Reach Control on Simplices by Piecewise Affine Feedback

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Abstract

We study the reach control problem for affine systems on simplices, and the focus is on cases when it is known that the problem is not solvable by continuous state feedback. We examine from a geometric viewpoint the structural properties of the system which make continuous state feedbacks fail. This structure is encoded by so-called *reach control indices*, which are defined and developed in the paper. Based on these indices, we propose a subdivision algorithm and associated piecewise affine feedback. The method is shown to solve the reach control problem in all remaining cases, assuming it is solvable by open-loop controls.

I. Introduction

This paper studies the *reach control problem* on simplices. The problem is for an affine system defined on a simplex to reach a prespecified facet of the simplex in finite time. The overall concept of the problem and its setting were introduced in [6] and further developed in [7], [8], [14], [4]. The significance of the problem stems from its capturing the essential features of reachability problems for control systems: the presence of state constraints and the notion of trajectories reaching a goal in a guided and finite-time manner. The problem fits within a larger family of reachability problems; namely, to reach a target set $\mathcal{X}_f$ with state constraint in a set $\mathcal{X}$, denoted as $\mathcal{X} \xrightarrow{\mathcal{X}} \mathcal{X}_f$. Two sample reachability problems are as follows. First, consider the reachability problem: starting at any initial state in a bounded set Q, reach a target set Q_t while avoiding an unsafe region Q_u. The problem can be formulated as $\mathcal{X} \xrightarrow{\mathcal{X}} Q_t$, where $\mathcal{X} = Q - Q_u$. A typical example of this problem is motion planning of multiple vehicles. Second, consider the problem of temporal logic control. For example, let three areas of interest in the state space be denoted by Q_0, Q_1, Q_2 such that $Q_1, Q_2 \subset Q_0$. The temporal logic specification $\Box Q_0 \wedge \Diamond(Q_1 \wedge (Q_1 \mathcal{U} Q_2))$, interpreted in natural language, says "Stay always in Q_0 and reach Q_1, then stay in Q_1 until eventually reaching Q_2." This problem can be broken down into two reachability problems $(Q_0 - Q_2) \xrightarrow{(Q_0 - Q_2)} Q_1$ and $Q_1 \xrightarrow{Q_1} Q_2$. In the present context, we assume that the state constraints give rise to a state space that is triangulable [9]; then the reachability specification is converted to a sequence of reachability problems on simplices of the triangulation. The reader is referred to [1], [4], [6], [7], [8], [14], [10] for further motivations, including how the studied problem arises in fundamental problems concerning hybrid systems.

In [4] it was shown that affine feedback and continuous state feedback are equivalent from the point of view of solvability of the reach control problem (RCP). The approach is based, fundamentally, on fixed point theory. The latter allows to deduce that continuous state feedbacks always generate closed-loop equilibria inside the simplex when affine feedbacks do. The current paper departs from these findings, and using a geometric approach, we explore the system structure that gives rise to equilibria. This structure is encoded in so-called *reach control indices*. The first goal of this paper is to elucidate these indices, and to show how they isolate closed-loop equilibria. The second goal is to use the indices to obtain a subdivision of the simplex and an associated piecewise affine feedback to solve RCP in those cases when the problem is not solvable by continuous state feedback. It is shown that RCP is solvable by piecewise affine feedback if it is solvable by open-loop controls.

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The paper is organized as follows. In Section II we review the existing results on the reach control problem. In Section III we give necessary conditions for solvability by open-loop controls. These then shape the assumptions that are made to define the reach control indices, which are developed in Section IV. Next, in Section V, a subdivision method and associated piecewise affine feedback are proposed to solve RCP when continuous state feedback does not. The main result is presented in Section V showing the relationship between solvability via open-loop controls and solvability via piecewise affine feeback. Examples are presented in Section VI.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The complement of $\mathcal{K}$ is $\mathcal{K}^c := \mathbb{R}^n \setminus \mathcal{K}$, the closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. For a vector $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). For a matrix $A \in \mathbb{R}^{n\times n}$, the notation $A \succ 0$ ($A \succeq 0$) means $a_{ij} > 0$ ($a_{ij} \geq 0$) for $1 \leq i, j \leq n$. Notation **0** denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\mathscr{B}$ denotes the open unit ball, and $\overline{\mathscr{B}}$ denotes its closure. The notation co$\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation sp$\{y_1, y_2, \ldots\}$ denotes the span of vectors $y_i \in \mathbb{R}^n$. Symbol $\mathbb{U}$ denotes a control type: we consider open-loop controls, continuous state feedback, affine feedback, and (discontinuous) piecewise affine feedback. Finally, $T_{\mathcal{S}}(x)$ denotes the Bouligand tangent cone to set $\mathcal{S}$ at a point x [5].

II. Background

We give a brief survey of relevant results on RCP. The reader may consult [7] for further explanations on the invariance conditions (2)-(3) and affine feedbacks, while [4] provides illustrations of the geometric constructs used.

Consider an n-dimensional simplex[1] with vertex set $V := \{v_0, v_1, \ldots, v_n\}$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ (the facet is indexed by the vertex it does not contain). Let h_i, $i = 0, \ldots, n$ be the unit normal vector to each facet $\mathcal{F}_i$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet* of $\mathcal{S}$. Define the index sets $I := \{1, \ldots, n\}$ and $I_i := I \setminus \{i\}$ (note $I_0 = I$). For $i \in I \cup \{0\}$, define the closed, convex cone

$$\mathcal{C}_i := \left\{ \; y \in \mathbb{R}^n \;\; : \;\; h_j \cdot y \leq 0, \;\; j \in I_i \; \right\}.$$

We'll write cone$(\mathcal{S}) := \mathcal{C}_0$ since $\mathcal{C}_0$ is the tangent cone to $\mathcal{S}$ at v_0. We consider the affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad\qquad x \in \mathcal{S}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and rank$(B) = m$. Let $\phi_u(t, x_0)$ denote the trajectory of (1) starting at x_0 under some control law u.

Problem 1 (Reach Control Problem (RCP))**.** *Consider system (1) defined on $\mathcal{S}$. Find a feedback control $u(x)$ such that:*

(i) *For every $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T+\gamma)$.*
(ii) *There exists $\varepsilon > 0$ such that for all $x \in \mathcal{S}$, $\quad \|Ax + a + Bu(x)\| > \varepsilon$.*
(iii) *Feedback $u(x)$ satisfies the invariance conditions* (3) *on $\mathcal{F}_0$.*

Remark 1. *Condition (ii) is a robustness condition that rules out the possibility of equilibria on $\mathcal{F}_0$ even if trajectories starting in $\mathcal{S}$ reach $\mathcal{F}_0$ in finite time. Condition (iii) only takes effect at points $x \in \mathcal{F}_0$ that also belong to one or more restricted facets $\mathcal{F}_i$, $i \in I$. It is placed for practical reasons to avoid trajectories "spraying out" outside of the set* cone$(\mathcal{S})$*, and it can be removed if so desired. It can be easily shown that conditions (ii) and (iii) hold automatically if condition (i) is met using affine or continuous state feedback [7]. Thus, results on affine feedbacks [8], [14] and continuous state feedbacks [4] remain valid. However, a distinction arises when studying solvability of RCP by open-loop and discontinuous controls, when conditions (ii) and (iii) cannot be deduced from (i).*

[1]An n-dimensional simplex is the convex hull of $n + 1$ affinely independent points.

Definition 2. *A point $x_0 \in \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ and using control type $\mathbb{U}$, denoted by $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a control u of type $\mathbb{U}$ such that properties (i)-(iii) of Problem 1 hold. We write $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by control type $\mathbb{U}$ if for every $x_0 \in \mathcal{S}$, $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$ with control of type $\mathbb{U}$.*

Definition 3. *We say the invariance conditions are solvable at vertex v_i if there exists $u_i \in \mathbb{R}^m$ such that*

$$h_j \cdot (Av_i + a + Bu_i) \le 0\,, \qquad\qquad j \in I_i\,. \tag{2}$$

We say the invariance conditions are solvable if they are solvable at all vertices of $\mathcal{S}$.

The inequalities (2) are called the *invariance conditions*. They guarantee trajectories cannot exit from the facets $\mathcal{F}_i$, $i \in I$, and are used to construct affine feedbacks [7]. For general state feedbacks, stronger conditions (also called invariance conditions) are needed.

Definition 4. *We say a state feedback $u(x)$ satisfies the invariance conditions if for all $j \in I$ and $x \in \mathcal{F}_j$,*

$$h_j \cdot (Ax + Bu(x) + a) \le 0\,. \tag{3}$$

For Problem 1 the following necessary and sufficient conditions have been established for the case of affine feedback.

Theorem 5. *[8], [14] Given the system* (1) *and an affine feedback $u(x) = Kx + g$, where $K \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$, and $u_0 = u(v_0), \ldots, u_n = u(v_n)$, the closed-loop system satisfies $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ if and only if*

(a) *The invariance conditions* (2) *are satisfied.*
(b) *There is no closed-loop equilibrium in $\mathcal{S}$.*

Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Define $\mathcal{O} := \{\ x \in \mathbb{R}^n \ :\ Ax + a \in \mathcal{B}\}$ and $\mathcal{G} := \mathcal{S} \cap \mathcal{O}$. Associated with $\mathcal{G}$ is its vertex index set $I_{\mathcal{G}} := \{i \mid v_i \in V \cap \mathcal{G}\}$. In the remainder of the paper we make an important assumption concerning the placement of $\mathcal{O}$ with respect to $\mathcal{S}$. The reader is referred to [4] for the motivation for and a method of triangulation of the state space that achieves this assumption. See also [9].

Assumption 6. *Simplex $\mathcal{S}$ and system* (1) *satisfy the following condition: if $\mathcal{G} \neq \emptyset$, then $\mathcal{G}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \le \kappa \le n$.*

In [4] several simple cases in which affine feedbacks exist were identified.

Theorem 7 ([4])**.** *Suppose $\mathcal{G} = \emptyset$. If the invariance conditions* (2) *are solvable, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.*

Theorem 8 ([4])**.** *Suppose Assumption 6 holds and $\mathcal{G} \neq \emptyset$. If the invariance conditions* (2) *are solvable and $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq \mathbf{0}$, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.*

The primary conclusion of [4] is that RCP is solvable by affine feedback if and only if it is solvable by continuous state feedback. The goal of this paper is to solve RCP in cases where continuous state feedback cannot be used. Based on [4], the cases to be studied are captured by the following assumptions.

Assumption 9. *Simplex $\mathcal{S}$ and system* (1) *satisfy the following conditions.*

(A1) $\mathcal{G} = \mathrm{co}\{v_1, \ldots, v_{\kappa+1}\}$*, with $0 \le \kappa < n$.*
(A2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$.
(A3) *The maximum number of linearly independent vectors in any set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ (with only one vector for each $\mathcal{B} \cap \mathcal{C}_i, i \in I_{\mathcal{G}}$) is $\widehat{m}$ with $0 \le \widehat{m} \le \kappa + 1$.*

Assumption (A1) rules out the application of Theorem 7, and it enforces that $v_0 \notin \mathcal{O}$. The latter requirement is because when $v_0 \in \mathcal{O}$ and (A2) holds, then RCP is not solvable. Assumption (A2) rules out the application of Theorem 8. Finally, (A3) introduces a new condition in terms of the variable $\widehat{m}$, which necessarily satisfies $\widehat{m} \leq \kappa + 1$. When $\widehat{m} = \kappa + 1$, an affine feedback solves RCP, as stated below. The remaining cases when $\widehat{m} \leq \kappa$ are the topic of this paper.

Theorem 10 ([4]). *Suppose Assumption 9 holds. If the invariance conditions* (2) *are solvable and* $\widehat{m} = \kappa+1$*, then* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by affine feedback.*

Finally, in the sequel we make use of the following family of matrices. Let $1 \leq p \leq q \leq \kappa+1$, $b_i \in \mathcal{B} \cap \mathcal{C}_i$, and define $H_{p,q} := [h_p \cdots h_q]$, $Y_{p,q} := [b_p \cdots b_q]$, and $M_{p,q} := H_{p,q}^T Y_{p,q}$. We say a matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \leq 0$ for all $i \neq j$ [2]. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in I_{\mathcal{G}}$, each $M_{p,q}$ is a $\mathscr{Z}$-matrix. Also under the condition that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$, certain matrices of the form $M_{p,q}$ will be shown in Lemma 18 to be nonsingular $\mathscr{M}$-matrices. A complete characterization of nonsingular $\mathscr{M}$-matrices is found in [2], Ch. 6.

III. Necessary Conditions

In this section we investigate necessary conditions for solvability of RCP using open-loop controls. We say that a function $\mu : [0, \infty) \to \mathbb{R}^m$ is an *open-loop control* if it is bounded on any compact interval and it is measurable. By Caratheodory's theorem solutions of (1) using open-loop controls exist and are unique. First, we show that condition (i) of RCP alone is sufficient to conclude that the invariance conditions are solvable.

Theorem 11. *Suppose there exist open-loop controls such that condition (i) of RCP holds. Then the invariance conditions* (2) *are solvable.*

Proof: Define $\mathcal{Y}(x) := \big\{ Ax+Bw+a \mid w \in \mathbb{R}^m \big\}$. Let $x_0 \in \mathcal{S} \setminus \mathcal{F}_0$. By assumption there exists $\mu(t)$ and a time $T > 0$ such that $\phi_\mu(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$. Since $\mu(t)$ is an open-loop control, there exists $c > 0$ such that $\|\mu(t)\| \leq c$, for all $t \in [0, T]$. Consider the set $\mathcal{Y}_c(x) := \big\{ Ax + Bw + a \mid w \in \mathbb{R}^m, \|w\| \leq c \big\}$. It is easy to show that both $x \mapsto \mathcal{Y}_c(x)$ and $x \mapsto \mathcal{Y}(x)$ are upper semicontinuous. Now take a sequence $\{t_i \mid t_i \in (0, T]\}$ with $t_i \to 0$. Since $\{y \in \mathcal{Y}_c(x) \mid x \in \mathbb{R}^n\}$ is bounded on $\mathcal{S}$, there exists $M > 0$ such that $\|\phi_\mu(t_i, x_0) - x_0\| \leq M t_i$. Therefore $\{\frac{\phi_\mu(t_i, x_0) - x_0}{t_i}\}$ is a bounded sequence, and there exists a convergence subsequence (with indices relabeled) such that $\lim_{i \to \infty} \frac{\phi_\mu(t_i, x_0) - x_0}{t_i} =: v$. Since $\phi_\mu(t_i, x_0) \in \mathcal{S}$, by the definition of the Bouligand tangent cone, $v \in T_{\mathcal{S}}(x_0)$. Now we show $v \in \mathcal{Y}(x_0)$.

We have

$$\frac{\phi_\mu(t_i, x_0) - x_0}{t_i} = \frac{1}{t_i} \int_0^{t_i} \big[A\phi_\mu(\tau, x_0) + B\mu(\tau) + a \big] d\tau \, . \tag{4}$$

Let $y_0 := Ax_0 + B\mu(0) + a \in \mathcal{Y}(x_0)$ and $y(\tau) := A\phi_\mu(\tau, x_0) + B\mu(\tau) + a \in \mathcal{Y}(\phi_\mu(\tau, x_0))$. By the upper semicontinuity of $x \mapsto \mathcal{Y}(x)$, given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x_0 - \phi_\mu(\tau, x_0)\| < \delta$, then $\|y_0 - y(\tau)\| < \epsilon$. This implies, for i sufficiently large and for all $\tau \in [0, t_i]$, $\|y_0 - y(\tau)\| < \epsilon$. This can be rewritten as: for i sufficiently large and $\forall \tau \in [0, t_i]$, $y(\tau) = y_0 + p(\tau)$ for some function $p(\tau)$ satisfying $\|p(\tau)\| < \epsilon$. Thus, for i sufficiently large $\frac{1}{t_i} \int_0^{t_i} y(\tau) d\tau = y_0 + \frac{1}{t_i} \int_0^{t_i} p(\tau) d\tau$. However $\left\| \frac{1}{t_i} \int_0^{t_i} p(\tau) d\tau \right\| \leq \frac{1}{t_i} \int_0^{t_i} \|p(\tau)\| d\tau < \epsilon$. We conclude, for i sufficiently large, $\frac{1}{t_i} \int_0^{t_i} \big[A\phi_\mu(\tau, x_0) + B\mu(\tau) + a \big] d\tau \in \mathcal{Y}(x_0) + \epsilon \mathscr{B}$. Using (4), for i sufficiently large we have $\frac{\phi_\mu(t_i, x_0) - x_0}{t_i} \in \mathcal{Y}(x_0) + \epsilon \mathscr{B}$. Since $\mathcal{Y}(x_0)$ is a closed subset of $\mathbb{R}^n$, $v \in \mathcal{Y}(x_0) + \epsilon \overline{\mathscr{B}}$, and since ϵ is arbitrary, $v \in \mathcal{Y}(x_0)$. We conclude that $\mathcal{Y}(x_0) \cap T_{\mathcal{S}}(x_0) \neq \emptyset \, , \qquad x_0 \in \mathcal{S} \setminus \mathcal{F}_0$. Since $T_{\mathcal{S}}(v_0) = \text{cone}(\mathcal{S})$, and $T_{\mathcal{S}}(x) = \mathcal{C}_i$ for $x \in (v_0, v_i)$, it follows that the invariance conditions are solvable at v_0 and along simplex edges $(v_0, v_i), i \in I$.

Now consider $v_i, i \in I$. If $v_i \in \mathcal{O}$, then the invariance conditions are solvable by selecting $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = 0$. Instead suppose $v_i \notin \mathcal{O}$. Suppose by way of contradiction that $\mathcal{Y}(v_i) \cap \mathcal{C}_i = \emptyset$. Then $\mathcal{Y}(v_i)$ and $\mathcal{C}_i$ are non-empty disjoint polyhedral convex sets in $\mathbb{R}^n$. By Corollary 19.3.3 of [13], they are strongly separated. That is, there exists $\epsilon > 0$ such that $\inf_{y \in \mathcal{Y}(v_i), z \in \mathcal{C}_i} \|y - z\| > \epsilon$. By the upper semicontinuity of $x \mapsto \mathcal{Y}(x)$, there exists $\delta > 0$ such that if $\|x - v_i\| < \delta$, then $\mathcal{Y}(x) \subset \mathcal{Y}(v_i) + \frac{\epsilon}{2} \mathscr{B}$. In particular, taking $x \in (v_0, v_i)$, we get $\mathcal{Y}(x) \cap \mathcal{C}_i = \emptyset$, a contradiction. ■

We know that $Av_i + a \in \mathcal{B}$ for vertices $v_i \in \mathcal{G}$. Thus, Theorem 11 says that if RCP is solvable by open-loop controls, then $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, $i \in I_{\mathcal{G}}$. The next result says that, moreover, the zero vector cannot be the only element of $\mathcal{B} \cap \mathcal{C}_i$, $i \in I_{\mathcal{G}}$.

Theorem 12. *If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls, then $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}$, $i \in I_{\mathcal{G}}$.*

Proof: Consider vertex $v_i \in \mathcal{O}$. Suppose by way of contradiction that $\mathcal{B} \cap \mathcal{C}_i = \mathbf{0}$. Since $Av_i + a \in \mathcal{B}$, there exists $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = 0$. By condition (ii) of RCP and by Theorem 11, there exists $\varepsilon > 0$ such that for all $x \in \mathcal{S} \setminus \mathcal{F}_0$, there exists $\mu_x \in \mathbb{R}^m$ such that $Ax + B\mu_x + a \in T_{\mathcal{S}}(x)$ and $\|Ax + B\mu_x + a\| > \varepsilon$. By continuity there exists $\delta > 0$ such that if $\|x - v_i\| < \delta$, then $\|Ax + Bu_i + 1\| < \varepsilon/2$. Thus, it must be that $\|B(\mu_x - u_i)\| > \varepsilon/2$ for all $x \in \mathcal{S} \setminus \mathcal{F}_0$ satisfying $\|x - v_i\| < \delta$. Since $\mathcal{B} \cap \mathcal{C}_i = \mathbf{0}$ and $\mathcal{C}_i$ is a closed cone, there exists $\alpha > 0$ such that if $b \in \mathcal{B}$ satisfies $\|b\| > \varepsilon/2$, then $b + \alpha\mathscr{B} \notin \mathcal{C}_i$. In particular, we can choose $x \in (v_0, v_i)$ sufficiently close to v_i such that $\|Ax + a + Bu_i\| < \min\{\alpha, \epsilon/2\}$. Then $Ax + Bu_x + a = (Ax + a + Bu_i) + B(u_x - u_i) \notin \mathcal{C}_i = T_{\mathcal{S}}(x)$, a contradiction. ■

IV. Reach Control Indices

The reach control indices are defined in the situation when it is known that RCP is not solvable by continuous state feedback but it is still solvable by open-loop control. Assumption 9 specifies there is a maximal set of linearly independent vectors in $\mathcal{B}$ available to vertices in $\mathcal{G}$ in terms of the variable $\widehat{m}$. We say that any set of $\widehat{m}$ linearly independent vectors with at most one vector from each cone $\mathcal{B} \cap \mathcal{C}_i, i \in I_{\mathcal{G}}$, is a *maximal set with respect to* $\mathcal{G}$. Without loss of generality (by reordering indices) let such a maximal set be $\{b_1, \ldots, b_{\widehat{m}}\}$ and define

$$\widehat{\mathcal{B}} := \text{sp}\{b_1, \ldots, b_{\widehat{m}} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\} . \tag{5}$$

By the maximality of $\widehat{\mathcal{B}}$, $\mathcal{B} \cap \mathcal{C}_j \subset \widehat{\mathcal{B}} \cap \mathcal{C}_j$ for all $j = \widehat{m} + 1, \ldots, \kappa + 1$. Thus $\mathcal{B} \cap \mathcal{C}_j = \widehat{\mathcal{B}} \cap \mathcal{C}_j$, $j = \widehat{m} + 1, \ldots, \kappa + 1$.

Assumption 13. *Simplex $\mathcal{S}$ and system* (1) *satisfy the following conditions.*

(R1) $\mathcal{G} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$, *where* $0 \leq \kappa < n$.
(R2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$.
(R3) $\widehat{\mathcal{B}} = \text{sp}\{b_1, \ldots, b_{\widehat{m}} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$, *where* $\widehat{m} < \kappa + 1$.
(R4) $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}\,, \quad i \in I_{\mathcal{G}}$.

Let $p := \kappa + 1 - \widehat{m}$. Consider $\mathcal{B} \cap \mathcal{C}_{\widehat{m}+1}$. By Assumption (R4), $\mathcal{B} \cap \mathcal{C}_{\widehat{m}+1} \neq \mathbf{0}$, so there exists $2 \leq r_1 \leq \widehat{m} + 1$ such that w.l.o.g. (reordering indices $1, \ldots, \widehat{m}$)

$$\mathcal{B} \cap \mathcal{C}_{\widehat{m}+1} \subset \text{sp}\{b_1, \ldots, b_{r_1 - 1}\} \tag{6}$$

and $\text{sp}\{b_1, \ldots, b_{r_1-1}\}$ is the smallest such subspace generated by the basis $\{b_1, \ldots, b_{\widehat{m}}\}$. We define $\mathcal{V}_{\widehat{m}+1} := \text{sp}\{b_1, \ldots, b_{r_1-1}\}$ to be the *container subspace* for the cone $\mathcal{B} \cap \mathcal{C}_{\widehat{m}+1}$. It is easily shown that, relative to the basis $\{b_1, \ldots, b_{\widehat{m}}\}$, $\mathcal{V}_{\widehat{m}+1}$ is unique. The container subspace has the interpretation that every basis vector in a container subspace contributes to generating at least one vector in the cone $\mathcal{B} \cap \mathcal{C}_{\widehat{m}+1}$. The following establishes that one can always find a vector in $\mathcal{B} \cap \mathcal{C}_{\widehat{m}+1}$ that depends on all the basis vectors of its container subspace.

Lemma 14. *Suppose Assumption 13 and* (6) *hold. There exists $b_{\widehat{m}+1} \in \mathcal{B} \cap \mathcal{C}_{\widehat{m}+1}$ such that*

$$b_{\widehat{m}+1} = c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}\,, \qquad c_i \neq 0, \quad i = 1, \ldots, r_1 - 1\,. \tag{7}$$

Lemma 15. *Suppose Assumption 13 and* (6) *hold. Then $\mathcal{B} \cap \mathcal{C}_i = \widehat{\mathcal{B}} \cap \mathcal{C}_i$, $i = 1, \ldots, r_1 - 1, \widehat{m} + 1$.*

Proof: First, it is clear that $\widehat{\mathcal{B}} \cap \mathcal{C}_i \subset \mathcal{B} \cap \mathcal{C}_i$. Now we show the converse. Consider w.l.o.g. $i = 1$. Suppose there exists $\tilde{b}_1 \in \mathcal{B} \cap \mathcal{C}_1$ and $\tilde{b}_1 \notin \widehat{\mathcal{B}} \cap \mathcal{C}_1$. Let $b_{\widehat{m}+1}$ be as in (7). Then $\{\tilde{b}_1, b_2, \ldots, b_{\widehat{m}}, b_{\widehat{m}+1}\}$ is a linearly independent set containing one more vector than $\widehat{\mathcal{B}}$, a contradiction. ■

It is useful at this stage to swap the indices $\widehat{m}+1$ and r_1 in order that cones that share directions in $\widehat{\mathcal{B}}$ can be listed consecutively. Then (6) and (7) become respectively

$$\mathcal{B} \cap \mathcal{C}_{r_1} \subset \text{sp}\{b_1, \ldots, b_{r_1-1}\} \tag{8}$$

$$(\exists b_{r_1} \in \mathcal{B} \cap \mathcal{C}_{r_1}) \quad b_{r_1} = c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}\,, \quad c_i \neq 0\,. \tag{9}$$

We can now write

$$\widehat{\mathcal{B}} = \text{sp}\{b_1, \ldots, b_{r_1-1}, \overline{b}_{r_1}, b_{r_1+1}, \ldots, b_{\widehat{m}+1}\} = \text{sp}\{b_2, \ldots, b_{\widehat{m}+1}\}\,. \tag{10}$$

The overbar on $\overline{b}_{r_1}$ indicates it depends on all the previous $r_1 - 1$ vectors in the list. For this reason, $\{b_2, \ldots, b_{\widehat{m}+1}\}$ are linearly independent.

Lemma 16. *Suppose Assumption 13 and* (8)-(9) *hold. Then the coefficients in* (9) *satisfy* $c_i < 0$, $i = 1, \ldots, r_1 - 1$.

Proof: Suppose w.l.o.g. (by reordering indices $\{1, \ldots, r_1 - 1\}$), there exists $1 \leq \rho \leq r_1 - 1$ such that $c_i > 0$ for $i = 1, \ldots, \rho$ and $c_i < 0$ for $i = \rho + 1, \ldots, r_1 - 1$. Consider the vector $\beta := b_{r_1} - c_{\rho+1} b_{\rho+1} - \cdots - c_{r_1-1} b_{r_1-1} = c_1 b_1 + \cdots + c_\rho b_\rho$. Notice that $\beta \neq 0$ since $\{b_1, \ldots, b_\rho\}$ are linearly independent. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in \{1, \ldots, r_1\}$, we have $h_j \cdot \beta = h_j \cdot \big(b_{r_1} - c_{\rho+1} b_{\rho+1} - \cdots - c_{r_1-1} b_{r_1-1}\big) \leq 0$, $i = 1, \ldots, \rho, r_1 + 1, \ldots, n$. Also $h_j \cdot \beta = h_j \cdot \big(c_1 b_1 + \cdots + c_\rho b_\rho\big) \leq 0$, $i = \rho + 1, \ldots, n$. In sum, $h_j \cdot \beta \leq 0$, $i \in I$; that is, $\beta \in \mathcal{B} \cap \text{cone}(\mathcal{S})$. By Assumption (R2), $\beta = 0$, a contradiction. ■

The dependency of cones on a limited number of vectors in $\widehat{\mathcal{B}}$ places restrictions on the orientation of those vectors with respect to $\mathcal{S}$.

Lemma 17. *Suppose Assumption 13 and* (8)-(9) *hold. Then*

$$h_j \cdot b_i = 0\,, \qquad i = 1, \ldots, r_1\,, \qquad j \in I \setminus \{1, \ldots, r_1\}\,. \tag{11}$$

Proof: Let b_{r_1} be as in (9). Since $b_{r_1} \in \mathcal{B} \cap \mathcal{C}_{r_1}$, $h_j \cdot b_{r_1} = h_j \cdot \big(c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}\big) \leq 0$, $j \in I \setminus \{1, \ldots, r_1\}$. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$ and, by Lemma 16, $c_i < 0$, every term in the sum is non-negative. The result immediately follows. ■

Lemma 18. *Suppose Assumption 13 and* (8)-(9) *hold. Then* $M_{1,r_1-1} \in \mathbb{R}^{(r_1-1)\times(r_1-1)}$ *is a nonsingular* $\mathscr{M}$*-matrix.*

Proof: First, we know M_{1,r_1-1} is a $\mathscr{Z}$-matrix because $h_j \cdot b_i \leq 0$, $j \neq i$, so the off-diagonal entries are non-positive. Second, we show $M_{1,r_1-1} = H^T_{1,r_1-1} Y_{1,r_1-1}$ is nonsingular. Suppose there exists $c \in \mathbb{R}^{r_1-1}$ such that $H^T_{1,r_1-1} Y_{1,r_1-1} c = 0$. Let $y := Y_{1,r_1-1} c$. By assumption $h_j \cdot y = 0$, $j = 1, \ldots, r_1 - 1$. Also by Lemma 17, $h_j \cdot y = 0$, $j = r_1 + 1, \ldots, n$. Thus, either $y \in \mathcal{B} \cap \text{cone}(\mathcal{S})$ or $-y \in \mathcal{B} \cap \text{cone}(\mathcal{S})$. By Assumption (R2), $y = 0$. However, $y = c_1 b_1 + \cdots + c_{r_1-1} b_{r_1-1}$ and $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent, so $c = 0$. We conclude that M_{1,r_1-1} is nonsingular.

Finally, we show M_{1,r_1-1} satisfies case (Q_{50}) of Theorem 6.2.3 of [2]. Suppose there exists $c \in \mathbb{R}^{r_1-1}$ with $c \neq 0$ and $c \succeq 0$ such that $M_{1,r_1-1} c \preceq 0$. Define the vector $\bar{y} = Y_{1,r_1-1} c \in \mathcal{B}$. Note that $\bar{y} \neq 0$ because $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent. Then $M_{1,r_1-1} c = H^T_{1,r_1-1} Y_{1,r_1-1} c = H^T_{1,r_1-1} \bar{y} \preceq 0$ implies $h_j \cdot \bar{y} \leq 0$ for $j = 1, \ldots, r_1 - 1$. Also, since $c_i \geq 0$ and $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $h_j \cdot \bar{y} = \sum_{i=1}^{r_1-1} c_i (h_j \cdot b_i) \leq 0$, $j = r_1, \ldots, n$. This implies $0 \neq \bar{y} \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, a contradiction. Therefore, M_{1,r_1-1} has the property that the only solution of the inequalities $c \succeq 0$ and $M_{1,r_1-1} c \preceq 0$ is $c = 0$. In sum, M_{1,r_1-1} is a nonsingular $\mathscr{Z}$-matrix satisfying Theorem 6.2.3 case (Q_{50}) of [2], so M_{1,r_1-1} is a nonsingular $\mathscr{M}$-matrix. ■

Remark 19. *An important feature of the formula* (9) *with* $c_i < 0$ *is that, using it, any* b_i, $i \in \{1, \ldots, r_1\}$, *can be expressed as a negative linear combination of the other vectors* $\{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{r_1}\}$. *This means Lemma 18 can be deduced for other combinations of* $r_1 - 1$ *indices, not just* $\{1, \ldots, r_1 - 1\}$. *In particular, if we begin with the singular matrix* $M_{1,r_1} \in \mathbb{R}^{r_1 \times r_i}$, *and if the* i*th row and column are removed, the resulting submatrix is a nonsingular* $\mathscr{M}$*-matrix.*

Next consider $\mathcal{B} \cap \mathcal{C}_{\widehat{m}+2}$. There exists a smallest subspace generated by the basis $\{b_2, \ldots, b_{\widehat{m}+1}\}$ that contains $\mathcal{B} \cap \mathcal{C}_{\widehat{m}+2}$. By independently reordering each index set $\{2, \ldots, r_1\}$ and $\{r_1+1, \ldots, \widehat{m}+1\}$ we have $\mathcal{B} \cap \mathcal{C}_{\widehat{m}+2} \subset \text{sp}\{b_{\rho_2}, \ldots, b_{\rho_2+r_2-2}\}$, for some $2 \leq \rho_2 \leq r_1+1$ and $r_1 \leq \rho_2 + r_2 - 2 \leq \widehat{m}+1$. Now we swap the indices $\rho_2 + r_2 - 1$ and $\widehat{m}+2$, so that $\mathcal{B} \cap \mathcal{C}_{\rho_2+r_2-1} \subset \text{sp}\{b_{\rho_2}, \ldots, b_{\rho_2+r_2-2}\}$. The next result shows that because of the restrictions placed on $\mathcal{B}$ by Lemma 17, the container subspaces of dependent cones split into independent subspaces, thus yielding a decomposition of $\mathcal{B}$ with respect to $\mathcal{G}$.

Lemma 20. *Suppose Assumption 13,* (8)*, and* (9) *hold. Then* $\rho_2 = r_1 + 1$.

Proof: Suppose by way of contradiction that $\rho_2 < r_1 + 1$. Applying Lemma 17 we obtain

$$h_j \cdot b_i = 0, \quad i = 1, \ldots, r_1, \quad j = r_1+1, \ldots, n \tag{12a}$$
$$h_j \cdot b_i = 0, \quad i = \rho_2, \ldots, \rho_2 + r_2 - 1, \quad j = 1, \ldots, \rho_2 - 1, \rho_2 + r_2, \ldots, n. \tag{12b}$$

Consider $M_{\rho_2,r_1} = H^T_{\rho_2,r_1} Y_{\rho_2,r_1}$, where $Y_{\rho_2,r_1} = [b_{\rho_2} \cdots b_{r_1}]$. By Lemma 18 and Remark 19, M_{2,r_1} is a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3 and the remarks thereafter [2], M_{ρ_2,r_1} is also a nonsingular $\mathscr{M}$-matrix. By Theorem 6.2.3 (case I_{28}) of [2] there exists $c' = (c'_{\rho_2}, \ldots, c'_{r_1})$ such that $c' \preceq 0$ and $M_{\rho_2,r_1} c' \prec 0$. Define $\beta := Y_{\rho_2,r_1} c' \neq 0$. The statement $H^T_{\rho_2,r_1} \beta = M_{\rho_2,r_1} c' \prec 0$ gives $h_j \cdot \beta < 0$, $j = \rho_2, \ldots, r_1$. Recall that by construction $\rho_2 + r_2 - 1 \geq r_1 + 1$. Therefore, by (12a)-(12b) $h_j \cdot \beta = h_j \cdot \left(c'_{\rho_2} b_{\rho_2} + \cdots + c'_{r_1} b_{r_1}\right) = 0$, $j = 1, \ldots, \rho_2 - 1, r_1 + 1, \ldots, n$. This implies $\beta \in \mathcal{B} \cap \text{cone}(\mathcal{S})$. By Assumption (R2) we obtain $\beta = 0$, a contradiction. ■

By iterating on Lemmas 17, 20, and our index swap, we can show that the decomposition obtained so far extends to all cones $\mathcal{B} \cap \mathcal{C}_i$ associated with $\mathcal{G}$. First, we note that the procedure generates a specially ordered list of velocity vectors associated with vertices in $\mathcal{G}$. This list has the form

$$\{b_1, \ldots, b_{r_1-1}, \overline{b}_{r_1}, b_{r_1+1}, \ldots, b_{r_1+r_2-1}, \overline{b}_{r_1+r_2}, \ldots, b_{r_1+\cdots+r_{p-1}+1}, \ldots, b_{r-1}, \overline{b}_r, b_{r+1}, \ldots, b_{\kappa+1}\}, \tag{13}$$

where $b_i \in \mathcal{B} \cap \mathcal{C}_i$ and $r := r_1 + \cdots + r_p$. The vectors that do not have an overbar are provided by (R3) (modulo the change of indices). The vectors that have overbars are provided by Lemma 14. Because of the index swapping and Lemma 20 each $\overline{b}_{r_1+\cdots+r_k}$ depends on all of the previous $r_k - 1$ vectors in the list. If the p vectors with overbars are removed, one obtains a maximal set with respect to $\mathcal{G}$ that spans $\widehat{\mathcal{B}}$. It is essentially the list in (R3) but has been reordered. Finally, the container subspace construction and Lemma 15 give

$$\widehat{\mathcal{B}} \cap \mathcal{C}_{r_1+\cdots+r_k} = \mathcal{B} \cap \mathcal{C}_{r_1+\cdots+r_k} \subset \mathcal{B}_k := \text{sp}\{b_{r_1+\cdots+r_{k-1}+1}, \ldots, b_{r_1+\cdots+r_k}\}, \quad k = 1, \ldots, p. \tag{14}$$

One can now extract a geometric consequence of our decomposition that has relevance to the usable directions in $\mathcal{B}$ at points in $\mathcal{G}$ that are not vertices. For example, at any point in $x \in \text{co}\{v_1, \ldots, v_{r_1}\}$, the only usable directions in $\mathcal{B}$ to satisfy the invariance conditions (3) are those already in $\mathcal{B}_1$. We state the result only for $k = 1$.

Lemma 21 ([4])**.** *Suppose Assumption 13 and* (14) *hold. There does not exist* $\beta \in \mathcal{B}$ *such that* $\{b_1, \ldots, b_{r_1-1}, \beta\}$ *are linearly independent and* $h_j \cdot \beta \leq 0$ *for* $j = r_1 + 1, \ldots, n$.

Theorem 22. *Suppose Assumption 13 holds. Then there exist integers* $r_1, \ldots, r_p \geq 2$ *and a decomposition of* $\mathcal{B}$ *into* p *subsets such that*

$$\mathcal{B} \cap \mathcal{C}_i \subset \text{sp}\{b_1, \ldots, b_{r_1}\}, \qquad i = 1, \ldots, r_1, \tag{15}$$
$$\mathcal{B} \cap \mathcal{C}_i \subset \text{sp}\{b_{r_1+1}, \ldots, b_{r_1+r_2}\}, \qquad i = r_1+1, \ldots, r_1+r_2 \tag{16}$$
$$\vdots \qquad\qquad \vdots$$
$$\mathcal{B} \cap \mathcal{C}_i \subset \text{sp}\{b_{r_1+\cdots+r_{p-1}+1}, \ldots, b_r\}, \qquad i = r_1 + \cdots + r_{p-1} + 1, \ldots, r. \tag{17}$$

Proof: W.l.o.g. we consider only (15). We must show $\mathcal{B} \cap \mathcal{C}_i \subset \text{sp}\{b_1, \ldots, b_{r_1}\}$, for $i = 1, \ldots, r_1$. (By (14) we already have $\mathcal{B} \cap \mathcal{C}_{r_1} \subset \text{sp}\{b_1, \ldots, b_{r_1-1}\}$). Consider any $i \in \{1, \ldots, r_1\}$ and any $\beta_i \in \mathcal{B} \cap \mathcal{C}_i =$

$\widehat{\mathcal{B}} \cap \mathcal{C}_i$ such that $\beta_i = c_1 b_1 + \cdots + c_{r_1} b_{r_1} + \beta$, where $c_i \in \mathbb{R}$ and $\beta \in \mathcal{B}$. We assume w.l.o.g. β is independent of $\{b_1, \ldots, b_{r_1}\}$, otherwise the c_i's can be redefined. From the invariance conditions associated with v_i and by Lemma 17, we have $h_j \cdot \beta_i = h_j \cdot (c_1 b_1 + \cdots + c_{r_1} b_{r_1} + \beta) = h_j \cdot \beta \leq 0$, $j = r_1 + 1, \ldots, n$. By Lemma 21, $\beta = 0$. Hence, for any $i \in \{1, \ldots, r_1\}$ and any $\beta_i \in \mathcal{B} \cap \mathcal{C}_i$, $\beta_i \in \text{sp}\{b_1, \ldots, b_{r_1}\}$, as desired. ■

The kth list in (15)-(17) has $r_k - 1$ linearly independent vectors of $\mathcal{B}$. We can say that $\mathcal{B}$ has been decomposed into p independent *cycles of dependency*. Also, because each of the excess p vertices in $\mathcal{G}$ has an associated non-zero velocity vector by (R4) depending on at least one (exclusive) vector in $\widehat{\mathcal{B}}$, we have $p \leq \widehat{m}$. Thus, in order for (R4) to hold it is necessary that $\widehat{m} \geq \frac{\kappa+1}{2}$. This condition is interpreted to say that RCP is only solvable if there are sufficient inputs. Since each of the p lists comprises $r_k - 1$ independent vectors in $\mathcal{B}$ and there are a total of $\widehat{m}$ such vectors, we also deduce that $r - p \leq \widehat{m}$.

The integers $\{r_1, \ldots, r_p\}$ are called the *reach control indices* of system (1) with respect to simplex $\mathcal{S}$. Their importance stems from their ability to isolate closed-loop equilibria when using continuous state feedback. Define for $k = 1, \ldots, p$, $m_k := r_1 + \cdots + r_{k-1} + 1$ and $\mathcal{G}_k := \text{co}\big\{v_{m_k}, \ldots, v_{m_k + r_k - 1}\big\}$.

Theorem 23 ([4]). *Suppose Assumption 13 holds. Let $u(x)$ be a continuous state feedback satisfying the invariance conditions* (3)*. Then each $\mathcal{G}_k$ contains an equilibrium of the closed-loop system.*

V. Piecewise Affine Feedback

The reach control indices catalog the degeneracies (caused by insufficient inputs) that lead to the appearance of equilibria in $\mathcal{S}$ whenever $p \geq 1$ and continuous state feedback is applied. Thus, any control method that overcomes the limits of continuous state feedback must confront this degeneracy and will necessarily draw upon the degrees of freedom in $\mathcal{B}$ provided to $\mathcal{G}$ which are inscribed by the indices. In this section we investigate the extent to which piecewise affine feedback can solve RCP, in cases when continuous state feedback cannot. We construct a triangulation [9] of the simplex $\mathcal{S}$ such that a sub-RCP is solvable for each simplex of the triangulation. We make the following standing assumptions.

Assumption 24. *Simplex $\mathcal{S}$ and system* (1) *satisfy the following conditions.*

(P1) $\mathcal{G} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$, *where* $0 \leq \kappa < n$.
(P2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$.
(P3) $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}\,, \quad i \in I_{\mathcal{G}}$.
(P4) $\exists\ \ \{r_1, \ldots, r_p\}$ *such that* (15)-(17) *hold.*
(P5) $\mathcal{B}_k \not\subset \mathcal{H}_0 := \{y \in \mathbb{R}^n \mid h_0 \cdot y = 0\}\,, \quad k = 1, \ldots, p$.

Conditions (P1)-(P2) define the problem setup as before. The necessity of (P3) was proved in Theorem 12, (P4) was proved in Theorem 22, and the necessity of (P5) is proved next.

Lemma 25. *If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls, then $\mathcal{B}_k \not\subset \mathcal{H}_0$ for each $k = 1, \ldots, p$.*

Proof: W.l.o.g. we consider only $k = 1$. Define $\widehat{\mathcal{F}}_0 := \text{co}\{v_1, \ldots, v_{r_1}\} \subset \mathcal{F}_0$. Let $x \in \widehat{\mathcal{F}}_0$. By condition (iii) of RCP, for any open-loop control values $\{\mu_x\}$ used to solve RCP (where μ_x is an open-loop control value used at x), we have $h_j \cdot (Ax + B\mu_x + a) \leq 0$, $x \in \mathcal{F}_j$, $j \in I$. By Lemmas 17 and 21, we also have that $h_j \cdot (Ax + B\mu_x + a) = 0$, $j = r_1 + 1, \ldots, n$. Suppose by way of contradiction that $\mathcal{B}_1 \subset \mathcal{H}_0$. Then $h_0 \cdot (Ax + B\mu_x + a) = 0$. Now we observe that for any $z \in \widehat{\mathcal{F}}_0$ $T_{\widehat{\mathcal{F}}_0}(z) = \{y \in \mathbb{R}^n \mid h_j \cdot y = 0, h_l \cdot y \leq 0, j = 0, r_1 + 1, \ldots, n, z \in \mathcal{F}_l\}$. We conclude that $Ax + B\mu_x + a \in T_{\widehat{\mathcal{F}}_0}(x)$, for all $x \in \widehat{\mathcal{F}}_0$. By Theorem 4.3.8 of [5], this implies $\widehat{\mathcal{F}}_0$ is a strongly invariant set, a contradiction to the statement that $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls. ■

Definition 26. *Given system* (1) *and a state feedback $u(x)$, we say $u(x)$ is a* piecewise affine feedback *if there exists a triangulation $\mathbb{T}$ of $\mathcal{S}$ such that for each n-dimensional $\mathcal{S}^j \in \mathbb{T}$, there exist $K^j \in \mathbb{R}^{m \times n}$ and $g^j \in \mathbb{R}^m$ such that $u(x) = K^j x + g^j$, $x \in \mathcal{S}^j$.*

Remark 27. *This definition of piecewise affine feedback allows for discontinuities at the boundaries of simplices; moreover, the feedback is a multi-valued function, distinct from the usual notion of piecewise affine function in algebraic topology [11]. Resolving what control value to use at points lying in more than one simplex is treated as a problem of implementation. The artifact of a* discrete supervisory controller *[12] will be introduced to convert the multi-valued function to a single-valued feedback.*

We now explain informally an inductive procedure for subdividing $\mathcal{S}$ in order that RCP can be solved by piecewise affine feedback. First, in Lemma 28 we show that because of condition (P5), each simplex $\mathcal{G}_k$, for $k = 1, \dots, p$, has a vertex (among $\{v_{m_k}, \dots, v_{m_k+r_k-1}\}$) with $b_i \in \mathcal{B} \cap \mathcal{C}_i$ pointing out of $\mathcal{S}$. By convention, we reorder indices so this vertex is the first one in each list $\{v_{m_k}, \dots, v_{m_k+r_k-1}\}$. We make a subdivision of $\mathcal{S}$ by placing a new vertex v' along the edge (v_0, v_{m_k}). In particular, at the first iteration we would have $v' \in (v_0, v_1)$, and we form two simplices $\mathcal{S}^1$ and $\mathcal{S}'$ as in Figure 1. Lemma 30 shows that because $b_{m_k} \in \mathcal{B} \cap \mathcal{C}_{m_k}$ points out of $\mathcal{S}$ at v_{m_k} and because the invariance conditions for $\mathcal{S}$ are solvable at v_0, a convexity argument (precisely, (20)) gives that v' can be placed along (v_0, v_{m_k}) so that $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq \mathbf{0}$. Then in Lemma 31 one applies Theorem 8 to obtain that RCP is solved for $\mathcal{S}^1$. Essentially $\mathcal{S}^1$ can be removed from further consideration, and the induction step is repeated with $\mathcal{S}$ replaced by the remainder $\mathcal{S}'$. To guarantee that the induction is sound, one must show that $\mathcal{S}'$ inherits the relevant properties of $\mathcal{S}$, especially condition (P5). This is done in Lemma 32.

Lemma 28. *Suppose Assumption 24 holds. Then w.l.o.g. (by reordering indices)* $h_0 \cdot b_{m_k} > 0$, $k = 1, \dots, p$.

Proof: We prove the result only for $k = 1$. If for any $j \in \{1, \dots, r_1\}$, $h_0 \cdot b_j > 0$, then the proof is finished. Instead suppose that for all $i \in \{1, \dots, r_1\}$, $h_0 \cdot b_i \leq 0$. Using (P5) and by reordering the indices $1, \dots, r_1$, assume $h_0 \cdot b_{r_1} < 0$. By Lemmas 14 and 16 there exists $b_1 \in \mathcal{B} \cap \mathcal{C}_1$ such that $b_1 = c_2 b_2 + \dots + c_{r_1} b_{r_1}$ with $c_i < 0$. Thus we obtain $h_0 \cdot b_1 = h_0 \cdot (c_2 b_2 + \dots + c_{r_1} b_{r_1}) \geq c_{r_1} h_0 \cdot b_{r_1} > 0$. ■

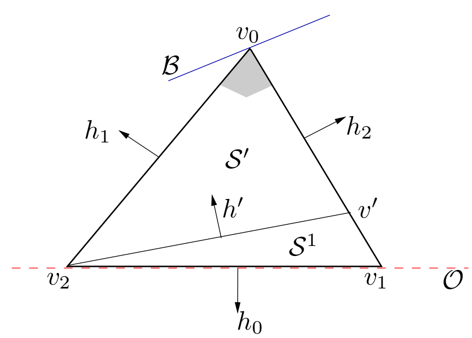


Fig. 1. Subdivision into two simplices $\mathcal{S}'$ and $\mathcal{S}^1$.

Following Lemma 28, suppose that b_1 satisfies $h_0 \cdot b_1 > 0$. We consider any point v' in the open segment (v_0, v_1). That is, let $\lambda \in (0, 1)$ and define

$$v' = \lambda v_1 + (1 - \lambda) v_0 \,. \tag{18}$$

Now define the following simplices in $\mathcal{S}$:

$$\begin{aligned} \mathcal{S}' &= \text{co}\{v_0, v', v_2, \dots, v_n\} \\ \mathcal{S}^1 &= \text{co}\{v', v_1, v_2, \dots, v_n\} \,. \end{aligned}$$

Also define the new exit facet for $\mathcal{S}'$ by $\mathcal{F}_0' := \text{co}\{v', v_2, \dots, v_n\}$. See Figure 1. The following lemma provides a formula for the normal vector h' of $\mathcal{F}_0'$.

Lemma 29. *Let* $h_0 = -\gamma_1 h_1 - \dots - \gamma_n h_n$ *with* $\gamma_i > 0$, *and let* $\lambda \in (0, 1)$. *Then the normal vector to* $\mathcal{F}_0'$

pointing out of $\mathcal{S}^1$ is

$$h' = \gamma_1 h_1 + \lambda \sum_{j=2}^{n} \gamma_j h_j = \gamma_1(1-\lambda)h_1 - \lambda h_0 \,. \tag{19}$$

Lemma 30. *Suppose Assumption 24 holds. There exists $v' \in (v_0, v_1)$, such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq \mathbf{0}$. Moreover, $b_1 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^1)$ with $h' \cdot b_1 < 0$.*

Proof: Observe that $\text{cone}(\mathcal{S}^1) = \{y \in \mathbb{R}^n \mid h' \cdot y \leq 0\,, h_j \cdot y \leq 0, \quad j \in \{2, \cdots, n\}\}$. We show there is an interval of values for λ such that $0 \neq b_1 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^1)$, where we assume the index ordering of Lemma 28. First, since $b_1 \in \mathcal{B} \cap \mathcal{C}_1$ we know $h_j \cdot b_1 \leq 0$ for $j \in \{2, \ldots, n\}$. We must only show that there exists $\lambda \in (0,1)$ such that $h' \cdot b_1 < 0$. From Lemma 29 we have

$$h' \cdot b_1 = \gamma_1(1-\lambda)h_1 \cdot b_1 - \lambda h_0 \cdot b_1 \,. \tag{20}$$

Since $h_1 \cdot b_1 > 0$ (because $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$) and $h_0 \cdot b_1 > 0$ (by Lemma 28), it is clear from (20) that we can select $\lambda = \lambda'$ sufficiently close to 1 such that $h' \cdot b_1 < 0$. Setting $v' = \lambda' v_1 + (1-\lambda')v_0$, we get $b_1 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^1)$. ∎

Lemma 31. *Suppose Assumption 24 holds and let v' be as in Lemma 30. If the invariance conditions for $\mathcal{S}$ are solvable, then $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback.*

Proof: By Lemma 30, we have $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq 0$. We show that the invariance conditions are solvable for $\mathcal{S}^1$. First, consider the vertex v'. By assumption there exist control inputs $u_0, u_1 \in \mathbb{R}^m$ such that the invariance conditions for $\mathcal{S}$ at v_0 and v_1 are satisfied, i.e. $y_0 := Av_0 + Bu_0 + a \in \text{cone}(\mathcal{S})$ and $y_1 := Av_1 + Bu_1 + a \in \mathcal{B} \cap \mathcal{C}_1$. In particular, $h_j \cdot y_i \leq 0$ for $i = 0, 1$ and $j = 2, \ldots, n$. Now by Lemma 30, there exists $\lambda \in (0,1)$ such that with $v' := \lambda v_1 + (1-\lambda)v_0$, $h' \cdot b_1 < 0$ and $h_j \cdot b_1 \leq 0$ for $j = 2, \ldots, n$. Let w_1 be such that $b_1 = Bw_1$. Set $\epsilon_1 > 0$ and let $u' := \lambda u_1 + (1-\lambda)u_0 + \epsilon_1 w_1$. $y' := Av' + Bu' + a = \lambda y_1 + (1-\lambda)y_0 + \epsilon_1 b_1$. Thus, $h_j \cdot y' \leq 0$ for $j = 2, \ldots, n$ and for $\epsilon_1 > 0$ sufficiently large, $h' \cdot y' < 0$. That is, the invariance conditions for $\mathcal{S}^1$ are solvable at v'.

Next consider v_1. Since the invariance conditions for $\mathcal{S}^1$ at v_1 are identical to those for $\mathcal{S}$ at v_1, and since the latter are by assumption solvable, the former are also solvable. Finally, consider vertices v_i, $i = 2, \ldots, n$. There exist control inputs $u_i \in \mathbb{R}^m$ such that $y_i := Av_i + Bu_i + a$ satisfy $h_j \cdot y_i \leq 0$ for $j = 2, \ldots, i-1, i+1, \ldots, n$. As above let w_1 be such that $b_1 = Bw_1$. Set $\epsilon_1 > 0$ and let $u_i' := u_i + \epsilon_1 w_1$. Then the closed-loop vector field for $\mathcal{S}^1$ at v_i is $y_i' = Av_i + Bu_i' + a = y_i + \epsilon_1 b_1$. Thus, $h_j \cdot y_i' \leq 0$ for $j = 2, \ldots, i-1, i+1, \ldots, n$ and for $\epsilon_1 > 0$ sufficiently large, $h' \cdot y_i' < 0$. That is, the invariance conditions for $\mathcal{S}^1$ are solvable at v_i. In sum, we can apply Theorem 8 to obtain that $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback. ∎

Lemma 32. *Suppose Assumption 24 holds and let v' be as in Lemma 30. If the invariance conditions for $\mathcal{S}$ are solvable then*

(i) *The invariance conditions for $\mathcal{S}'$ are solvable.*
(ii) $(-h') \cdot b_{m_k} > 0\,, \quad k = 1, \ldots, p.$

Proof: First we prove (i). By assumption the invariance conditions are solvable for $\mathcal{S}$, and since the invariance conditions for $\mathcal{S}'$ are identical (the only facet that changed for $\mathcal{S}'$ is $\mathcal{F}_0$, which plays no role in invariance conditions), they are also solvable for $\mathcal{S}'$. Next we prove (ii). First we have $(-h') \cdot b_{m_1} > 0$ by Lemma 30. Second, since $b_{m_k} \in \mathcal{B} \cap \mathcal{C}_{m_k}$, we have $h_1 \cdot b_{m_k} \leq 0$, for $k = 2, \ldots, p$. Also by Lemma 28, $h_0 \cdot b_{m_k} > 0$, for $k = 2, \ldots, p$. Thus using (19), $(-h') \cdot b_{m_k} = -\gamma_1(1-\lambda)h_1 \cdot b_{m_k} + \lambda h_0 \cdot b_{m_k} > 0$, $k = 2, \ldots, p$. ∎

We have demonstrated the first step of a triangulation procedure that partitions $\mathcal{S}$ into simplices on which sub-reach control problems are solvable. Now we present a triangulation algorithm that iterates on the presented subdivision method. It consists of p iterations, one for each set $\{v_{m_k}, \ldots, v_{m_k+r_k-1}\}$, $k =$

$1, \ldots, p$. The notation $\mathcal{S}^k := \text{co}\{v', v_1, \ldots, v_n\}$ is understood to mean that all $n+1$ vertices of $\mathcal{S}^k$ are assigned simultaneously in the order presented. The vertices of $\mathcal{S}^k$ are later identified as $\{v_0^k, \ldots, v_n^k\}$. The algorithm generates simplices $\mathcal{S}^1, \ldots, \mathcal{S}^{p+1}$ starting from the given simplex $\mathcal{S}$. At the kth iteration, the current declaration of $\mathcal{S}$ is split into a lower simplex $\mathcal{S}^k$ and an upper simplex. The lower simplex is then "thrown away" and the remainder - the upper simplex - is declared to be $\mathcal{S}$ with vertices called $\{v_0, \ldots, v_n\}$ (overloading the vertices of the previous $\mathcal{S}$). In this way each iterate mimics the first subdivision developed in the discussion above.

Subdivision Algorithm:

1. Set $k = 1$.
2. Select $v' \in (v_0, v_{m_k})$ such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^k) \neq \mathbf{0}$, where $\mathcal{S}^k := \text{co}\{v', v_1, \ldots, v_n\}$.
3. Set $\mathcal{S} := \text{co}\{v_0, v_1, \ldots, v_{m_k - 1}, v', v_{m_k + 1}, \ldots, v_n\}$.
4. If $k < p$, set $k := k + 1$ and go to step 2.
5. Set $\mathcal{S}^{p+1} := \mathcal{S}$.

Let $\mathcal{F}_0^k = \text{co}\{v_1^k, \ldots, v_n^k\}$ denote the exit facet of $\mathcal{S}^k = \text{co}\{v_0^k, \ldots, v_n^k\}$. The triangulation generated by the algorithm has the property that $\mathcal{S}^k \cap \mathcal{S}^{k-1} = \mathcal{F}_0^k$, $k = 2, \ldots, p+1$, and closed-loop trajectories follow paths through simplices with decreasing indices. Thus, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ is achieved by implementing affine controllers that achieve $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$ for $k = 1, \ldots, p+1$. In order to guarantee that switching occurs in the proper sequence (with decreasing simplex indices), and to avoid chattering caused by measurement errors, a *discrete supervisor* should accompany the implementation of the piecewise affine feedback. The supervisor has two functions:

(i) Once the piecewise affine controller has switched to simplex $\mathcal{S}^k$, then all affine controllers for $\mathcal{S}^j$, $j > k$, are disabled.
(ii) At a point $x \in \mathcal{S}$ belonging to more than one simplex $\mathcal{S}^j$, the controller for the simplex with the higher index is used.

Theorem 33. *Suppose Assumption 24 holds. If the invariance conditions for $\mathcal{S}$ are solvable, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by piecewise affine feedback.*

Proof: Form the triangulation $\{\mathcal{S}^1, \ldots, \mathcal{S}^{p+1}\}$ of $\mathcal{S}$ based on the Subdivision Algorithm. First we show by induction that $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$ by affine feedback for $k = 1, \ldots, p$. For the initial step, by assumption the invariance conditions are solvable for $\mathcal{S}$ and by Lemma 28, $h_0 \cdot b_{m_k} > 0$ for $k = 1, \ldots, p$. Thus, by Lemma 31, $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback. Now assume that at the jth step the invariance conditions are solvable for $\mathcal{S}$ and $h_0 \cdot b_{m_k} > 0$ for $k = 1, \ldots, p$. Then by Lemma 31, $\mathcal{S}^j \xrightarrow{\mathcal{S}^j} \mathcal{F}_0$ by affine feedback. Now consider the $(j+1)$th step. By the algorithm $\mathcal{S} := \text{co}\{v_0, v_1, \ldots, v_{m_j - 1}, v', v_{m_j + 1}, \ldots, v_n\}$ and $h_0 = -h'$, where v' and h' are provided by the jth step. By Lemma 32, the invariance conditions are solvable for $\mathcal{S}$ and $h_0 \cdot b_{m_k} > 0$ for $k = 1, \ldots, p$. Then by Lemma 31, $\mathcal{S}^{j+1} \xrightarrow{\mathcal{S}^{j+1}} \mathcal{F}_0$ by affine feedback.

Next consider $\mathcal{S}^{p+1}$. We observe that $\mathcal{S}^{p+1}$ and $\mathcal{S}$ share the same invariance conditions since they only differ in their exit facets, so the invariance conditions for $\mathcal{S}^{p+1}$ are solvable. Now let $\mathcal{G}^{p+1} := \mathcal{S}^{p+1} \cap \mathcal{O}$. Then by the algorithm, $\mathcal{G}^{p+1} = \text{co}\{v_2, \ldots, v_{m_2 - 1}, v_{m_2 + 1}, \ldots, v_{m_p - 1}, v_{m_p + 1}, \ldots, v_{\kappa+1}\}$. We can see that the algorithm has removed the p vertices $v_{m_1}, v_{m_2}, \ldots, v_{m_p}$ from $\mathcal{G}$, and this has the effect to break up the dependencies of $\mathcal{B}$ associated with $\mathcal{G}$. There remain $\widehat{m}$ linearly independent vectors in $\mathcal{B}$ associated with $\mathcal{G}^{p+1}$ (an $(\widehat{m} - 1)$-dimensional simplex) given by $\{b_2, \ldots, b_{m_2 - 1}, b_{m_2 + 1}, \ldots, b_{m_p - 1}, b_{m_p + 1}, \ldots, b_{\kappa+1}\}$. Therefore, we can apply Theorem 10 to obtain $\mathcal{S}^{p+1} \xrightarrow{\mathcal{S}^{p+1}} \mathcal{F}_0^{p+1}$.

Next we verify conditions (ii) and (iii) of RCP. Condition (ii) follows immediately because there are a finite number of affine feedbacks $u^k(x)$ each defined on a compact set $\mathcal{S}^k$ that does not contain an equilibrium. For (iii) we must verify that the invariance conditions for $\mathcal{S}$ hold on the vertices of $\mathcal{F}_0$. The exit facet of $\mathcal{S}^{p+1}$ is $\mathcal{F}_0^{p+1} = \{v_0^1, v_{m_1+1}, \ldots, v_{m_1+r_1-1}, \ldots, v_0^p, v_{m_p+1}, \ldots, v_{m_p+r_p-1}, v_{r+1}, \ldots, v_n\}$. The invariance conditions for $\mathcal{S}^{p+1}$ are identical to those for $\mathcal{S}$ and the controller for $\mathcal{S}^{p+1}$ takes precedence over controllers for simplices with lower index. This implies that the invariance conditions of $\mathcal{S}$ hold at all vertices of $\mathcal{F}_0^{p+1}$. The only vertices of $\mathcal{F}_0$ that are not in $\mathcal{F}_0^{p+1}$ are $v_{m_1}, v_{m_2}, \ldots, v_{m_p}$. For these vertices we have: $v_{m_1} \in \mathcal{S}^1$, $v_{m_2} \in \mathcal{S}^1 \cap \mathcal{S}^2$,...,$v_{m_p} \in \mathcal{S}^1 \cap \cdots \cap \mathcal{S}^p$. We use the controller for the simplex with the highest index. Now the invariance conditions for $\mathcal{S}^k$ at v_{m_k} are precisely those for $\mathcal{S}$. We can see this because the invariances conditions for v_{m_k} do not include the normal vector h^k.

Finally, we must prove that trajectories progress through simplices with decreasing indices (thereby guaranteeing that the supervisor cannot block). Consider w.l.og. the boundary between $\mathcal{S}^1$ and $\mathcal{S}^2$ given by $\mathcal{F}_0^1 = \text{co}\{v', v_2, \ldots, v_n\}$, and let $u = K_1 x + g_1$ be the affine feedback obtained for $\mathcal{S}^1$. We show that for any $x_0 \in \mathcal{S}^1 \setminus \mathcal{F}_0^1$, closed-loop trajectories do not reach $\mathcal{F}_0^1$. This can be deduced from the proof of Lemma 31 where it is shown that the controls $\{u', u_2, \ldots, u_n\}$ can be selected so that $h' \cdot (Av' + Bu' + a) < 0$ and $h' \cdot (Av_i + Bu_i + a) < 0$, $i = 2, \ldots, n$. By convexity, $h' \cdot (Ax + B(K_1 x + g_1) + a) < 0$ for all $x \in \mathcal{F}_0^1$, from which the result easily follows. ■

The following is the main result of the paper.

Theorem 34. *Suppose Assumption 6 holds. Then the following are equivalent:*

1) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by piecewise affine feedback.*
2) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by open-loop controls.*

Proof: (1) $\Longrightarrow$ (2) is obvious.
(2) $\Longrightarrow$ (1) Suppose $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls. By Theorem 11, the invariance conditions are solvable. Let $\mathcal{G} := \mathcal{S} \cap \mathcal{O}$. If $\mathcal{G} = \emptyset$, then by Theorem 7, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose instead $\mathcal{G} \neq \emptyset$. If $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$, then by Theorem 8, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose instead $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. From Theorem 12, $v_0 \notin \mathcal{G}$, so by reordering indices, $\mathcal{G} = \text{co}\{v_1, \ldots, v_{\kappa+1}\}$, where $0 \leq \kappa < n$. Define $\widehat{\mathcal{B}} = \text{sp}\{b_1, \ldots, b_{\widehat{m}} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ where $\{b_1, \ldots, b_{\widehat{m}}\}$ is a maximal set with respect to $\mathcal{G}$. If $\kappa < \widehat{m}$, then by Theorem 10, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose instead $\kappa \geq \widehat{m}$. By Theorem 12, $\mathcal{B} \cap \mathcal{C}_i \neq \mathbf{0}$ for $i \in I_{\mathcal{G}}$. Then (R1)-(R4) of Assumption 13 hold. The reach control indices can be defined, yielding a decomposition of $\mathcal{B}$ into $\mathcal{B}_1, \ldots, \mathcal{B}_p$. Lemma 25 gives $\mathcal{B}_k \not\subset \mathcal{H}_0$ for $k = 1, \ldots, p$. Then (P1)-(P5) of Assumption 24 hold. By Theorem 33, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by piecewise affine feedback. ■

VI. Examples

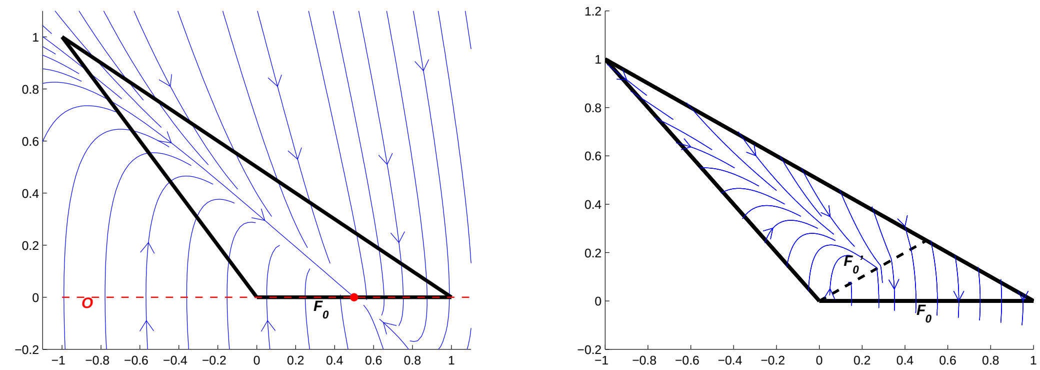
A. Example 1

Consider the longitudinal dynamics of a mobile robot that transports materials from one end of a mining tunnel to another:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Safety constraints on both position x_1 and velocity x_2 determine a polyhedral state space within which the robot dynamics evolve, while also satisfying a liveness requirement to transport materials efficiently. The polyhedral state space is triangulated according to Assumption 6. We focus on the reach control problem for a specific simplex of the triangulation: consider the simplex $\mathcal{S}$ determined by vertices $v_0 = (-1, 1)$, $v_1 = (1, 0)$ and $v_2 = (0, 0)$. It can be verified that $\mathcal{O} = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$, $\mathcal{G} = \text{co}\{v_1, v_2\}$, $\kappa = 1$, and $\widehat{m} = m = 1$. Also $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. By the results of [4], RCP is not solvable by continuous state feedback. For example, suppose we choose control values $u_0 = -\frac{3}{4}$, $u_1 = -1$, and $u_2 = 1$ to satisfy the invariance conditions (2). By the method in [7], this yields an affine feedback $u = \begin{bmatrix} -2 & -3.75 \end{bmatrix} x + 1$. Simulation of the closed-loop system is shown in Figure 2(a). We observe there exists a closed-loop equilibrium point on $\mathcal{G}$. Now we show the problem is solvable by piecewise affine feedback.

Let $b_1 = (0, -1) \in \mathcal{B} \cap \mathcal{C}_1$. Since $h_0 = (0, -1)$, we have $h_0 \cdot b_1 > 0$, verifying Lemma 28. Next, we choose $v' = (0.5, 0.25)$ along the simplex edge (v_0, v_1) such that from Lemma 29, $h' = (-0.25, 0.5)$. Then



(a) (b)

Fig. 2. Closed-loop vector fields using (a) affine feedback and (b) piecewise affine feedback.

$h' \cdot b_1 < 0$ and $b_1 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^1)$, verifying Lemma 30. Let $\mathcal{S}^1 := \text{co}\{v', v_1, v_2\}$, $\mathcal{S}^2 := \text{co}\{v_0, v', v_2\}$, and $\mathcal{F}_0' = \text{co}\{v', v_2\}$. To satisfy the invariance conditions for $\mathcal{S}^1$ we choose control inputs at the vertices to be $u' = -1$, $u_1 = -1$, and $u_{12} = -1$. Similarly, for $\mathcal{S}^2$ we choose $u_0 = -\frac{3}{4}$, $u' = -1$, and $u_{22} = 1$. The piecewise affine feedback is

$$u(x) := \begin{cases} \begin{bmatrix} 0 & 0 \end{bmatrix} x - 1\,, & x \in \mathcal{S}^1 \\ \begin{bmatrix} -2.0833 & -3.833 \end{bmatrix} x + 1\,, & x \in \mathcal{S}^2\,. \end{cases}$$

By Theorem 8, $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ using $u(x)$. Because $\mathcal{G}^2 := \mathcal{S}^2 \cap \mathcal{O} = \{v_2\}$, we have $\widehat{m}^2 = 1$ and $\kappa^2 = 0$ for $\mathcal{S}^2$. By Theorem 8, $\mathcal{S}^2 \xrightarrow{\mathcal{S}^2} \mathcal{F}_0'$ using $u(x)$. The closed-loop vector field is shown in Figure 2(b), where it is clear that RCP is solved.

B. *Example 2*

Consider the simplex $\mathcal{S}$ in $\mathbb{R}^4$ defined by the vertices $v_0 = (0, 0, 0, 0)$, $v_1 = (1, 0, 0, 0)$, $v_2 = (0, 1, 0, 0)$, $v_3 = (0, 0, 1, 0)$, and $v_4 = (0, 0, 0, 1)$. Consider the system

$$\dot{x} = \begin{bmatrix} -3 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \\ -3 & -3 & -3 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & -2 \\ 0 & 1 \\ -2 & 0 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We compute $\mathcal{O} = \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 - 1 = 0\}$. Thus, $\mathcal{G} = \mathcal{F}_0$, and we note that $\kappa = 3$, $\widehat{m} = m = 2$, and $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$. By the results of [4], RCP is not solvable by continuous state feedback. Now we show it is solvable by piecewise affine feedback. First we examine the structure of $\mathcal{B}$ (note that indices are not reordered, as is the convention in our proofs). We find by inspect that $b_1 := (-2, 1, 0, 0) \in \mathcal{B} \cap \mathcal{C}_1$, $b_3 := (0, 0, -2, 1) \in \mathcal{B} \cap \mathcal{C}_3$, and $\mathcal{B} = \text{sp}\{b_1, b_3\}$. Therefore, $\mathcal{B}$ splits into two dependent cycles with respect to $\mathcal{G}$. In particular, $b_2 := -b_1 \in \mathcal{B} \cap \mathcal{C}_2$ and $b_4 := -b_3 \in \mathcal{B} \cap \mathcal{C}_4$. Thus, $r_1 = 2$ and $r_2 = 2$.

1) First subdivision: In the first iteration $\mathcal{S}$ is subdivided into simplices $\mathcal{S}^1$ and $\mathcal{S}'$. Since $b_2 \cdot h_0 > 0$, we choose $v' = (0, 0.75, 0, 0) \in (v_0, v_2)$ such that we obtain the condition $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq \mathbf{0}$. Hence $\mathcal{S}' = \text{conv}\{v_0, v_1, v', v_3, v_4\}$ and $\mathcal{S}^1 = \text{conv}\{v', v_1, v_2, v_3, v_4\}$. In order to satisfy the invariance conditions for $\mathcal{S}^1$ the control inputs at the vertices of $\mathcal{S}^1$ are chosen as $u' = (-1, -2)$, $u_{11} = (-1, -2)$, $u_{12} = (-1, -2)$,

$u_{13} = (-1, -2)$, and $u_{14} = (1, 0)$. This yields an affine feedback

$$u(x) := \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \qquad x \in \mathcal{S}^1 .$$

For $\mathcal{S}^1$ the invariance conditions are solvable and $\mathcal{B} \cap \text{cone}(\mathcal{S}^1) \neq \mathbf{0}$, so by Theorem 8, $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ using $u(x)$. For $\mathcal{S}'$ we have $\mathcal{G}' := \mathcal{S}' \cap \mathcal{O} = \text{co}\{v_1, v_3, v_4\}$. Since $\kappa' = 2$ and $m = 2$, RCP is not solvable by continuous state feedback on $\mathcal{S}'$, and further subdivision of $\mathcal{S}'$ is required.

2) Second subdivision: Consider the simplex $\mathcal{S}' = \text{co}\{v_0, v_1, v', v_3, v_4\}$, where $v' \in (v_0, v_2) = (0, 0.75, 0, 0)$ and the exit facet is $\mathcal{F}_0'$ =conv$\{v_1, v', v_3, v_4\}$. We subdivide $\mathcal{S}'$ into simplices $\mathcal{S}^3$ and $\mathcal{S}^2$ and use a piecewise affine feedback law to solve RCP on $\mathcal{S}'$. It is clear that $b_4 \cdot h_0' > 0$ and therefore we can choose $v'' \in (v_0, v_4)$ such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^2) \neq \mathbf{0}$. One choice is $v'' := (0, 0, 0, 0.8)$. Let $\mathcal{S}^3 = \text{co}\{v_0, v_1, v', v_3, v''\}$ and $\mathcal{S}^2 = \text{co}\{v'', v_1, v', v_3, v_4\}$. It can be verified that $b_4 \in \mathcal{B} \cap \text{cone}(\mathcal{S}^2)$. To satisfy the invariance conditions for $\mathcal{S}^2$ we choose $u'' = (-4, 0.6)$, $u_{21} = (-5, -1)$, $u' = (-1, -2)$, $u_{23} = (-5, -1)$, and $u_{24} = (-3, 1)$. To satisfy the invariance conditions for $\mathcal{S}^3$ we choose $u_0 = (0, 0)$, $u_{31} = (-1, 0)$, $u' = (-1, -2)$, $u_{33} = (0, -1)$, and $u'' = (-4, 0.6)$. This yields a piecewise affine feedback

$$u = \begin{cases} \begin{bmatrix} -1 & -1.33 & 0 & -5 \\ 0 & -2.66 & -1 & 0.75 \end{bmatrix} x, & x \in \mathcal{S}^3 \\ \begin{bmatrix} 3 & 9.33 & 3 & 5 \\ 0 & -1.33 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} -8 \\ -1 \end{bmatrix}, & x \in \mathcal{S}^2 . \end{cases}$$

For $\mathcal{S}^2$ the invariance conditions are solvable and $\mathcal{B} \cap \text{cone}(\mathcal{S}^2) \neq \mathbf{0}$, so by Theorem 8, $\mathcal{S}^2 \xrightarrow{\mathcal{S}^2} \mathcal{F}_0'$ using $u(x)$. For $\mathcal{S}^3$ we have $\mathcal{G}_3 = \mathcal{S}^3 \cap \mathcal{O} = \text{co}\{v_1, v_3\}$. Since $\kappa^3 = 1$ and $\widehat{m}^3 = 2$, by Theorem 10, $\mathcal{S}^3 \xrightarrow{\mathcal{S}^3} \mathcal{F}''$ using $u(x)$. Indeed, $\{b_1, b_3 \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ is a linearly independent set associated with $\mathcal{G}_3$.

VII. Conclusion

The paper studies the reach control problem on simplices, and we investigate cases when the problem is not solvable by continuous state feedback. It is shown that the class of piecewise affine feedbacks is sufficient to solve the problem in all cases of interest; namely, those cases when the problem is solvable by open-loop controls.

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