STRUCTURAL STABILITY OF PIECEWISE SMOOTH SYSTEMS

MIREILLE E. BROUCKE, CHARLES C. PUGH, AND SLOBODAN N. SIMIĆ†

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Abstract

A.F. Filippov has developed a theory of dynamical systems that are governed by piecewise smooth vector fields [2]. It is mainly a local theory. In this article we concentrate on some of its global and generic aspects. We establish a generic structural stability theorem for Filippov systems on surfaces, which is a natural generalization of Mauricio Peixoto's classic result [12]. We show that the generic Filippov system can be obtained from a smooth system by a process called pinching. Lastly, we give examples. Our work has precursors in an announcement by V.S. Kozlova [6] about structural stability for the case of planar Filippov systems, and also the papers of Jorge Sotomayor and Jaume Llibre [8] and Marco Antonio Teixiera [17], [18].

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1. Introduction

Imagine two independently defined smooth vector fields on the 2-sphere, say X_+ and X_-. While a point p is in the Northern hemisphere let it move under the influence of X_+, and while it is in the Southern hemisphere, let it move under the influence of X_-. At the equator, make some intelligent decision about the motion of p. See Figure 1. This will give an orbit portrait on the sphere. What can it look like?

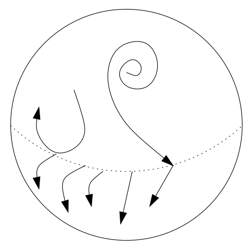


Figure 1. A piecewise smooth vector field on the 2-sphere.

How do perturbations affect it? How does it differ from the standard vector field case in which $X_+ = X_-$? These topics will be put in proper context and addressed in Sections 2-7. In Section 8 we discuss the phenomenon of Zeno trajectories and in Section 9 we discuss pinching. Section 10 concludes with some examples from circuit theory, mechanics, and control theory.

2. Definitions and Results

Throughout the paper we make the following **Standing Assumptions.**

- M is a smooth, orientable, boundaryless, compact surface.
- $K \subset M$ is a fixed, smoothly embedded, finite 1-complex. The angle between edges of K meeting at a vertex is non-zero. That is, K has no cusps.
- The connected components of $M \setminus K$ are denoted $G_1, \dots, G_k$, and those that abut across an edge of K are pairwise distinct. That is, K locally separates M.

Definition 2.1. A **piecewise C^r vector field** X on M, $1 \le r \le \infty$, is a family $\{X_i\}$ where X_i is a C^r vector field defined on the closure of the i^{th} connected component $\bar{G}_i$ of $M \setminus K$, $i = 1, \dots, k$. The X_i are referred to as **branches** of X.

In this definition we have used the standard convention from smooth analysis that for a function defined on a non-open domain D, being of class C^r means being extendable to a C^r function defined on an open set containing D. The same applies to vector fields. Thus X_i extends to a C^r vector field defined on a neighborhood of $\bar{G}_i$. In fact, using a smooth cutoff function one can extend X_i to a C^r vector field defined on all of M. In the case of piecewise analytic vector fields the same extension convention applies although X_i need not extend to a global analytic vector field.

Example. Suppose that T is a smooth triangulation of M and K is the 1-skeleton of T. The triangulation gives each 2-simplex T_i an induced linear structure and we can choose an affine vector field X_i on it. This is a situation investigated by Leon Chua and Robert Lum [1]. See Figure 2.

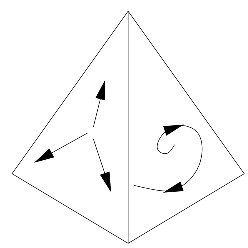


FIGURE 2. A piecewise affine vector field on a tetrahedral sphere.

It is pictorially clear that a piecewise smooth vector field X has some kind of orbit portrait and structural stability should mean that perturbing X leaves the orbit portrait unchanged topologically. We proceed to spell this out.

A point $q \in K$ is of one of the following four types.

(a) q is a vertex of K.
(b) q is a **tangency point**: it is not a vertex and at least one of the two branches of X is tangent to K at q. (This includes the possibility that a branch vanishes at q.)
(c) q is a **crossing point**: it is not a vertex, the two branches of X are transverse to K at q, and both point to the same side of K.
(d) q is an **opposition point**: it is not a vertex and the two branches of X oppose each other in the sense that they are transverse to K at q but they point to opposite sides of K.

At an opposition point there is a unique strictly convex combination

$$X^*(q) = \lambda X_i(q) + (1 - \lambda) X_j(q)$$

tangent to K at q.

Definition 2.2. X^* is the **sliding field**. If $X^*(q) \neq 0$, q is a **sliding point**, while if $X^*(q) = 0$, it is a **singular equilibrium**.

X^* indicates the direction a point should slide along K. See Figure 3. The sliding field is defined on a relatively open subset of K, which

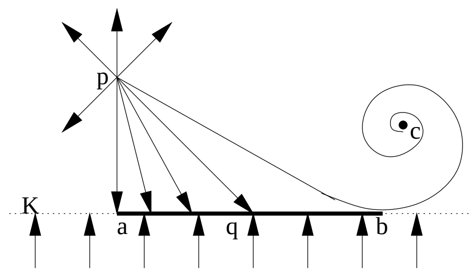


FIGURE 3. Points q slide along K from a to b under the influence of X^*.

includes neither the vertices of K, nor the points of K at which a branch of X is tangent to K. For at a tangency point, the branches of X do not oppose each other – transversality fails there.

Definition 2.3. A **singularity** of X is a singular equilibrium, a tangency point, or a vertex of K.

Definition 2.4. A **regular orbit** of X is a piecewise smooth curve $\gamma \subset M$ such that $\gamma \cap G_i$ is a trajectory of X_i, $\gamma \cap K$ consists of crossing points, and γ is maximal with respect to these two conditions. A **singular orbit** of X is a smooth curve $\gamma \subset K$ such that γ is either an orbit of X^* or a singularity.

Evidently, M decomposes into the disjoint union of orbits, each being regular or singular. They form the **phase portrait** of X. The only periodic or recurrent orbits on K are equilibria.

Remark. Filippov takes a different point of view. He defines an X-trajectory as an amalgam of what we call orbits (and what Kozlova calls quasi-curves), the amalgamation of γ to β at q being permitted if γ arrives at its forward endpoint q in finite time, while β departs from q in finite time. (See also Section 5.) Amalgamation only occurs at singularities. In Figure 3 there are many X-trajectories through q. One is the singular X-orbit (a, b) plus the tangency point $\{b\}$ plus the regular X-orbit from b to the focus c. A second is the regular X-orbit from the regular source p to q plus the forward singular orbit $[q, b)$ plus $\{b\}$ plus the regular X-orbit from b to c. In contrast, our convention is that there is only one X-orbit through q, namely (a, b). *Our principle is to amalgamate orbits as little as possible.*

The set of all piecewise C^r vector fields on M is denoted $\mathcal{X}_K^r$. It is a Banach space with respect to the norm

$$\|X\|_{C^r} = \max_i \|X_i\|_{C^r}.$$

Definition 2.5. An **orbit equivalence** is a homeomorphism $h : M \to M$ that sends X-orbits to X'-orbits where $X, X' \in \mathcal{X}_K^r$. The orbit equivalence must preserve the sense (i.e., the direction) of the orbits and send K to itself. If X has a neighborhood $\mathcal{U} \subset \mathcal{X}_K^r$ such that each $X' \in \mathcal{U}$ is orbit equivalent to X then X is **structurally stable**.

Remark. Filippov and Kozlova define structural stability in terms of X-trajectories, i.e., amalgamated orbits. This gives a stronger requirement on the equivalence homeomorphism, so in principle Filippov and Kozlova get fewer structurally stable systems than we do. However, it is easy to see that the orbit equivalence we construct respects amalgamation and hence their definition of structural stability is equivalent to ours for surfaces. It seems likely that the two definitions are always equivalent.

Definition 2.6. An orbit $\gamma(t)$ **departs** from $q \in K$ if $\lim_{t\to 0^+} \gamma(t) = q$. Arrival at q is defined analogously.

Definition 2.7. An **unstable separatrix** is a regular orbit such that either

- its α-limit set is a regular saddle point, or
- it departs from a singularity of X.

A **stable separatrix** is defined analogously. If a separatrix is simultaneously stable and unstable it is a **separatrix connection**. If unstable separatrices arrive at the same point of K they are **related**.

Proposition 2.8. *The branches of the generic $X \in \mathcal{X}_K^r$ have the following properties:*

(a) *They are Morse-Smale.*
(b) *None of them vanishes at a point of K.*
(c) *They are tangent to K at only finitely many points, none of which is a vertex of K, and distinct branches are tangent to K at distinct points.*
(d) *They are non-colinear except at a finite number of points, none of which is a vertex.*
(e) *Properties (a)-(d) are stable (i.e., robust) under small perturbations of X.*

See Section 3 for the proof. Here are our main results about structural stability of piecewise smooth systems.

Theorem A. *The generic $X \in \mathcal{X}^r_K$ is structurally stable.*

Theorem B. *The following conditions characterize structural stability of $X \in \mathcal{X}^r_K$.*

- *The conditions listed in Proposition 2.8.*
- *Hyperbolicity of all periodic orbits.*
- *No separatrix connections or relations.*
- *Only trivially recurrent orbits.*

See Section 7 for the proofs of Theorems A and B. Note that the orbits referred to in Theorem B include those that cross K.

3. Proof of Proposition 2.8.

To prove Proposition 2.8 and to analyze generic singularities we introduce a smooth coordinate chart ϕ in M along an edge $E \subset K$ such that $\phi(E)$ is an interval on the x-axis, say $\phi(E) = [-1,1]$. In this coordinate system the branches X_i, X_j are expressed as vector fields defined on the closed upper half plane and closed lower half plane. On the x-axis we have

$$(1) \qquad \begin{aligned} X_i(x,0) &= f_i(x)\left(\frac{\partial}{\partial x}\right) + g_i(x)\left(\frac{\partial}{\partial y}\right) \\ X_j(x,0) &= f_j(x)\left(\frac{\partial}{\partial x}\right) + g_j(x)\left(\frac{\partial}{\partial y}\right) \end{aligned}$$

Definition 3.1. A function $f : [a,b] \to \mathbb{R}$ has **generic zeros** if

(a) $f(a) \neq 0 \neq f(b)$, and
(b) $f(x) = 0$ implies $f'(x) \neq 0$.

Lemma 3.2. *The generic C^r function $f : [-1,1] \to \mathbb{R}$ has generic zeros.*

Proof. This is a special case of the Thom Transversality Theorem, for (b) is equivalent to 0 being a regular value of f. □

Proof of Proposition 2.8. Peixoto's Genericity Theorem, applied on the surface with smoothly cornered boundary $\bar{G}_i$, states that the generic X_i is Morse-Smale, which is assertion (a) of Proposition 2.8.

As above, introduce a smooth coordinate chart ϕ in which an edge of K is $[-1,1]$ on the x-axis. Express the branches of X as in (1).

Elaboration of Lemma 3.2 shows that the generic pair of C^r functions $[-1,1] \to \mathbb{R}$ has no common zero. This implies that the generic X has no zeros on K, which is assertion (b) of Proposition 2.8. For the generic X, $g_i(x) = 0$ at only a finite number of points, all of them different from ± 1, and at these zeros, $g_i'(x) \neq 0$. Further, g_i and g_j have no common zeros. This means that generically X_i and X_j are tangent to K at only finitely many points, none of them vertices, and never are they tangent to K at a common point, which is assertion (c) of Proposition 2.8.

Consider any $x_0 \in [-1, 1]$. By (c), either $g_i(x_0) \neq 0$ or $g_j(x_0) \neq 0$, say it is the latter. There is an interval $I \subset [-1, 1]$ containing x_0 on which $g_j \neq 0$. That is, X_j is not tangent to K there. Using a flowbox chart for X_j at I, we may assume that $f_j(x) = 0$ and $g_j(x) > 0$ for all $x \in I$. (In the flowbox coordinates, X_j points straight upwards.) Then colinearity of X_i and X_j occurs when $f_i(x) = 0$. By Lemma 3.2, for the generic X this happens only finitely often, never at an endpoint of I, which is assertion (d) of Proposition 2.8 on the subinterval I. Compactness of $[-1, 1]$ completes the proof of (d).

Assertion (e), stability of (a) - (d) under small perturbations of X, follows from openness of Morse-Smale systems and openness of transversality. □

4. Singular Equilibria

The sliding field X^* is defined at non-vertex points $q \in K$ where X_i, X_j oppose each other. These opposition points form a relatively open set in K. With respect to the smooth chart ϕ as in Section 3 and the expression for X_i, X_j along K in (1), we see that g_i and g_j are non-zero and have opposite signs. Since the sliding field is the unique strictly convex combination

$$X^* = \lambda X_i + (1 - \lambda) X_j$$

tangent to K, its vertical component $\lambda g_i(x) + (1 - \lambda) g_j(x)$ is zero. This gives

$$\lambda = \frac{g_j(x)}{g_j(x) - g_i(x)}. \tag{2}$$

Note that this denominator is never zero at opposition points.

Proposition 4.1. *The sliding field X^* is of class C^r. For the generic X, the zeros of X^* are hyperbolic sources or sinks along K.*

Proof. The expression for λ given in (2) is C^r so X^* is C^r. Zeros of X^* occur when X_i and X_j are colinear. Changing the chart ϕ to a flowbox chart ψ for X_j makes $f_j \equiv 0$. Then colinearity occurs if $f_i(x) = 0$. For the generic X, Lemma 3.2 states that $f_i'(x) \neq 0$ when $f_i(x) = 0$. Thus, with respect to the chart ψ, a zero of X^* is a hyperbolic source when $f_i'(x) > 0$ and a hyperbolic sink when $f_i'(x) < 0$. □

Proposition 4.2. *For the generic X, a singularity is either*

- *a singular saddle,*

- *a singular sink,*
- *a singular source,*
- *a singular saddle node,*
- *a singular grain,*
- *a vertex of K.*

(See the proof for the definitions.) The singularities are finite in number and stable (robust) under small perturbations of X.

Proof. Let $q \in K$ be a singular equilibrium or tangency point of X. By Proposition 2.8 genericity implies that at least one branch of X is transverse to K at q. Say it is X_j. In the flowbox chart as in the proof of Proposition 2.8, $f_j \equiv 0$, $g_j = 1$, and we may take $q = (0, 0)$.

We write a matrix to describe the singularity at q as

$$S = \begin{bmatrix} f_i(0) & g_i(0) \\ \\ f_i'(0) & g_i'(0) \end{bmatrix}.$$

Observe that $g_i(0) \leq 0$, or if $g_i(0)$ is positive then X_i and X_j both point upwards across the x-axis at q, and q is regular, not singular. Genericity implies that the first row of S contains exactly one 0, and neither column is a zero column. Using the symbols $-$, 0, $+$, $\pm$, $*$ to denote an entry that is negative, zero, positive, non-zero, or one whose sign is irrelevant, we get four topologically distinct cases for S. See Figure 4.

Case 1. $S = \begin{bmatrix} 0 & - \\ + & * \end{bmatrix}$. The fact that $f_i(0) = 0$ implies that X_i, X_j are colinear at q and there is a unique X_i-orbit that arrives at the origin from above. Since $f_i'(0)$ is positive this gives a **singular saddle** with regular stable separatrices.

Case 2. $\begin{bmatrix} 0 & - \\ - & * \end{bmatrix}$. The fact that $f_i(0) = 0$ implies that X_i, X_j are colinear at q and there is a unique X_i-orbit that arrives at the origin from above. Since $f_i'(0)$ is negative this gives a **singular sink** with regular stable separatrices. See Figure 4.

Case 3. $\begin{bmatrix} \pm & 0 \\ * & + \end{bmatrix}$. Since $g_i(0) = 0$ and $g_i'(0)$ is positive, we get a **singular saddle node** with three regular separatrices, two stable and one unstable. The sign of $f_i(0)$ only affects whether the arrows point right or left.

Case 4. $\begin{bmatrix} \pm & 0 \\ * & - \end{bmatrix}$. Since $g_i(0) = 0$ and $g_i'(0)$ is negative, we get a **singular grain** with one regular stable separatrix. The sign of $f_i(0)$ only affects whether the arrows point right or left.

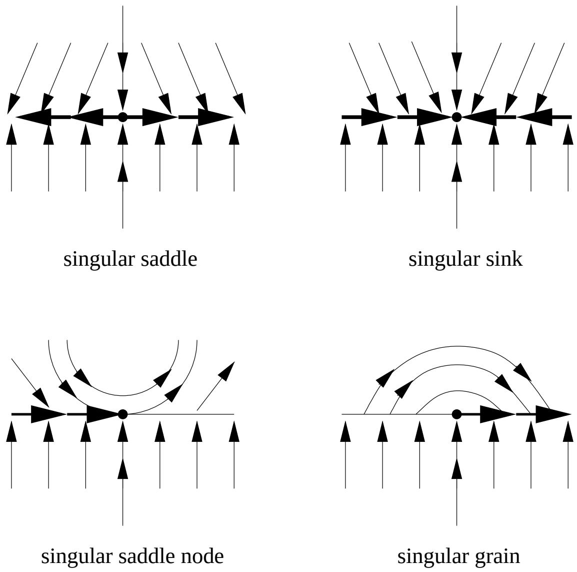


FIGURE 4. The four generic non-vertex singularities. The heavy lines are X^*-orbits.

Four homeomorphic pictures, but with the orientations of all orbits reversed, are achieved by starting with the field X_j pointing straight down instead of straight up. Finiteness of the singular equilibria and tangency points is a consequence of (c), (d) in Proposition 2.8. Stability under perturbation of X follows from openness of transversality. □

5. Singular Sectors

The qualitative behavior of the generic X at a vertex of K involves concepts from the Poincaré-Bendixson Sector Theorem. See Chapter 7 of [5].

Definition 5.1. Let v be a vertex of K. A **base orbit** of X at v is an orbit, singular or regular, that arrives at v or departs from v, but is not v itself.

By definition, a vertex is an orbit, but because we do not amalgamate orbits, it is not part of the base orbit.

Remark. Hartman uses the term "base solution" in much the same way when analyzing the sector structure at an isolated fixed point of a planar flow [5]. In the flow case, the base solution converges to the fixed point as time tends to $\pm\infty$, in contrast to arriving to it or departing from it in finite time.

Proposition 5.2. *The generic X has at least one base orbit at each vertex v. One of the following two possibilities occurs.*

(a) *There exists no singular base orbit, in which case X has a **singular focus** at v: all orbits near v are regular and either all arrive at v or all depart from v.*

(b) *There exists a singular base orbit, in which case there exist at most $2n$ base orbits at v, where n is the number of edges of K at v.*

Proof. Because K locally separates M, $n \geq 2$. We draw a small circle C around v and refer to the component of G_i inside C as the **corner** V_i of G_i, $i = 1, \ldots, n$. (In this section we reserve the term "sector" mainly for dynamically defined regions at v.) The angle between the edges of V_i at v is the **aperture** of V_i. We choose C small – it encloses no singular equilibria or tangency points, and the edges of K at v are arcs from C to v.

Suppose that the aperture of some corner V_i is $\geq \pi$. Generically $X_i(v)$ is non-zero and is parallel to neither edge of V_i. This gives a base orbit as shown in Figure 5.

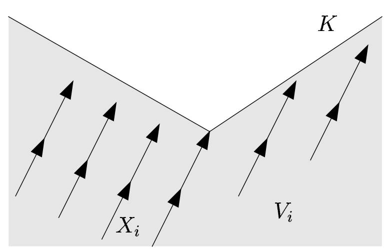


FIGURE 5. Generically, a base orbit exists in a corner with aperture $\geq \pi$.

From now on we assume that the aperture of all corners is $< \pi$. In particular, $n \geq 3$.

Proposition 2.8 states that the generic X is not tangent to an edge of K at v. Thus, when the circle C is small and $1 \leq i \leq n$, either

(c) X_i points inward across both edges of V_i, or points outward across both edges.

(d) X_i points inward across one edge of V_i and outward across the other.

Thus, each corner contains at most one base orbit in its interior. See Figure 6.

Assume that there is no singular base orbit at v. Then X_i never opposes X_{i+1} across the common edge $V_i \cap V_{i+1}$, and we get an arc α

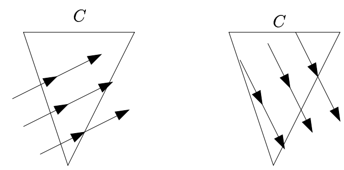


FIGURE 6. X_i is transverse to the edges of V_i.

of a regular X-orbit that starts at $a \in E_1$, an edge of V_1 and continues through each of the corners at v until it comes back to E_1, say at the point a'.

The vectors $X_i(v)$ are fixed, and so are the apertures of the corners V_i. The arc α is an amalgam of nearly straight segments, $\alpha_1, \dots, \alpha_n$. The exterior angle θ_i between α_i and α_{i+1} tends to a definite non-zero limit Θ_i as the circle C shrinks to v. Generically $\Theta = \sum_{i=1}^{n} \Theta_i \neq 2\pi$, and thus the points at which α crosses E_1 are distinct. If $\Theta > 2\pi$ then a' lies closer than a to v along E_1. In fact, for a constant $c < 1$,

$$|a' - v| \leq c|a - v|.$$

The same constant c works for all a sufficiently close to v. See Figure 7. The length of time it takes for α to make the circuit through the corners

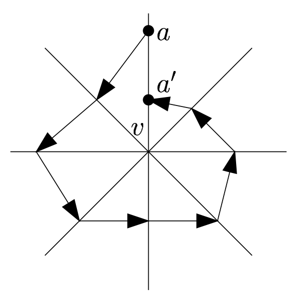


FIGURE 7. A singular focus.

at v is proportional to $|a - v|$. For $X_i(z)$ tends to the non-zero vector $X_i(v)$ as $z \to v$ in V_i. Thus the X-orbit through a arrives at v (in finite time), and is a base orbit. In fact the local orbit portrait at v is that of a focus where all orbits arrive at v.

If $\Theta < 2\pi$ it is the opposite. All orbits near v depart from v. This completes the proof of assertion (a), lack of a singular base orbit implies a singular focus.

Now assume that there is at least one singular base orbit β. It is an edge of adjacent corners V_i, V_{i+1} such that X_i and X_{i+1} oppose each other across β. No base orbit can spiral around v because it is blocked by β. Thus, the only possible base orbits are the n edges of K at v (they would be singular base orbits) plus n regular base orbits, one interior to each corner. See Figure 6. This completes the proof of assertion (b), the existence of one singular base orbit implies there are at most $2n$ base orbits at v. □

Definition 5.3. A **singular sector** is a region S bounded by an arc of the small circle C at v together with two consecutive base orbits. S must contain no other base orbits.

Corollary 5.4. *The orbit portrait for the generic X inside a singular sector is either singular hyperbolic, singular parabolic, or singular elliptic.*

Proof. The proof amounts to inspecting Figure 8. See also [5], Chapter 7, section 8.

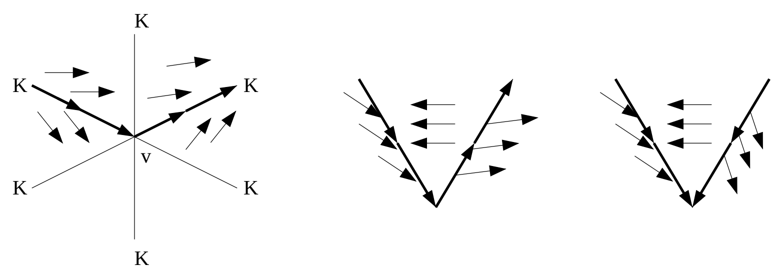


FIGURE 8. Singular sectors: hyperbolic, elliptic, and parabolic.

6. Separatrix Connections

In this section we examine various forms of separatrix connections, and in particular we show how to break some of them. Examples of separatrices are shown in Figure 4. Note that a singular sink q has stable separatrices – they are the unique *regular* orbits that arrive at q. Likewise a singular source has unstable separatrices. A base orbit is also a separatrix.

Remark. It is easy to see that structural stability fails if there are separatrix connections or separatrix relations. Also, singular orbits never connect singular saddles, so there is no need to exclude them by a genericity argument. See Proposition 6.4 and Figure 9.

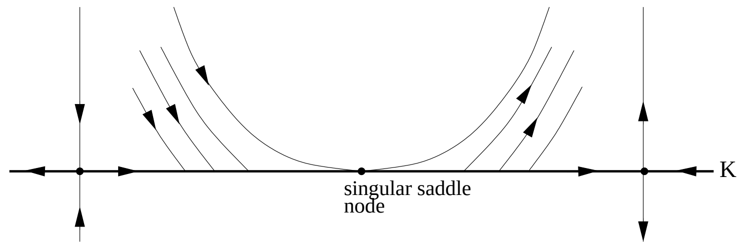


FIGURE 9. Why there are no singular saddle connections: a singular saddle node intervenes along K.

Proposition 6.1. *For the generic X, no separatrix connections or relations occur in a neighborhood of K. Moreover there are restrictions on how the equilibria appear along K.*

Proof. The separatrices are regular, finite in number, and locally leave the neighborhood of K. Thus they have no local connections or relations.

The restrictions are as follows. The generic X has a finite set S of singularities, so $K \setminus S$ is divided into finitely many relatively open segments, each of which consists of regular points or is an X^*-orbit. The endpoints of the regular segments can be any combination of vertices, singular nodes, and singular grains, but can not be singular saddles, singular sources, or singular sinks.

The endpoints of an X^*-orbit μ can be any of the seven kinds of singularities, but not all combinations are possible. For example if the α-limit of μ is a singular saddle then it can arrive at a vertex, a singular sink, or a singular saddle node, but it cannot arrive at a singular saddle, a singular source, or a singular grain. □

Definition 6.2. An unstable separatrix is **tame** if its ω-limit set is a regular point sink, a regular periodic orbit sink, or if it arrives at a sliding point. Time reversal gives the corresponding definition for stable separatrices.

Proposition 6.3. *If a separatrix is tame then it stays tame under small perturbations of X.*

Proof. This is clear enough.

Proposition 6.4. *A separatrix connection between a singularity and a singularity or regular saddle point can be broken by a small perturbation of X. (The two new separatrices are tame.) Also, separatrix relations can be broken by small perturbations of X, and once broken, they stay broken under subsequent sufficiently small perturbations of X.*

Proof. This is easy because it is essentially local. See Figure 10.

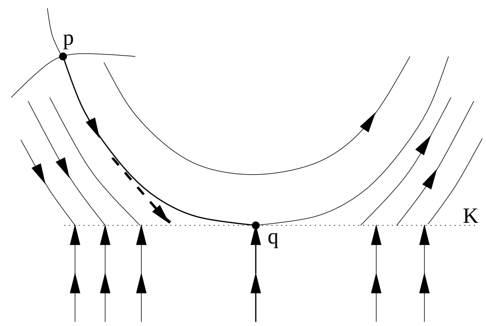


FIGURE 10. Breaking the separatrix γ which connects the regular saddle p to the singular saddle node q.

Remark. A separatrix connection that joins regular saddles can also be broken, but, as in the case of flows, the robustness is harder to prove because the situation is global.

Corollary 6.5. *The generic X has no separatrix connection between a singularity and a singularity or regular saddle point. This remains true for small perturbations of X.*

7. Proof of Theorems A and B

We follow the presentation of Peixoto's Genericity Theorem in [11] and [12]. The key issue is non-trivial recurrence. The idea is that if X has recurrent orbits then you can keep connecting separatrices until there are no more left to connect, and then you can tame one more separatrix. Induction finishes the proof except in the special case in which there are no separatrices to connect. This case arises for the torus with an irrational or a Denjoy flow.

Proposition 7.1 (Existence of a Circle Transversal). *Assume that the X-orbit through p is non-trivially recurrent. Then it is regular and through p there passes a smooth Jordan curve J everywhere transverse to X.*

Proof. Singular orbits are singularities or sliding curves. They are not non-trivially recurrent. The existence of J is proved in the same way as for flows. See [11], page 145. Although the orbit arc from p to a closest return may have a few corners where it crosses K, the construction is unaffected. ∎

The first return or Poincaré map P, is naturally defined by the X-orbit that leaves J at y and next returns to J at $P(y)$. Transversality implies that the domain of definition of P is an open subset $D \subset J$. It consists of open intervals I or it equals J. The X-orbits through I are regular, and we get an open-sided strip from I to $P(I)$. Let a be an endpoint of I. Its orbit does not return to J but it does stay on the boundary of the strip. Thus the forward orbit of a ends at a point a'. For the generic X, a' can be a regular saddle point, a singular saddle node, or a vertex. Thus a lies on a separatrix that does not return to J after leaving a. There are only finitely many separatrices and therefore D is either J or a finite union of intervals I whose endpoints lie on stable separatrices that leave J and go directly to regular saddles, singular saddle nodes, or vertices.

Proposition 7.2. *Assume that the Poincaré map is defined on the whole closed transversal J and some points of J are non-trivially recurrent. Then M is the torus, all points are regular non-equilibria, and a small perturbation of X produces a periodic orbit.*

Proof. The proof is purely topological. See [11], pages 145-146. □

Proposition 7.3. *Assume that the Poincaré map is defined on $D \neq J$ and a point $p \in D$ is non-trivially recurrent. Then there exists a regular unstable separatrix that accumulates at p. There exists a small perturbation that leaves all existing separatrix connections intact and produces one more.*

Proof. See [11], pages 141 and 147. □

Proposition 7.4. *If X has only trivial recurrence then the ω-limit set of an orbit of X is either empty (the orbit arrives at a singularity or sliding point in finite time), or is a regular equilibrium, or is a finite **graphic cycle** that consists of separatrix connections.*

Proof. This is the same as Proposition 2.3 of [11], page 139. □

Proposition 7.5. *A graphic cycle that is the ω-limit of an unstable separatrix σ can be perturbed to produce a regular periodic orbit that tames σ.*

Proof. This is the same as the flow case since none of the separatrices is singular. □

Proof of Theorems A and B. Applying the preceding propositions repeatedly eventually tames all the separatrices and eliminates non-trivial recurrence. Then the periodic orbits are perturbed to become hyperbolic without introducing new recurrence or separatrix connections. This gives an X that satisfies the conditions in Theorem B. It is structurally stable by the same considerations as in [11] and [13].

On the other hand, if X violates one of the conditions in Theorem B then it can be approximated by an X' for which all the conditions are true and, as for flows, this gives a contradiction to orbit equivalence. □

Remark. David Wallwork, a graduate student at Berkeley in the 60's, pointed out that Peixoto's treatment of the non-orientable case of structural stability is incomplete. To this day, the best results in that direction are due to Carlos Gutiérrez [4], who shows that if the non-orientable surface is the projective plane, the Klein bottle, or the torus with a cross cap then Peixoto's theorem is true. Our Theorems A and B remain true in these cases.

Similarly, if γ is a non-trivially recurrent orbit of the Filippov system X through p then X can be C^1 approximated by an X' having a periodic orbit through p, for the Poincaré map on which the proof relies is a local C^1 diffeomorphism since no singular orbits are involved. The corners that the regular orbits experience as they cross K do not make the Poincaré map piecewise C^1. Rather, it is completely C^1 because it is merely the composite of the Poincaré map from J to K and from K to J. Thus the C^1 closing lemma proof in [14] goes through word for word. The rest of the proofs of Theorems A and B in the non-orientable C^1 case are the same as in [15].

Remark. Finally, consider the situation in [1]. The surface M is triangulated and the Filippov system is affine on each simplex. It is easy to see that piecewise affine Filippov systems exhibit all the behavior of the generic smooth nonlinear Filippov systems, at least at K, and that their genericity properties at K are the same. It seems likely that this continues to hold for the global genericity properties, including structural stability.

8. Zeno Trajectories

A **Zeno trajectory** is an amalgamated orbit (a trajectory in the sense of Filippov) whose tangent field undergoes an infinite number of vector field switches in a finite length of time.

Theorem 8.1. *Generically, Zeno trajectories occur only at the vertices of K, and they do so because of only two phenomena:*

- *singular foci*
- *singular elliptic sectors.*

Proof. Let $\zeta(t)$ be a Zeno trajectory, $0 \leq t < b$, and let $X(t)$ be the branch of X or the sliding field tangent to ζ at time t. There is a succession of times $0 < t_1 < t_2 < \cdots < b$ such that

- $\zeta(t_k) \in K$ and $t_k \to b$ as $k \to \infty$.
- $X(t)$ changes branches or sliding fields at $t = t_k$ and only then.

Continuity implies that

$$v = \lim_{t \to b} \zeta(t)$$

exists, and that v is either a singular equilibrium or a vertex. Infinite switching does not occur in the neighborhood of a generic singular equilibrium. (In contrast, the Fuller phenomenon [3, 7] in which all trajectories are Zeno occurs in the degenerate case of two non-distinct tangency points.) So v is a vertex.

If v is a singular focus then there is nothing to prove, so we may assume that v has base orbits.

Consider an X-orbit that departs from v. It is an unstable base orbit. If it is singular then it either absorbs nearby orbits as t increases, or it emits them. See Figure 11. Thus there exists a minimum return time

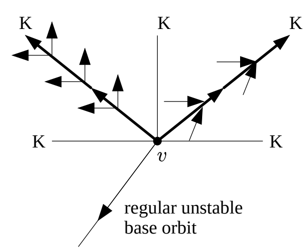


FIGURE 11. An unstable, singular base orbit absorbs or emits nearby orbits.

$\tau > 0$ for X-trajectories that depart from v along regular unstable base orbits or singular absorbing base orbits. Since ζ is a Zeno orbit, it must contain infinitely many disjoint arcs α contained in emitting, singular base orbits departing from v. Such an α quickly amalgamates to a short emitted, regular X-orbit β that then amalgamates to a short, stable, absorbing, singular base orbit ω. Furthermore, since $\tau > 0$ is fixed, eventually ζ consists entirely of a succession of such short 3-legged elliptic X-trajectories $\alpha\beta\omega$ in singular elliptic sectors at v. $\square$

Theorem 8.2. *Generically, a singular elliptic sector E has aperture $< \pi$ and is contained in a larger sector of aperture $< \pi$ bounded by edges of K or regular base orbits.*

Proof. After a smooth change of coordinates we can assume that E is the sector shown in Figure 12. A subsequent linear change of variables

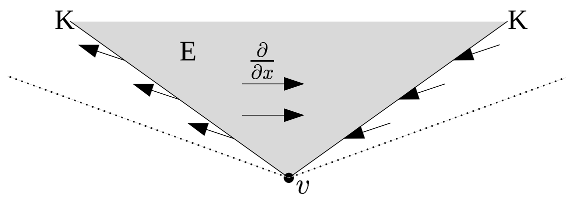


FIGURE 12. A symmetric representation of the singular elliptic sector E.

lets us assume that the branch X_E of X in E is $\partial/\partial x$.

The branch of X to the right of E, X_R, points down and into E. Thus, there is a regular base orbit of X_R in the upper half plane to the right of E, or some edge of K intervenes. Similarly there is a regular base orbit or edge of K in the upper half plane to the left of E. □

The **multiplicity** of K at v is the number of edges ending there.

Theorem 8.3 (Non-existence of Zeno trajectories)**.** *The generic X has no Zeno trajectories if either*

(a) *X has only two branches and is defined on the 2-sphere,*
(b) *X has no singular foci and at each vertex of K the multiplicity of K plus the number of regular base orbits is < 5.*

Proof. (a) Suppose not: there exists a Zeno trajectory ζ on S^2 where the complex K is a Jordan curve dividing the S^2 into the two regions G_1, G_2 that support the branches of X. The complex K contains finitely many vertices and is the common boundary of G_1 and G_2. Since X is generic, it is not tangent to K at the vertices, and there is at least one regular base orbit at each vertex. Thus there are no singular foci.

By Theorem 8.1 there is a singular elliptic sector. By Theorem 8.2 we can assume it lies in a larger sector S of aperture $< \pi$ bounded by regular base orbits or edges of K. Since the multiplicity of K at v is 2, there are no edges of K which can intervene, and it must be the case that S is bounded by regular base orbits, say of branches X_R and X_L to the right and left of E. These branches have conflicting $\partial/\partial x$ -components, which implies the existence of a third edge of K at v, a contradiction.

(b) Suppose not: there is a Zeno orbit ζ at a non-focus vertex v, and the multiplicity of K at v plus the number of regular base orbits there is < 5. As in part (a) there is a singular elliptic sector E at v

which is responsible for the Zeno behavior of ζ, and it is contained in a larger sector S of aperture $< \pi$, bounded by regular base orbits or edges of K. The sector complementary to S has aperture $> \pi$ and therefore contains a regular base orbit or an edge of K. In either case this implies that the number of edges of K at v plus the number of regular base orbits there is ≥ 5, a contradiction. □

Corollary 8.4. *The generic Filippov system on the* 2*-sphere investigated in* [17] *has no Zeno trajectories.*

Proof. The assumption in [17] is that K is the equator. □

9. Pinching

In this section we show that the generic $X \in \mathcal{X}_K^r$ can be obtained from some $X' \in \mathcal{X}_K^r$ by collapsing a neighborhood of K to K. We call the process pinching.

Definition 9.1. $P : M \to M$ is a **pinching map** to K if

- P is a continuous surjection.
- There exist open sets $G'' \subset G' \subset G$ such that both G' and G'' have k connected components, one in each G_i, and P sends G' diffeomorphically onto G.
- P is the identity map on G''.
- P is a retraction of $M \setminus G'$ onto K.

Definition 9.2. P **pinches** a smooth vector field $X' \in \mathcal{X}^r$ to a Filippov vector field $X \in \mathcal{X}_K^r$ if it is a pinching map and

- P is a topological equivalence between $X'|_{G'}$ and $X|_G$,
- for each X^*-orbit γ there is an arc γ' of an X'-orbit, called its **parent**, such that P sends γ' homeomorphically onto γ, preserving the orientation.

If P pinches X' to X we refer to X' as a **resolution** of X.

In the previous definition, by an orbit of $X'|_{G'}$ we mean the maximal arc of an X'-orbit (in M) which is contained in G'. Similarly for $X|_G$.

Definition 9.3. A **local resolution** of X is a resolution of $X|_U$, where U is a neighborhood of some $p \in M$ and G, G', G'', K are replaced by their intersections with U.

We can now state the main result of this section.

Theorem 9.4. *For the generic $X \in \mathcal{X}_K^r$ there exists a resolution. Singularities of X are resolved as follows.*

- *a singular saddle becomes a saddle;*

- *a singular sink becomes a sink;*
- *a singular source becomes a source;*
- *a singular saddle node becomes a collection of regular orbits;*
- *a singular grain becomes a collection of regular orbits;*
- *a vertex of K becomes a degenerate equilibrium point.*

Proof. In outline, we construct local resolutions of X along K and glue them together. First we treat non-vertex singularities.

Suppose that p is a singular saddle of $X \in \mathcal{X}^r_K$. Introduce smooth xy-coordinates at p in which K is the x-axis, the separatrix is the y-axis, and X has saddle type boundary behavior at the boundary of the square B, as shown in Figure 13.

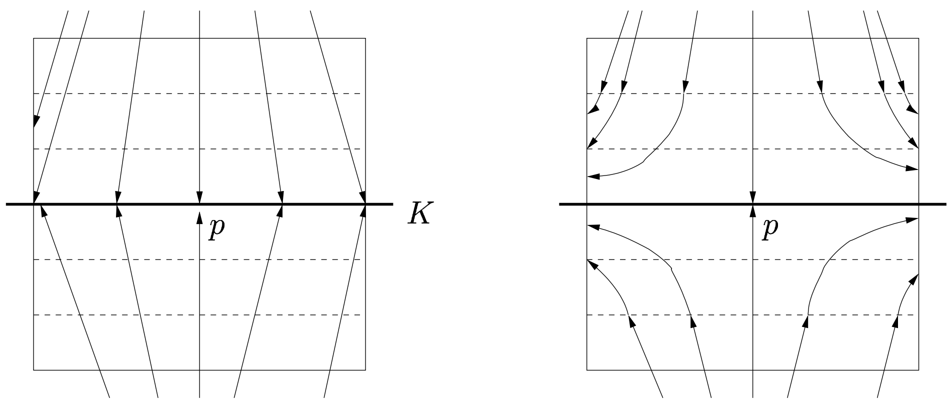


FIGURE 13. Resolving a singular saddle in a pinching box.

We refer to B as a **pinching box**. We scale the coordinates so that the pinching box corresponds to $[-1,1] \times [-1,1]$ and p is the origin.

Fix a diffeomorphism $\psi : [0,1] \to [1/3,1]$ such that $\psi(y) = y$ for all $y \geq 2/3$ and $\psi'(y) = 1$ for all y near 0. Define a retraction

$$\phi : [-1,1] \times [0,1] \to [-1,1] \times [1/3,1]$$
$$(x,y) \mapsto (x, \psi(y)).$$

Then $\phi_* X$ is a smooth vector field X^+ on $[-1,1] \times [1/3,1]$, and it has saddle type boundary behavior there. On the rectangle $[-1,1]\times[2/3,1]$, $X = X^+$. The same construction done below the x-axis produces a vector field X^-.

Figure 13 shows how to extend $X^- \cup X^+$ to a smooth vector field X' locally resolving X. The pinching map is essentially ϕ^{-1}, namely,

$$P(x,y) = \begin{cases} (x, \psi^{-1}(y)) & \text{if } 1/3 \leq y \leq 1 \\ (x, 0) & \text{if } -1/3 \leq y \leq 1/3 \\ (x, -\psi^{-1}(-y)) & \text{if } -1 \leq y \leq -1/3. \end{cases}$$

In keeping with the G, G', G'' notation we refer to

$$B' = \{(x, y) \in B : |y| \geq 1/3\} \qquad B'' = \{(x, y) \in B : |y| \geq 2/3\}.$$

Then P sends the interior of B' diffeomorphically to $B \setminus K$, and it leaves B'' pointwise fixed.

Figure 14 shows how to resolve a singular sink.

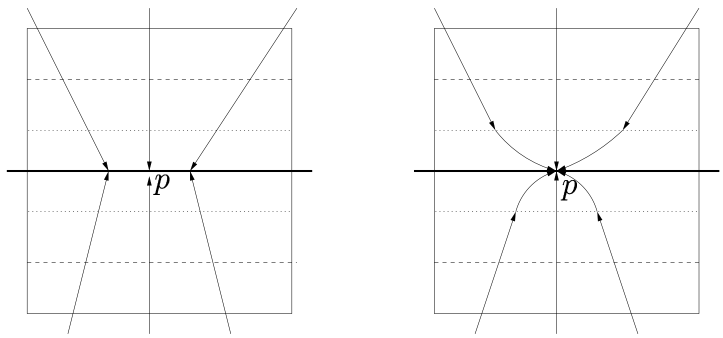


FIGURE 14. Resolving a singular sink.

Resolution of a singular source is the same with arrows reversed. Figure 15 shows how to resolve a singular saddle node, and Figure 16 shows how to resolve a singular grain.

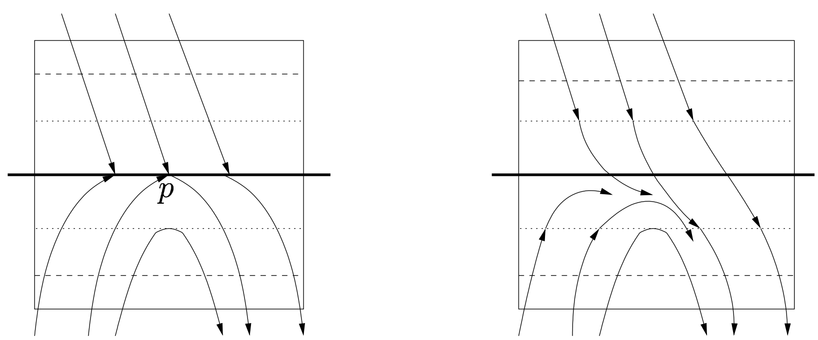


FIGURE 15. Resolving a singular saddle node.

Before treating vertices, we consider sliding segments σ between consecutive pinching boxes along K. The X-orbits near σ form a singular flowbox, which we resolve as a smooth flowbox. See Figure 17.

There is the possibility that X points across a segment σ on K between two pinching boxes. This is also a singular flowbox, and we resolve it to a smooth flowbox as shown in Figure 18.

Note. We can choose the pinching map on the flowbox to agree with the adjacent pinching maps and thus extend them smoothly to a tube along K.

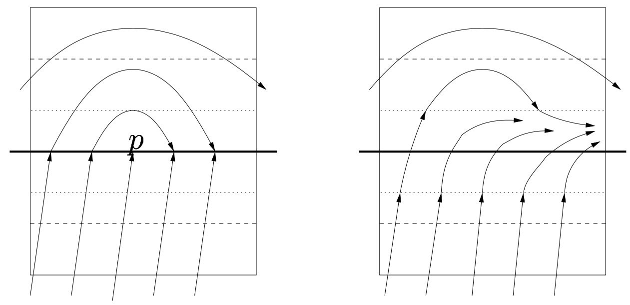


FIGURE 16. Resolving a singular grain.

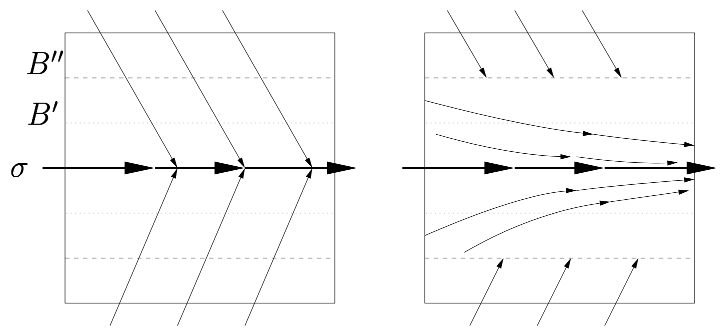


FIGURE 17. Resolving a singular flowbox along a sliding orbit. Note that the X^*-orbit remains an X'-orbit, so it has a parent.

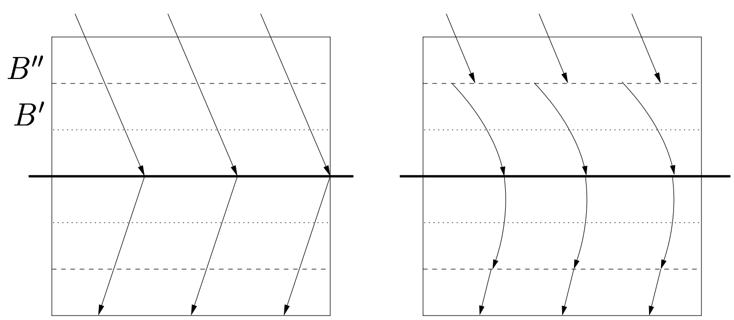


FIGURE 18. Resolving a singular flowbox along a crossing segment of K.

Now we turn to a vertex v of K.

Introduce smooth polar coordinates in a **pinching disc** D at v such that the edges of K at v are rays. For the generic $X \in \mathcal{X}^r_K$, we can assume there are no singularities other than v in D. Thus, in D the branches of X are never zero, and never in direct opposition at K. The pinching disc divides into finitely many closed sectors S; each is bounded by adjacent edges of K at v and an arc of ∂D. In each S we draw the bisector β, and subsectors $S'' \subset S' \subset S$ as shown in Figure 19. (We make the subsectors nearly equal to S.)

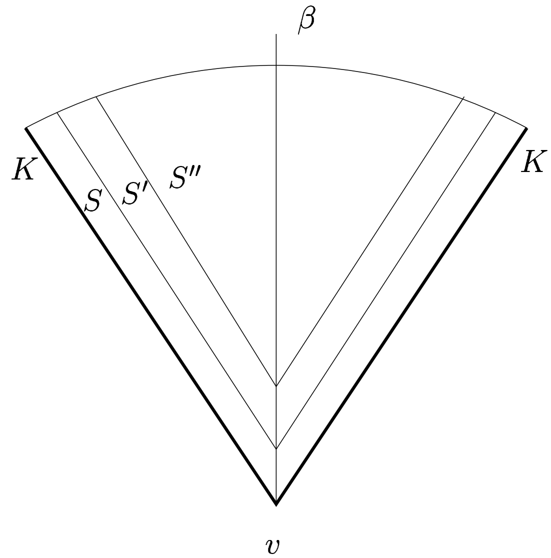


FIGURE 19. Subsectors of S.

There is a diffeomorphism $\phi : S \to S'$ that leaves S'' pointwise fixed. (Since S is closed, a diffeomorphism is actually defined on some open set that contains S.) The existence of ϕ is clear from Figure 19 and can also be verified explicitly with some effort, using the polar coordinates.

We define X_1 on S' as $\phi_*(X^S)$, where X^S is the appropriate branch of X restricted to S. We define P on S' as ϕ^{-1}. By construction P is a topological conjugacy from $X_1|_{\Delta'}$ to $X|_{D \setminus K}$, where Δ' is the union of S'-sectors. It remains to extend X_1 to the rest of D so that sliding orbits have parents. To do so, the following differential topology lemmas are useful.

Lemma 9.5. *Every smooth, non-vanishing vector field defined on the boundary of a disc D extends to a smooth vector field on D that vanishes exactly once.*

Proof. Let Y_0 be the given vector field defined on ∂D. Extend it to a smooth vector field Y_1 defined on a neighborhood of ∂D. Let A be an annular neighborhood of ∂D in D. If A is thin, Y_1 does not vanish on A. Pinch the inner boundary A_0 of A to a point. Since $A/A_0 \approx D \setminus 0$, this gives a smooth vector field Y_2 on $D \setminus 0$ with the prescribed boundary value Y_0. If $\lambda : D \to [0, 1]$ is smooth, positive on $D \setminus 0$, equal to 1 in a neighborhood of ∂D, and sufficiently flat at 0 then $Y = \lambda Y_2$ is the extension of Y_0 that we sought. □

The next lemma lets us prescibe some "base solutions" too.

Lemma 9.6. *Let Y_0 be a smooth non-vanishing vector field defined on ∂D, and suppose that $R_1, \dots, R_k$ are smooth arcs in D from 0 to $z_1, \dots, z_k$ in ∂D such that*

- *Except at 0, the R_j are disjoint.*
- *Except at z_j, R_j is interior to D.*
- *At z_j, Y_0 is tangent to R_j and transverse to ∂D.*

(Except at z_j, ∂D need not be smooth.) Then there is a smooth extension Y of Y_0 on D that vanishes only at 0 and $R_1, \dots, R_k$ are Y-orbits.

Proof. The construction in the proof of Lemma 9.5 gives an extension Y_2 of Y_0 for which there is a trajectory T_j joining 0 and z_j. Thus, $\{R_1, \dots, R_k\}$ and $\{T_1, \dots, T_k\}$ are webs of smooth arcs in $D \setminus 0$ such that R_j and T_j are tangent to Y_0 at ∂D. From this data alone, it follows that there is a diffeomorphism $h : D \setminus 0 \to D \setminus 0$ carrying the T-web to the R-web, and preserving Y_0. The vector field $Y_3 = h_* Y_2$, when multiplied by a sufficiently flat λ, is the extension of Y_0 we sought. □

Now we return to the pinching disc D, and the sub-disc D' obtained by excising its S'-subsectors. Define a vector field Y_0 on $\partial D'$ as follows. On the boundaries of the S'-subsectors, let Y_0 be X_1. The rest of $\partial D'$ consists of arcs $\Gamma_j = a_j b_j$ of ∂D crossed by edges E of K. See Figure 20.

If $\Gamma_j \cap E = z_j$ and z_j is a sliding point, set $Y_0(z_j) = X^*(z_j)$, and define Y_0 on the rest of Γ_j by smooth interpolation between the values X_1 near a_j, $X^*(z_j)$, and the values of X_1 near b_j. If z_j is not a sliding point, define Y_0 on Γ_j by smooth interpolation between the values of X_1 near a_j and b_j. Since X is generic and D contains no singularities except v, this produces a smooth non-vanishing vector field Y_0 on $\partial D'$.

If z_j is a sliding point let R_j be the ray of K from v to z_j. Applying Lemma 9.6 gives a smooth extension of Y_0 on D' that vanishes only at v and has base solutions R_j. Together with X_1 as previously constructed off D', this gives a smooth vector field X' on D that pinches to X, and each sliding orbit in D has a parent X'-orbit.

Finally we glue together the local resolutions as follows. Along an edge of K we have made local resolutions of X in pinching boxes. Between them are segments of sliding orbits of X and segments transverse to X. These we resolve as flowboxes.

Let G'' be the complement of all pinching boxes and pinching discs. It is an open subset of M, and its closure is disjoint from K. As noted above, because we can adjust the definition of the local pinching map inside pinching flowboxes, we can make the union of the local pinching

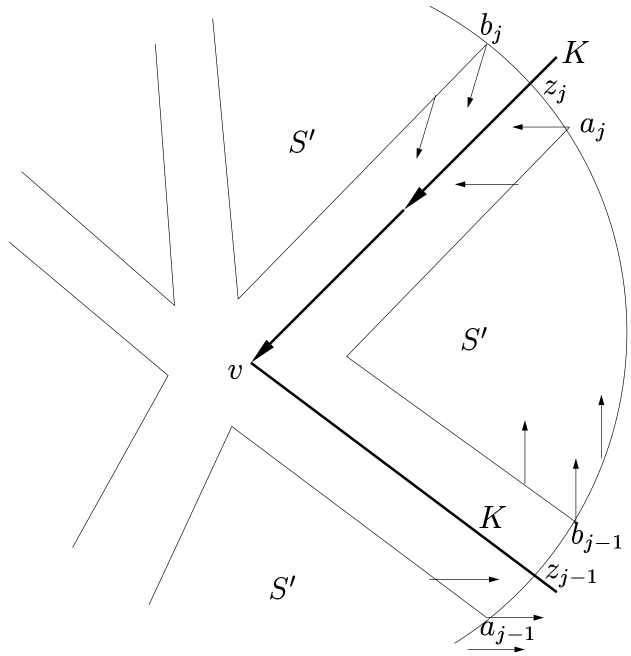


FIGURE 20. Defining the field Y_0 on $\Gamma_j = a_j b_j$.

maps, P, smooth. Its domain of definition is

$$G' = \text{Interior}\Big(\bigcup_p B'_p \cup \bigcup_\sigma B'_\sigma \cup \bigcup_{S'_v} S'_v \cup G''\Big),$$

where p ranges through the non-vertex singularities of X, σ ranges through the segments of K between the pinching boxes and the pinching discs, v ranges through the vertices of K, and S'_v ranges through the S'-subsectors of the pinching disc at v. Under P, G' is sent diffeomorphically onto $M \setminus K$, and $X'|_{G'}$-orbits are sent to $X|_G$-orbits. By construction, P leaves G'' pointwise fixed and every X^*-orbit has an X' parent. □

Remark. A strengthening of Theorem 9.4 would assert that there is a resolution X' of X such that every X-orbit, not merely every $X|_G$-orbit, has a parent X'-orbit. This would require local resolutions to respect the X-Poincaré maps defined between the boundaries of the pinching boxes. For that is how to show that X-orbits, such as periodic orbits which cross K, have X'-parents.

10. Filippov Automata

Piecewise smooth dynamical systems arise in applications in computer science, mechanics, and control theory. In this section we describe

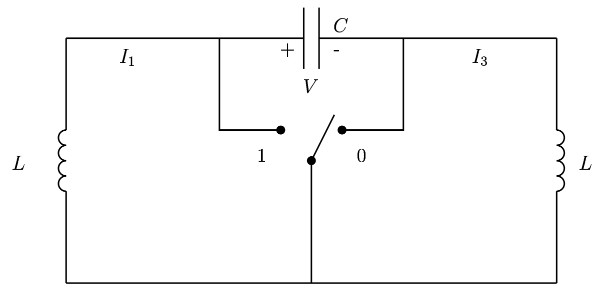


FIGURE 21. Switched circuit with dynamics on $SO(3)$.

a model that captures non-smooth phenomena, and we give several examples.

A *Filippov automaton* is a dynamical system defined by the triple

$$H = (M, K, D)$$

M and K are as before. Let $I = \{1, \dots, k\}$. The map $D : I \to \mathcal{X}^r$ assigns a C^r vector field to each $\bar{G}_i$. There is one location of the automaton for each $i \in I$. The set of edges $E \subset I \times I$ of H is determined by K. There exists an edge $e = (i, i') \in E$ if $\bar{G}_i$ intersects $\bar{G}_{i'}$ at a non-vertex. This intersection is referred to as the **e**nabling condition of edge e.

10.1. **Switched Circuits.** Consider the example depicted in Figure 21 of a switched circuit whose dynamics evolve on the Lie group $SO(3)$ [16]. The switch is used to transfer energy from one inductor to another. Let u be the switch position taking values $\{0, 1\}$. The dynamics are given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ f &= \omega x_2 \frac{\partial}{\partial x_1} - \omega x_1 \frac{\partial}{\partial x_2} \\ g &= -\omega x_2 \frac{\partial}{\partial x_1} + \omega(x_1 + x_3) \frac{\partial}{\partial x_2} - \omega x_2 \frac{\partial}{\partial x_3}, \end{aligned}$$

where $x_i = \sqrt{L} I_i$ for $i = 1, 3$, $x_2 = \sqrt{C} V$, and

$$\omega = \frac{1}{\sqrt{LC}}.$$

One can show that the stored energy $E = \frac{1}{2} x^T x$ is an invariant of the flow, so that for unit norm initial conditions, the state evolves on S^2.

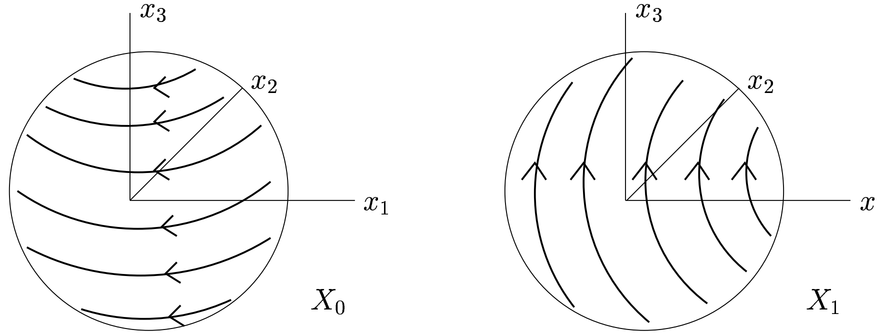


FIGURE 22. Vector fields X_0 and X_1 on S^2.

For $u = 0$, the vector field is $X_0 = f$. For $u = 1$, the vector field is

$$X_1 = \omega x_3 \frac{\partial}{\partial x_2} - \omega x_2 \frac{\partial}{\partial x_3}.$$

The orbit portraits are shown in Figure 22.

Suppose that we want to obtain an energy transfer between the inductors while keeping the capacitor voltage constant. This can be achieved using a piecewise smooth vector field. We define a 1-complex K that lies in the circle $x_2 = k$, $k \in (-1, 0)$ and consists of the points p_j and arcs l_j, for $j = 1, \ldots, 4$, as depicted in Figure 23(a). We know that the sliding field on arc l_1 is given by

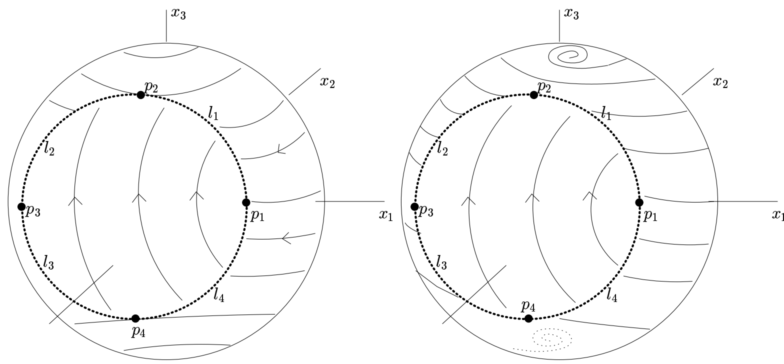
$$X^* = \lambda X_0 + (1 - \lambda) X_1.$$

Using the fact that X^* must be tangent to l_1 we obtain

$$X^* = \frac{\omega x_2}{x_1 + x_3} \left[x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} \right].$$

The resulting piecewise smooth flow is depicted in Figure 23(a). Vertices p_1 and p_3 are singular grains while p_2 and p_4 are singular saddle nodes. Edges l_1 and l_3 consist of sliding points while edges l_2 and l_4 consist of regular points. There is a separatrix connection between p_1 and p_3 by a regular orbit which is a base orbit of p_1 and p_3; there is a separatrix connection between p_2 and itself by a regular base orbit (hence, a homoclinic connection), and there is a separatrix connection between p_2 and p_4. There are also two regular equilibria at the north and south pole which are centers. The system is clearly not structurally stable.

Suppose that we perturb the flows X_0 and X_1 so that instead of two centers, each has one regular source node and one regular sink node;



(a) Non-structurally stable (b) Structurally stable

FIGURE 23. Piecewise smooth flows on S^2

namely,

$$\begin{aligned} X_0 &= (\omega x_2 - x_1 x_3)\frac{\partial}{\partial x_1} - (\omega x_1 + x_2 x_3)\frac{\partial}{\partial x_2} + (1 - x_3^2)\frac{\partial}{\partial x_3} \\ X_1 &= (1 - x_1^2)\frac{\partial}{\partial x_1} + (\omega x_3 - x_1 x_2)\frac{\partial}{\partial x_2} - (\omega x_2 + x_1 x_3)\frac{\partial}{\partial x_3}. \end{aligned}$$

Using the same 1-complex K as before, we obtain the sliding vector field on l_1 and l_3 given by

$$X^* = \frac{(1+\omega^2)x_2}{\omega x_3 - x_1 x_2 + \omega x_1 + x_2 x_3}\left[x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}\right]$$

Vertices p_1 and p_3 are singular grains while p_2 and p_4 are singular saddle nodes as before. Edges l_1 and l_3 are sliding points and edges l_2 and l_4 are regular points, as before. There are no longer separatrix connections between p_1 and p_3, between p_2 and itself, and p_2 and p_4. All regular equilibria are generic. Hence this system is structurally stable.

10.2. **Pendulum with Nonlinear Friction.** Consider a pendulum that has positive damping when it rotates in one direction and negative damping in the other. If θ is the angle the pendulum makes with the vertical then we have the dynamics

$$\ddot{\theta} + \operatorname{sgn}(\dot{\theta})[a + b\dot{\theta}] + c\theta = 0.$$

a, b, and c are positive constants. The state space of the system is a cylinder. We choose the 1-complex K to lie in the circle $\dot{\theta} = 0$ and consist of the end points $(0,0)$ and $(\pi, 0)$ and the two half-circles that

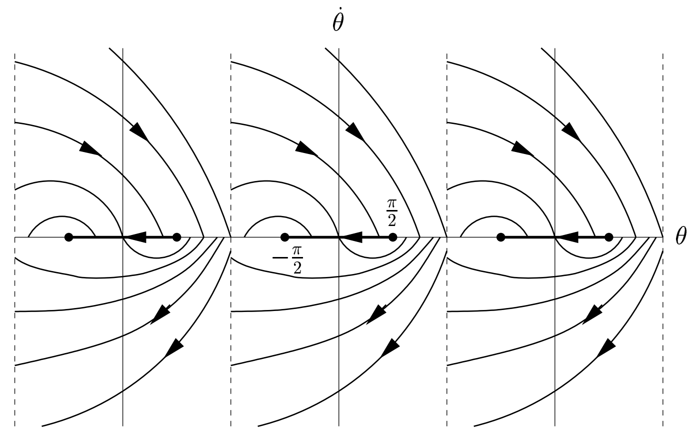


FIGURE 24. Piecewise smooth flow X on the cylinder.

connect them. Fixing $a, b, c > 0$ we obtain a vector field in the upper half cylinder with a stable equilibrium point at $(\frac{-\pi}{2}, 0)$, while the vector field in the lower half cylinder has an unstable equilibrium point at $(\frac{\pi}{2}, 0)$. (Thus, these points are half foci for X, not singular grains.) The situation is depicted in Figure 24. The open segment $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\dot\theta = 0$ consists of sliding points. A base orbit which is also a separatrix arrives at the points $(-\frac{\pi}{2}, 0)$ and at $(\frac{\pi}{2}, 0)$. Figure 24 shows trajectories in the negative $\dot\theta$ half-cylinder that never reach the sliding field. This corresponds to the pendulum swinging in the negative direction increasingly fast. Trajectories that start in the positive $\dot\theta$ half-cylinder eventually reach K, and this corresponds to the pendulum switching to swinging in the negative direction. Some of them reach the sliding field and others continue in the negative $\dot\theta$ direction. The system is not structurally stable, since it fails the conditions for genericity, namely regular equilibria in K. We can perturb the system by moving the regular foci away from K to obtain a structurally stable system.

10.3. **Control theory.** A control system is an ODE $\dot x = f(x, u)$ where the control parameter u varies in $\mathbb{R}^m$ and the space variable x varies in $\mathbb{R}^n$. Hence, it is an m-parameter family of ODE's. The set

$$f^{-1}(0) = \{(x, u) \in \mathbb{R}^{n+m} \mid f(x, u) = 0\},$$

is called the **e**quilibrium set of the control system. A control system is said to be **c**ontinuously stabilizable at $(x^*, u^*) \in f^{-1}(0)$ if there exists a continuous function $u = u(x)$ such that $u(x^*) = u^*$, and x^* is a globally asymptotically stable (g.a.s.) equilibrium point of the closed-loop system $\dot x = f(x, u(x))$. We give examples showing how a piecewise

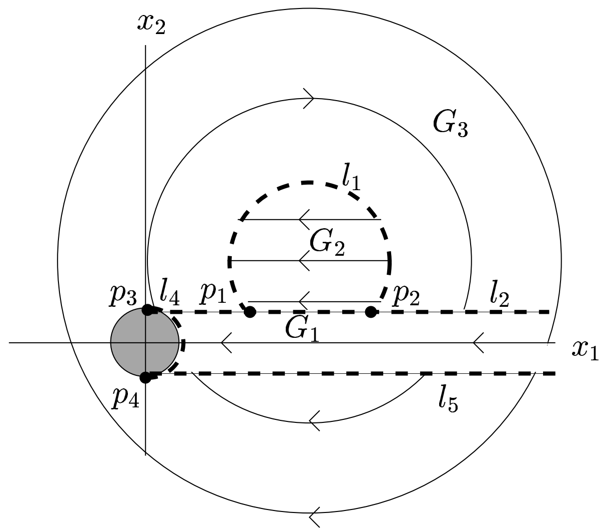


FIGURE 25. Non-structurally stable control system. The complex K is depicted with dotted lines.

smooth control that gives rise to a piecewise smooth vector field can be used to render an equilibrium point g.a.s.

Suppose that, given a control system with $n = 2$ and $m = 1$, we select two continuous controls u_1 and u_2 such that we obtain the vector fields

$$\begin{aligned} X_0 &= -\frac{\partial}{\partial x_1} \\ X_1 &= (x_2 - 2)\frac{\partial}{\partial x_1} + (-x_1 + 4)\frac{\partial}{\partial x_2}. \end{aligned}$$

We want to find a 1-complex K such that the trajectories of the Filippov system corresponding to (X_0, X_1, K) go to a disk around the origin. We propose the complex K shown in Figure 25 consisting of points $p_1, \ldots, p_4$ and the segments $l_1, \ldots, l_6$. Segments l_3 connecting p_1 and p_2, and l_6 connecting p_3 and p_4 are not labeled. Note that the definition of a complex is generalized to include segments that go to infinity. X_0 is applied in components G_1 and G_2 while X_1 applies everywhere else. Because all points of K are tangency points the system is not structurally stable.

If we modify the vector fields:

$$\begin{aligned} X_0 &= -\frac{\partial}{\partial x_1} - 0.1\frac{\partial}{\partial x_2} \\ X_1 &= (0.1x_1 + x_2 - 2)\frac{\partial}{\partial x_1} + (-x_1 + 0.1x_2 + 4)\frac{\partial}{\partial x_2}. \end{aligned}$$

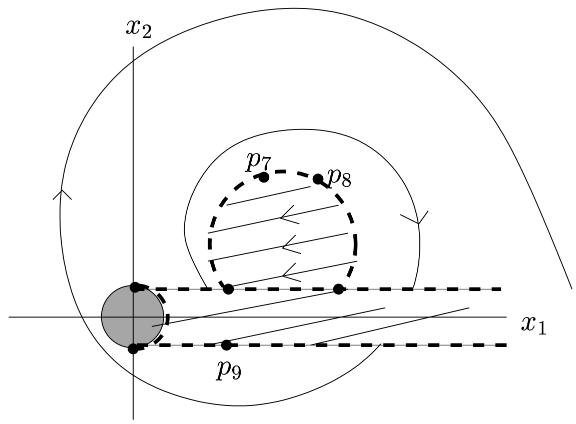


FIGURE 26. Structurally stable control system.

we obtain the phase portrait of Figure 26. Points p_7 and p_9 are singular grains and p_8 is a singular source. Points on l_1 to the right of p_8, on l_1 between p_7 and p_8, on l_5 to the left of p_9, and on l_4 are sliding points. Points on l_1 to the left of p_7, on l_5 to the the right of p_9, and on l_2 and l_3 are crossing points. This system is structurally stable.

A topological obstruction to global stabilizability arises when the graph of a continuous feedback $(x, u(x))$ stabilizing the system has at least two points of intersection with the equilibrium set [9]. Consider the system [10]

$$X = \sin(x_1^2 + x_2^2)\frac{\partial}{\partial x_1} + u\frac{\partial}{\partial x_2}.$$

The equilibrium set is

$$\{\ (x_1, x_2, u) : u = 0, x_1^2 + x_2^2 = n\pi, n = 0, 1, 2, \dots\ \}.$$

If we apply the control $u(x) = -x_2$ derived using the linearization of the system at the point $(\sqrt{\pi}, 0)$, the graph $(x, u(x))$ intersects the circle of radius $\sqrt{\pi}$ at the point $(-\sqrt{\pi}, 0)$ as well.

To overcome the obstruction, rather than insisting on a continuous control, we use a piecewise continuous control. We define the 1-complex K as depicted in Figure 27 and define u to be a continuous function in each component G_i. K consists of the points p_1, p_2, p_3, and p_4 lying in the circle of radius $\sqrt{\pi}$ and the arcs that connect them, and similarly for the the circles of radius $\sqrt{n\pi}$, $n = 2, 3, \dots$. All points on the arcs are sliding points except singular saddle nodes left of p_3 and left of p_4. The point p_2 is a singular source and p_1 is a singular sink. The system is not structurally stable because there is a singular separatrix between p_2 and p_1.

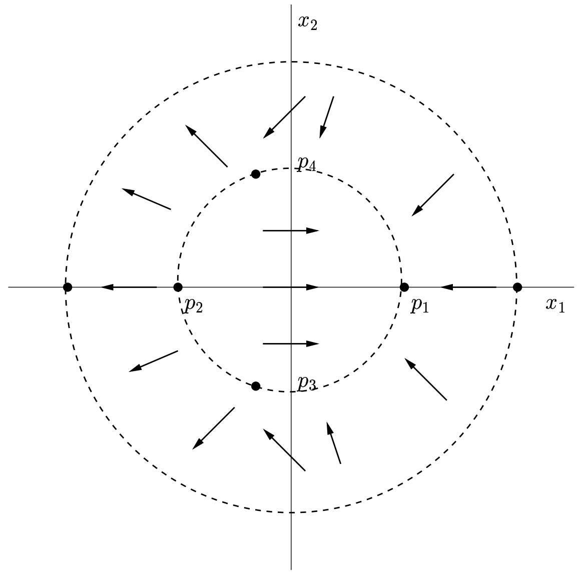


FIGURE 27. Piecewise smooth control system.

Consider the inverted pendulum

$$X = x_2 \frac{\partial}{\partial x_1} + (-\sin x_1 + u)\frac{\partial}{\partial x_2}.$$

We want to make the point $p_6 = (\pi, 0)$ globally asymptotically stable. After linearizing the system about p_6, we choose a linear controller $u(x) = -x_2$. The equilibrium set is

$$\{ \ (x_1, x_2, u) : x_2 = 0, \sin x_1 = u \ \}.$$

The graph $(x, u(x))$ intersects the equilibrium set at the points $(n\pi, 0)$, $n = 0, 1, 2, \dots$ so p_6 is not g.a.s. Instead we use a piecewise continuous control

$$u = \begin{cases} \frac{1}{2} & x_1 \in [0, \pi), \quad x_2 \in [0, 1) \\ 0 & x_1 \in [0, \pi), \quad x_2 \in (-1, 0) \\ -\frac{1}{2} & x_1 \in (-\pi, 0), \quad x_2 \in (-1, 0) \\ 0 & x_1 \in (-\pi, 0), \quad x_2 \in [0, 1) \\ -2 & x_2 \geq 1 \\ 2 & x_2 \leq -1 \end{cases}$$

Associated with this control is the one-complex K, shown in Figure 28 consisting of the points $p_1, \dots, p_9$ and the segments that connect them. There are three tangency points: t_1 is a singular saddle node, and t_2 and t_3 are grains. Points on the segments between p_1 and t_1 and between p_9 and p_4 are sliding points. Points between t_2 and t_3 are

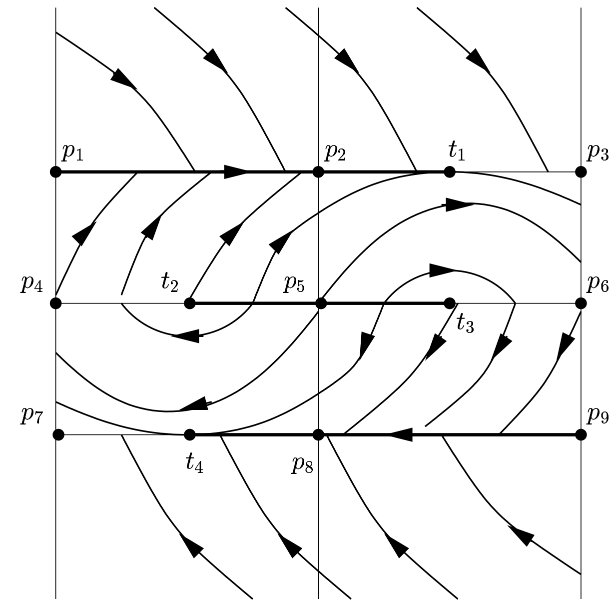


FIGURE 28. Inverted pendulum with piecewise continuous control.

singular equilibria, and all others are regular points. The system is not structurally stable because of the segment of singular equilibria. The point p_6 has been rendered g.a.s. because all trajectories arrive in a neighbhorhood of p_6 at which point the locally asymptotically stable linear controller can be used.

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Mathematics Department, University of California, Berkeley, CA 94720

EECS, University of California, Berkeley, CA 94720
E-mail address: `pugh@math.berkeley.edu, mire,simic@eecs.berkeley.edu`