

Thus, $h_j \cdot y_i' \leq 0$ for $j = 2, \ldots, i-1, i+1, \ldots, n$ and for $\epsilon_1 > 0$ sufficiently large, $h' \cdot y_i' < 0$. That is, the invariance conditions for $\mathcal{S}^1$ are solvable at v_i. In sum, we can apply Theorem 2.3 to obtain that $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback. ■

Lemma 4.6: Suppose Assumption 4.2 holds. If the invariance conditions for $\mathcal{S}$ are solvable then

(i) The invariance conditions for $\mathcal{S}'$ are solvable.
(ii) $(-h') \cdot b_{m_k} > 0\,, \quad k = 2, \ldots, p$.

Proof: First we prove (i). By assumption the invariance conditions are solvable for $\mathcal{S}$, and since the invariance conditions for $\mathcal{S}'$ are identical (the only facet that changed for $\mathcal{S}'$ is $\mathcal{F}_0$, which plays no role in invariance conditions), they are also solvable for $\mathcal{S}'$.

Next we prove (ii). Since $b_{m_k} \in \mathcal{B} \cap \mathcal{C}_{m_k}$, we have $h_1 \cdot b_{m_k} \leq 0$, for $k = 2, \ldots, p$. Also by Lemma 4.2, $h_0 \cdot b_{m_k} > 0$, for $k = 2, \ldots, p$. Thus, using (8), for $k = 2, \ldots, p$ we have

$$(-h') \cdot b_{m_k} = -\gamma_1 (1-\lambda) h_1 \cdot b_{m_k} + \lambda h_0 \cdot b_{m_k} > 0\,.$$

■

We have demonstrated the first step of a triangulation procedure that partitions $\mathcal{S}$ into sub-simplices on which sub-reach control problems are solvable. Now we present a triangulation algorithm that iterates on the presented subdivision method. It consists of p iterations, one for each set $\{v_{m_k}, \ldots, v_{m_k + r_k - 1}\}$, $k = 1, \ldots, p$. The notation $\mathcal{S}^k := \text{co}\{v', v_1, \ldots, v_n\}$ is understood to mean that all $n+1$ vertices of $\mathcal{S}^k$ are assigned simultaneously in the order presented. The vertices of $\mathcal{S}^k$ are later identified as $\{v_0^k, \ldots, v_n^k\}$. The algorithm generates subsimplices $\mathcal{S}^1, \ldots, \mathcal{S}^{p+1}$ starting from the given simplex $\mathcal{S}$. At the kth iteration, the current declaration of $\mathcal{S}$ is split into a lower simplex $\mathcal{S}^k$ and an upper simplex. The lower simplex is then "thrown away" and the remainder is declared to be $\mathcal{S}$ with vertices called $\{v_0, \ldots, v_n\}$ (overloading the vertices of the previous $\mathcal{S}$).

Subdivision Algorithm:

1) Set $k := 1$.
2) Select $v' \in (v_0, v_{m_k})$ such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^k) \neq \mathbf{0}$, where $\mathcal{S}^k := \text{co}\{v', v_1, \ldots, v_n\}$.
3) Set $\mathcal{S} := \text{co}\{v_0, v_1, \ldots, v_{m_k - 1}, v', v_{m_k + 1}, \ldots, v_n\}$.
4) If $k < p$, set $k := k+1$ and go to step 2.
5) Set $\mathcal{S}^{p+1} := \mathcal{S}$.

Let $\mathcal{F}_0^k = \text{co}\{v_1^k, \ldots, v_n^k\}$ denote the exit facet of $\mathcal{S}^k$. The triangulation generated by the algorithm has the property that

$$\mathcal{S}^k \cap \mathcal{S}^{k-1} = \mathcal{F}_0^k\,, \qquad k = 2, \ldots, p+1\,,$$

and closed-loop trajectories follow paths through sub-simplices with decreasing indices. Thus, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ is achieved by implementing affine controllers that achieve $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$ for $k = 1, \ldots, p+1$. In order to guarantee that switching occurs in the proper sequence (with decreasing simplex indices), and to avoid chattering caused by measurement errors, a supervisor should accompany the implementation of the piecewise affine feedback. The supervisor has two functions:

(i) Once the piecewise affine controller has switched to simplex $\mathcal{S}^k$, then all affine controllers for $\mathcal{S}^j$, $j > k$, are disabled.
(ii) The affine controller for $\mathcal{S}^j$ is released to $\mathcal{S}^{j-1}$ only after the closed-loop trajectory exits $\mathcal{S}^j$. Thus, at a point $x \in \mathcal{S}^j \cap \mathcal{S}^{j-1}$, the controller for the simplex with the higher index is used.

Theorem 4.2: Suppose Assumption 4.2 holds. If the invariance conditions for $\mathcal{S}$ are solvable, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by piecewise affine feedback.

Proof: Form the triangulation $\{\mathcal{S}^1, \ldots, \mathcal{S}^{p+1}\}$ of $\mathcal{S}$ based on the Subdivision Algorithm. To show that $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by piecewise affine feedback, we first show that $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$ by affine feedback for $k = 1, \ldots, p+1$.

Lemmas 4.4 and 4.5 depend on two properties of $\mathcal{S}$: condition (P6) and solvability of its invariance conditions. Let

$$\tilde{\mathcal{S}}^k := \text{co}\{v_0, v_1^k, \ldots, v_n^k\}\,.$$

Then $\mathcal{S}^k \subset \tilde{\mathcal{S}}^k$ and $\tilde{\mathcal{S}}^k$ takes the role of $\mathcal{S}$ in Lemmas 4.2, 4.4, and 4.5. Thus, we must verify that $\tilde{\mathcal{S}}^k$ inherits the needed properties of $\mathcal{S}$. However, Lemma 4.6 guarantees by an inductive argument that for each successor $\tilde{\mathcal{S}}^k$, the invariance conditions remain solvable and $(-h^k) \cdot b_{m_k} > 0$ for $k = 2, \ldots, p$. The latter statement means that Lemma 4.2 applies to each $\mathcal{S}^k$ (with h^k representing the kth iterate of h'); and this, in turn, means Lemmas 4.4 and 4.5 also apply to $\mathcal{S}^k \subset \tilde{\mathcal{S}}^k$. We conclude $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$ by affine feedback for $k = 1, \ldots, p$.

Next consider $\mathcal{S}^{p+1}$. By Lemma 4.6, the invariance conditions for $\mathcal{S}^{p+1}$ are solvable ($\mathcal{S}^{p+1}$ and $\mathcal{S}$ share the same invariance conditions since they only differ in their exit facets). Now let $\mathcal{G}^{p+1} := \mathcal{S}^{p+1} \cap \mathcal{O}$. Then by the algorithm,

$$\mathcal{G}_{p+1} = \text{co}\{v_2, \ldots, v_{m_2 - 1}, v_{m_2 + 1}, \ldots, v_{m_p - 1}, v_{m_p + 1}, \ldots, v_{\kappa + 1}\}\,.$$

We can see that the algorithm has removed the p vertices $v_1, v_{m_2}, \ldots, v_{m_p}$ and this has the effect to break up the dependencies of $\mathcal{B}$ associated with $\mathcal{G}$. There remain $\widehat{m}$ linearly indepedent vectors in $\mathcal{B}$ associated with $\mathcal{G}_{p+1}$ (an $(\widehat{m} - 1)$-dimensional simplex) given by

$$\{b_2, \ldots, b_{m_2 - 1}, b_{m_2 + 1}, \ldots, b_{m_p - 1}, b_{m_p + 1}, \ldots, b_{\kappa + 1}\}\,.$$

Therefore, we can apply Theorem 2.4 to obtain $\mathcal{S}^{p+1} \xrightarrow{\mathcal{S}^{p+1}} \mathcal{F}_0^{p+1}$.

Next we verify conditions (ii) and (iii) of RCP. Condition (ii) follows immediately because there are a finite number of affine feedbacks $u^k(x)$ each defined on a compact set $\mathcal{S}^k$ that does not contain an equilibrium. For (iii) we must verify that the invariance conditions for $\mathcal{S}$ hold on the vertices of $\mathcal{F}_0$. The exit facet of $\mathcal{S}^{p+1}$ is $\mathcal{F}_0^{p+1} = \text{co}\{v_0^1, v_{m_1+1}, \ldots, v_{m_1+r_1-1}, \ldots, v_0^p, v_{m_p+1}, \ldots, v_r, v_{r+1}, \ldots, v_n\}$. The invariance conditions for $\mathcal{S}^{p+1}$ are identical to those for $\mathcal{S}$ and the controller for $\mathcal{S}^{p+1}$ takes precedence over controllers for simplices with lower index. This implies that the invariance conditions of $\mathcal{S}$ hold at all vertices of $\mathcal{F}_0^{p+1}$. The only vertices of $\mathcal{F}_0$ that are not in $\mathcal{F}_0^{p+1}$ are $v_{m_1}, v_{m_2}, \ldots, v_{m_p}$. For these vertices we have: $v_{m_1} \in \mathcal{S}^1$, $v_{m_2} \in \mathcal{S}^1 \cap \mathcal{S}^2$,...,$v_{m_p} \in \mathcal{S}^1 \cap \cdots \cap \mathcal{S}^p$. We use the controller for the simplex with the highest index. Now the invariance conditions for $\mathcal{S}^k$ at v_{m_k} are precisely those for $\mathcal{S}$. We can see this because the invariances conditions for v_{m_k} do not include the normal vector h^k.

Finally, we must prove that trajectories progress through sub-simplices with decreasing indices (thereby guaranteeing that the supervisor cannot block). Consider w.l.og. the boundary between $\mathcal{S}^1$ and $\mathcal{S}^2$ given by $\mathcal{F}_0^1 = \text{co}\{v', v_2, \ldots, v_n\}$, and let $u = K_1 x + g_1$ be the affine feedback obtained for $\mathcal{S}^1$. We must show that for any $x_0 \in \mathcal{S}^1 \setminus \mathcal{F}_0^1$, closed-loop trajectories do not reach $\mathcal{F}_0^1$. This can be deduced from the proof of Lemma 4.5 where it is shown that the controls $\{u', u_2, \ldots, u_n\}$ can be selected so that

$$\begin{aligned} h' \cdot (Av' + Bu' + a) &< 0 \\ h' \cdot (Av_i + Bu_i + a) &< 0\,, \qquad i = 2, \ldots, n\,. \end{aligned}$$

By convexity, $h' \cdot (Ax + B(K_1 x + g_1) + a) < 0$ for all $x \in \mathcal{F}_0^1$, from which the result easily follows. ∎

V. Main Result

Theorem 5.1: Suppose Assumption 2.1 holds. Then the following statements are equivalent:

1) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by piecewise affine feedback.
2) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls.

Proof: (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1) Suppose $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls. By Theorem 3.1, the invariance conditions are solvable. Let $\mathcal{G} := \mathcal{S} \cap \mathcal{O}$. If $\mathcal{G} = \emptyset$, then by Theorem 2.2, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose instead $\mathcal{G} \neq \emptyset$. If $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$, then by Theorem 2.3, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose instead $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. From Theorem 3.2, $v_0 \notin \mathcal{G}$, so by reordering indices, $\mathcal{G} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$, where $0 \leq \kappa < n$. Define $\hat{\mathcal{B}} = \text{sp}\{b_1, \ldots, b_{\widehat{m}} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ where $\{b_1, \ldots, b_{\widehat{m}}\}$ is a maximal set with respect to $\mathcal{G}$. By Theorem 3.2, $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}$ for $i \in I_{\mathcal{G}}$. If $\kappa < \widehat{m}$, then by Theorem 2.4, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose instead $\kappa \geq \widehat{m}$. Then Assumption 4.1 holds. The reach control indices can be defined, yielding a decomposition of $\hat{\mathcal{B}}$ into $\hat{\mathcal{B}}_1, \ldots, \hat{\mathcal{B}}_p$. Lemma 4.1 gives $\mathcal{B}_k \not\subset \mathcal{H}_0$ for $k = 1, \ldots, p$. Thus, all conditions of Assumption 4.2 hold. By Theorem 4.2, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by piecewise affine feedback. ∎

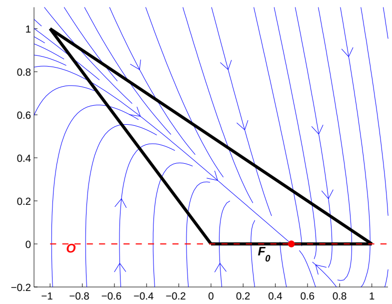

Fig. 2. Closed-loop vector field using affine feedback.

VI. Examples

A. Example 1

Consider a simplex $\mathcal{S}$ determined by vertices $v_0 = (-1, 1)$, $v_1 = (1, 0)$ and $v_2 = (0, 0)$, and consider the affine system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have

$$\mathcal{O} = \{x \mid x_2 = 0\}\,.$$

Hence $\mathcal{S} \cap \mathcal{O} = \mathcal{G} = \text{co}\{v_1, v_2\}$, $\kappa = 1$, and $m = 1$. Also we note that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. By the results of [3] the problem is not solvable by continuous state feedback. For example, suppose we choose control values at the vertices to satisfy the invariance conditions: $u_0 = -\frac{3}{4}$, $u_1 = -1$, and $u_2 = 1$. This yields an affine feedback

$$u = \begin{bmatrix} -2 & -3.75 \end{bmatrix} x + 1\,.$$

Simulation of the closed-loop system is shown in Figure 2. The vector field satisfies the invariance conditions; however, there exists an equilibrium point on $\mathcal{G} = \text{co}\{v_1, v_2\}$. Now we show the problem is solvable by piecewise affine feedback.

According to the Subdivision Algorithm, we choose $v' = (0.5, 0.25)$ so that $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq 0$. Then $\mathcal{S}^2 := \text{co}\{v_0, v', v_2\}$, $\mathcal{S}^1 := \text{co}\{v', v_1, v_2\}$, $\mathcal{F}_0' = \text{co}\{v', v_2\}$, and $h' = (-0.25, 0.5)$. To satisfy the invariance conditions for $\mathcal{S}^2$ we choose control inputs at the vertices to be $u_0 = -\frac{3}{4}$, $u' = -1$, and $u_{22} = 1$. To satisfy invariance conditions for $\mathcal{S}^1$ we choose control inputs at the vertices to be $u' = -1$, $u_1 = -1$, and $u_{12} = -1$. The piecewise affine feedback is

$$u = \begin{cases} \begin{bmatrix} 0 & 0 \end{bmatrix} x - 1\,, & x \in \mathcal{S}^1 \\ \begin{bmatrix} -2.0833 & -3.833 \end{bmatrix} x + 1\,, & x \in \mathcal{S}^2\,. \end{cases}$$

The closed-loop vector field is shown in Figure 3, where it is clear that RCP is solved.

B. Example 2

Consider the simplex $\mathcal{S}$ in $\mathbb{R}^4$ defined by the vertices $v_0 = (0, 0, 0, 0)$, $v_1 = (1, 0, 0, 0)$, $v_2 = (0, 1, 0, 0)$, $v_3 =$

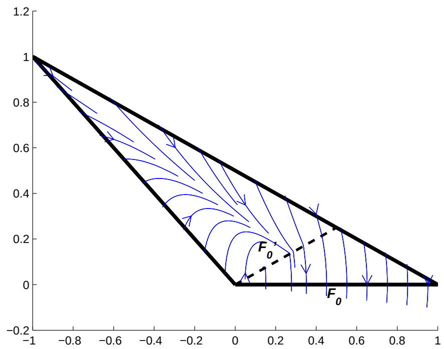

Fig. 3. Closed-loop vector field using piecewise affine feedback.

$(0, 0, 1, 0)$, and $v_4 = (0, 0, 0, 1)$. Consider also the affine dynamics on $\mathcal{S}$

$$\dot{x} = \begin{bmatrix} -3 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \\ -3 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & -2 \\ 0 & 1 \\ -2 & 0 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

We have

$$\mathcal{O} = \{x \mid x_1 + x_2 + x_3 + x_4 - 1 = 0\} .$$

Thus, $\mathcal{G} = \mathcal{F}_0$, and we note that $\kappa = 3$, $m = 2$, and $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. By the results of [3], RCP is not solvable by continuous state feedback. Now we show it is solvable by piecewise affine feedback.

First we examine the structure of $\mathcal{B}$ to reveal the reach control indices (note indices are not reordered). We find by inspect that $b_1 := (-2, 1, 0, 0) \in \mathcal{B} \cap \mathcal{C}_1$, $b_3 := (0, 0, -2, 1) \in \mathcal{B} \cap \mathcal{C}_3$, and $\mathcal{B} = \text{sp}\{b_1, b_3\}$. Therefore, $\mathcal{B}$ splits into two dependent cycles with respect to $\mathcal{G}$. In particular, $b_2 := -b_1 \in \mathcal{B} \cap \mathcal{C}_2$, $b_4 := -b_3 \in \mathcal{B} \cap \mathcal{C}_4$, $r_1 = 2$ and $r_2 = 2$.

1) First subdivision: In the first iteration $\mathcal{S}$ is subdivided into subsimplices $\mathcal{S}^1$ and $\mathcal{S}'$. Since $b_2 \cdot h_0 > 0$, we choose $v' = (0, 0.75, 0, 0) \in (v_0, v_2)$ such that we obtain the condition $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq 0$. Hence $\mathcal{S}'$ = conv$\{v_0, v_1, v', v_3, v_4\}$ and $\mathcal{S}^1$ = conv$\{v', v_1, v_2, v_3, v_4\}$.

In order to satisfy the invariance conditions for $\mathcal{S}^1$ the control inputs at the vertices of $\mathcal{S}^1$ are chosen as $u' = (-1, -2)$, $u_{11} = (-1, -2)$, $u_{12} = (-1, -2)$, $u_{13} = (-1, -2)$, and $u_{14} = (1, 0)$. This yields an affine feedback

$$u = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ -2 \end{bmatrix} , \qquad x \in \mathcal{S}^1 .$$

For $\mathcal{S}^1$ the invariance conditions are solvable and $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq 0$, so RCP on $\mathcal{S}^1$ is solvable. For $\mathcal{S}'$ we have $\mathcal{G}' := \mathcal{S}' \cap \mathcal{O} = \text{co}\{v_1, v_3, v_4\}$. Since $\kappa' = 2$ and $m = 2$, RCP is not solvable by continuous state feedback on $\mathcal{S}'$, and further subdivision of $\mathcal{S}'$ is required.

2) Second subdivision: Consider the subsimplex $\mathcal{S}' = \text{co}\{v_0, v_1, v', v_3, v_4\}$, where $v' \in (v_0, v_2) = (0, 0.75, 0, 0)$ and the exit facet is $\mathcal{F}_0'$ =conv$\{v_1, v', v_3, v_4\}$. We subdivide $\mathcal{S}'$ into subsimplices $\mathcal{S}^3$ and $\mathcal{S}^2$ and use a piecewise affine feedback law to solve RCP on $\mathcal{S}'$. It is clear that $b_4 \cdot h_0' > 0$ and therefore we can choose $v'' \in (v_0, v_4)$ such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^2) \neq \mathbf{0}$. One choice is $v'' := (0, 0, 0, 0.8)$. Let $\mathcal{S}^3 = \text{co}\{v_0, v_1, v', v_3, v''\}$ and $\mathcal{S}^2 = \text{co}\{v'', v_1, v', v_3, v_4\}$. It can be verified that $\mathcal{B} \cap \text{cone}(\mathcal{S}^2) \neq 0$. In order to satisfy the invariance conditions for $\mathcal{S}^2$ the control inputs at the vertices of $\mathcal{S}^2$ can be chosen as follows: $u'' = (-4, 0.6)$, $u_{21} = (-5, -1)$, $u' = (-1, -2)$, $u_{23} = (-5, -1)$, $u_{24} = (-3, 1)$. In order to satisfy the invariance conditions for $\mathcal{S}^3$ the control inputs at the vertices of $\mathcal{S}^3$ can be chosen as follows: $u_0 = (0, 0)$, $u_{31} = (-1, 0)$, $u' = (-1, -2)$, $u_{33} = (0, -1)$, and $u'' = (-4, 0.6)$. This yields a piecewise affine feedback

$$u = \begin{cases} \begin{bmatrix} -1 & -1.33 & 0 & -5 \\ 0 & -2.66 & -1 & 0.75 \end{bmatrix} x , & x \in \mathcal{S}^3 \\ \begin{bmatrix} 3 & 9.33 & 3 & 5 \\ 0 & -1.33 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} -8 \\ -1 \end{bmatrix} , & x \in \mathcal{S}^2 . \end{cases}$$

For $\mathcal{S}^2$ the invariance conditions are solvable and $\mathcal{B} \cap \text{cone}(\mathcal{S}^2) \neq 0$, so RCP on $\mathcal{S}^2$ is solvable. For $\mathcal{S}^3$ we have $\mathcal{G}_3 = \mathcal{S}^3 \cap \mathcal{O} = \text{co}\{v_1, v_3\}$. Since $\kappa_3 = 1$ and $m = 2$, RCP is solvable by affine feedback. Indeed, $\{b_1, b_3 \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ is a linearly independent set associated with $\mathcal{G}_3$.

VII. Conclusion

The paper studies the reach control problem on simplices, and we investigate cases when the problem is not solvable by continuous state feedback. It is shown that the class of piecewise affine feedbacks is sufficient to solve the problem in all cases of interest; namely, those cases when the problem is solvable by open-loop controls.

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