Proceedings of the 37th IEEE
Conference on Decision & Control
Tampa, Florida USA • December 1998

Regularity of solutions and homotopic equivalence for hybrid systems

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1 Introduction

In this paper we study the trajectories of hybrid systems evolving according to constant, convex inclusions and Lipschitz nonlinear inclusions. Two questions are addressed. First, we investigate the existence of continuous selections of trajectories with respect to the initial conditions. Second, previous work on timed automata and hybrid automata has examined equivalence relations on runs of the automaton that visit the same locations and regions of the state space. Here we examine an equivalence relation defined directly on the trajectories. With suitable conditions on the enabling regions and using a suitable metric, we construct a homotopy on the set of solutions and use the homotopy to form an equivalence relation on the trajectories. We show the relationship between region equivalence introduced in [1] and homotopic equivalence. The tools needed for studying homotopic equivalence are the same as for obtaining continuity with respect to initial conditions.

2 Preliminaries

2.1 Notation

We denote by $|\cdot|$ the Euclidean norm. $d(x, B)$ is the distance from a point x to set B defined by $d(x, B) = \inf_{y \in B} |x - y|$. The Hausdorff distance between two sets, denoted d_H is $d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$.

For an interval $I = [t_0, t_1]$ let $C(I)$ and $C_{ac}(I)$ denote the spaces of continuous and absolutely continuous functions $f : I \to \mathbb{R}^n$, endowed with the sup norm $\|f\|_\infty$ and the norm $\|f\|_{ac} = |f(t_0)| + \int_I |\dot{f}(s)| ds$, respectively. We denote by $\mathcal{L}^1(I)$ the Lebesgue integrable functions on I. χ_E is the characteristic function of the set E. Finally, $\mathcal{J}(\mathbb{R}^n)$ denotes the space of Lipschitz differential inclusions defined from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$. We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of all functions $f : \mathbb{R} \to \mathbb{R}^n$ that are left continuous, $\lim_{t \uparrow a} f(t) = f(a)$, and have limits from the right.

2.2 Timed automata

We review the definition and semantics of timed automata, since hybrid automata build upon this model. A timed automaton is a tuple

$$A = (Q, \Sigma, Q^0, E, J)$$

consisting of the following components. $Q = L \times \mathbb{R}^n$ is the state space consisting of a finite set L of control locations and a continuous variable $x \in \mathbb{R}^n$. The dynamics for each $l \in L$ are given by the translation vector field $\dot{x} = 1$. Σ is a finite observation alphabet. $Q^0 : L \to 2^{\mathbb{R}^n}$ is a function assigning an initial set of states for each location such that if the automaton is initialized in location l, then $x \in Q^0(l)$ at $t = 0$. $E \subset L \times \Sigma \times L$ is a transition relation defining a finite set of edges. $e = (l, \sigma, l')$ is a directed edge between a source location l and a target location l' with observation σ. We write $l \xrightarrow{\sigma} l'$ for $e = (l, \sigma, l') \in E$ where σ is the observation of e.

$J : E \to G \times R$ is a function assigning to each edge a guard condition and a reset condition. $G = \{g \mid g \subset \mathbb{R}^n\}$ is the set of guard conditions on the continuous states. $R = \{r\}$ is a set of reset maps, where $r : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued map. We use the notation $G(e) = g_e$ and $R(e) = r_e$. The enabling conditions are generated by the grammar:

$$g := x_i \leq c_i | x_i \geq c_i | x_i < c_i | x_i > c_i | g_1 \wedge g_2 | g_1 \vee g_2 ,$$

where $c_i \in \mathbb{Z}$. The reset condition is of the form $r_i(x) = [a_i, b_i]$, where r_i is the ith component of the reset map, and $a, b \in \mathbb{Z}$. The reset map initializes the ith clock non-deterministically to a value between a_i and b_i, where $a_i, b_i \in \mathbb{Z}$. When a clock is not reset $r_i(x) = x_i$.

2.3 Hybrid automata

The tuple $A = (Q, \Sigma, D, Q^0, Inv, E, J)$ denotes a hybrid automaton consisting of the following components:

State space The state space $Q = L \times \mathbb{R}^n$, with l a finite set of control locations.

Events Σ is a finite observation alphabet.

Differential Inclusions $D : L \to \mathcal{J}(\mathbb{R}^n)$ is a function assigning a Lipschitz differential inclusion to each location. We use the notation $D(l) = F_l$. For location l, the dynamics are given by $\dot{x} \in F_l(x)$, $F_l \in \mathcal{J}(\mathbb{R}^n)$.

Initial conditions $Q^0 : L \to 2^{\mathbb{R}^n}$ is a function assigning an initial set of states for each location. If the automaton is started in location l, then $x \in Q^0(l)$ at $t = 0$.

Invariant conditions $Inv : L \rightarrow 2^{\mathbb{R}^n}$ is a function assigning for each location, an invariant condition on the continuous states. The invariant condition restricts the region on which the continuous states can evolve for each location.

Control switches $E \subset L \times \Sigma \times L$ is a set of control switches. $e = (l, \sigma, l')$ is a directed edge between a source location l and a target location l' with observation σ.

Jump conditions $J : E \rightarrow G \times R$ is a function assigning to each edge a guard condition and a reset condition. $G = \{g \mid g \subset \mathbb{R}^n\}$ is the set of guard conditions on the continuous states. $R = \{r\}$ is the set of reset conditions, where $r : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is a set-valued map.

We assume at the outset that for each $e = (l, \sigma, l') \in E$, $g_e \subseteq Inv(l)$, $r_e(g_e) \subseteq Inv(l')$, and for each $l \in L$, $Q^0(l) \subseteq Inv(l)$.

2.3.1 Semantics: A state is a pair (l, x) satisfying $x \in Inv(l)$. The invariant can be used to enforce edges from location l. In location l the continuous state evolves according to the differential inclusion F_l. $\Sigma(l)$ denotes the set of events possible at $l \in L$ and $E(l)$ denotes the set of edges possible at $l \in L$. An edge is enabled when the discrete location is l and the continuous state satisfies $x \in g_e$, for $e \in E(l)$. When the transition $e = (l, \sigma, l')$ is taken, the event σ is recorded, the discrete location becomes l', and the continuous state is reset (possibly non-deterministically) to $x' := r_e(x)$.

For $\sigma \in \Sigma$ a σ*-step* is a tuple $\xrightarrow{\sigma} \subset Q \times Q$ and we write $q \xrightarrow{\sigma} q'$. Define $\varphi_t^l(x)$ to be a trajectory of F_l at l, starting from x and evolving for time t. For $t \in \mathbb{R}^+$, define a t*-step* to be the tuple $\xrightarrow{t} \subset Q \times Q$. We write $(l, x) \xrightarrow{t} (l', x')$ iff (1) $l = l'$, (2) at $t = 0$, $x' = x$, and (3) for $t > 0$, $x' = \varphi_t^l(x)$, where $\dot{\varphi}_t^l(x) \in F_l(\varphi_t^l(x))$. When we do not want to explicitly fix the duration of the t-step, we use the label $\mathcal{T}$.

A *timed word* of A is a finite or infinite sequence $\overline{\tau} = \tau_0\tau_1\tau_2\cdots$ of letters from $\Sigma \cup \mathbb{R}^+$; that is, each τ_i is either an observation of A or a non-negative real that denotes a duration of time between observations. The timed word $\overline{\tau}$ is *divergent* if $\overline{\tau}$ is infinite and $\sum\{\tau_i | \tau_i \in \mathbb{R}^+, i \in \mathbb{N}\} = \infty$. A *trajectory* π of A is a finite or infinite sequence

$$\pi : q_0 \xrightarrow{\tau_0} q_1 \xrightarrow{\tau_1} q_2 \xrightarrow{\tau_2} \cdots$$

where $q_0 \in Q^0$, and for all $i \geq 0$, we have $q_i \in Q$, $\tau_i \in \Sigma \cup \mathbb{R}^+$. The trajectory π accepts the timed word $\overline{\tau} = \tau_0\tau_1\cdots$ and π is called divergent if $\overline{\tau}$ is divergent. Finally, a *run* of A is the projection to the discrete part of a trajectory accepted by A; namely, a finite or infinite sequence $l_0, l_1, l_2, \ldots$ of admissible locations. Π is the set of all trajectories of A and is called the *trajectory language*. Π_0 denotes the set of trajectories of A defined on a finite time interval. Finally, Π_d is the set of all runs accepted by A.

In the sequel we frequently view the trajectory as progressing in steps. A *step* refers to a t-step followed by a σ-step. Associated with the kth step of a trajectory is the data $I^k = (t^k, t^{k+1}]$, the time interval of the step, where $\tau^k = t^{k+1} - t^k$ is the duration, and $q^k = (l^k, x^k(t))$, the state, where l^k is fixed over I^k and $x^k(t)$ satisfies $\dot{x}^k(t) \in F_{l^k}(x^k(t))$. Thus, the step can be represented as

$$(l^k, x^k) \xrightarrow{t} (l^k, x^k(t^{k+1})) \xrightarrow{\sigma} (l^{k+1}, x^{k+1}), \qquad (2.1)$$

where $x^k(t^{k+1})$ is the value of the continuous state before the reset.

We assume throughout a *non-zeno* condition: every trajectory of A admits a finite number of σ-steps in any bounded time interval.

2.3.2 Bisimulation: A *bisimulation* of A is a binary relation $\simeq \subset Q \times Q$ satisfying the additional condition: for all states $p, q \in Q$, if $p \simeq q$ and $\sigma \in \Sigma \cup \mathcal{T}$, then

(a) if $p \xrightarrow{\sigma} p'$, then $\exists q'$ such that $q \xrightarrow{\sigma} q'$ and $p' \simeq q'$

(b) if $q \xrightarrow{\sigma} q'$, then $\exists p'$ such that $p \xrightarrow{\sigma} p'$ and $p' \simeq q'$.

2.4 Skorohod metric for hybrid systems

Now we introduce a metric which yields a suitable topology for hybrid trajectories. The Skorohod metric was originally used in the study of stochastic processes with right (or left)-continuous sample paths, such as Poisson processes [3].

We define the metric on $\mathcal{D}(\mathbb{R}^n)$ denoted $d_s'(f, g)$, as follows. Given two functions $f : I_f \rightarrow \mathbb{R}^n$ and $g : I_g \rightarrow \mathbb{R}^n$ the *Skorohod metric* $d_s'(f, g)$ is the infimum of $\epsilon > 0$ for which there exists a strictly increasing, continuous, onto function $\kappa : I_f \rightarrow I_g$ such that

(a) $\sup_{t \in I_f} |\kappa(t) - t| \leq \epsilon$ and

(b) $\sup_{t \in I_f} |f(t) - g(\kappa(t))| \leq \epsilon$.

We define a metric on Π_0 that combines the Skorohod metric on the continuous parts of a pair of trajectories with the distance between the corresponding runs in the Cantor topology. The resulting metric space is denoted (Π_0, d_s).

Let $\pi, \pi' \in \Pi_0$ with $\pi = \{(l^k, x^k(t)\}_{k=0}^{m-1}$ and $x = \{x^k(t)\}_{k=0}^{m-1}$ referring to the entire (finite) continuous trajectory. Analogous terms can be defined for π'. Then the distance between π and π' is given by

$$d_s(\pi, \pi') = d_s'(x, x') + \sum_{k=0}^{r-1} \frac{1}{2^k} \mathbf{1}(l^k \neq l'^k) + \frac{|m - m'|}{2^r},$$

where $r = \min\{m, m'\}$ and $\mathbf{1}(\cdot)$ is the indicator function.

3 Regularity

Regularity, or equivalently, continuity with respect to initial conditions for hybrid systems with Lipschitz differential inclusions is established under a *transversality* condition, stated in Definition 3.1. Let π_0 be a trajectory starting from p_0. We show that if π_0 satisfies the transversality condition, and under mild assumptions on the automaton stated in Assumption 3.2, there exists a continuous selection of trajectories from Π_0 on a neighborhood of p_0.

Consider the problem

$$\dot{x} \in F(x), \qquad x(0) = \xi \tag{3.1}$$

on a time interval $[0, T]$, where ξ ranges over a compact $X \subset \mathbb{R}^n$ with diameter D. In addition, we assume F satisfies

Assumption 3.1. The inclusion satisfies:

(a) The values of F are closed, nonempty subsets of $\mathbb{R}^n$.

(b) There exists $K \in \mathbb{R}$ such that $d_H(F(x), F(x')) \leq K|x - x'|$.

Under Assumption 3.1, an absolutely continuous solution to (3.1) exists for each $\xi \in X$ [5]. Let $\xi_0 \in X$ and $x(\cdot)$ be a solution of (3.1) such that $x(0) = \xi_0$. It is shown in [6] that there exists a selection $\varphi_t(\xi)$ from the set of solutions of (3.1) (with the topology of C_{ac}), which is continuous in $\xi \in X$ and such that $\varphi_t(\xi_0) = x(t)$. Such a selection is found by constructing a sequence of approximate trajectories, $\{\varphi_t^m(\xi)\}_{m=0}^{\infty}$ which are shown to form a Cauchy sequence in the normed space C_{ac}. In particular, this sequence can be chosen to satisfy

$$\|\varphi^m(\xi) - \varphi^{m-1}(\xi)\|_{ac} \leq D\Big(\frac{(KT)^m}{m!} + \frac{e^{2KT}}{2^{m+1}}\Big).$$

Thus,

$$\|\varphi(\xi) - \varphi^0(\xi)\|_{ac} \leq D\big(e^{KT} + e^{2KT}\big) \tag{3.2}$$

where

$$\varphi_t^0(\xi) = \xi + \int_0^t \dot{\varphi}_s(\xi_0)ds \tag{3.3}$$

is the initial guess of the approximate trajectories. Thus, we obtain the estimate

$$\|\varphi(\xi) - \varphi(\xi_0)\|_{ac} \leq D\big(e^{KT} + e^{2KT} + 1\big). \tag{3.4}$$

Assumption 3.2. The automaton A satisfies the following:

(a) The inclusion $\dot{x} \in F_l(x)$ at each location l satisfies Assumption 3.1.

(b) For each $e \in E$, g_e is either a compact, n-dimensional smooth manifold with boundary, or an embedded $(n-1)$-dimensional submanifold.

(c) r_e is a lower semi-continuous reset map from $\mathbb{R}^n$ to the closed, convex subsets of $\mathbb{R}^n$.

The following definition is essential for our main result.

Definition 3.1. Let $e = (l, \sigma, l')$ and $x(t)$, $t \in [t_0, T]$ be a solution of $\dot{x} \in F_l(x)$ such that $x(T) \in \partial g_e$. We say that $x(t)$ is *transversal* to g_e at $x(T)$ if, for some $\epsilon > 0$,

(i) there exist a neighborhood V of $x(T)$, and local coordinates $u = (u_1, \ldots, u_n)$ centered at $x(T)$ mapping V onto $\tilde{V} \times (-\epsilon, \epsilon) \subset \mathbb{R}^{n-1} \times \mathbb{R}^n$ such that $u^{-1}(\tilde{V} \times \{0\}) \subset \partial g_e$. In addition, if g_e is n-dimensional, then $u_n(y) > 0$, $\forall y \in V \cap int(g_e)$.

(ii) the interval of definition of $x(t)$ can be extended to $[t_0, T + \epsilon]$, in such a manner that

$$\dot{x}(t) \cdot \nabla u_n(x(t)) \geq 1, \qquad \text{a.e. on } \{t : x(t) \in V\}.$$

We say that an m-step trajectory $\pi = \{(l^k, x^k(t))\}_{k=0}^{m-1}$, whose steps are denoted as in (2.1), is a *transversal trajectory* if for each k such that $x^k(t^{k+1}) \in \partial g_{e^k}$, $x^k(t)$ is transversal to g_{e^k} at $x^k(t^{k+1})$.

The following technical lemma is needed.

Lemma 3.1. *Let $\dot{x} \in F_l(x)$ be a Lipschitz inclusion satisfying Assumption 3.1, and let $x(t)$, $t \in [t_0, t_1]$ be a solution that is transversal to g_e, $e = (l, \sigma, l')$ at $x(t_1)$. Then there exist $t_1'' > t_1$, a neighborhood W of $x(t_0)$, and a continuous selection of solutions $\varphi : W \to C_{ac}([t_0, t_1''])$ of $\dot{\varphi} \in F_l(\varphi)$ such that*

(a) $\varphi_t(x(t_0)) = x(t)$,

(b) there exists $t_1' \in (t_0, t_1)$ such that, with u denoting the coordinates in Definition 3.1, $\forall \xi \in W$,

$$\dot{\varphi}_t(\xi) \cdot \nabla u_n(\varphi_t(\xi)) \geq \frac{1}{2}, \qquad a.e. \text{ on } [t_1', t_1''].$$

(c) there exists a continuous $\tau : W \to [t_1', t_1'']$, satisfying $\tau\big(x(t_0)\big) = t_1$, such that $\varphi_{\tau(\xi)}(\xi) \in \partial g_e$, $\forall \xi \in W$.

Proof: By the transversality assumption there exists an open neighborhood V of $x(t_1)$ and coordinates $u : V \to \tilde{V} \times (-\epsilon, \epsilon) \subset \mathbb{R}^{n-1} \times \mathbb{R}$ such that x can be extended to $[0, t_1 + \epsilon]$ and $\dot{x}(t) \cdot \nabla u_n(x(t)) \geq 1$, a.e. on $\{t : x(t) \in V\}$. Since ∇u_n is continuous, there exists an open set $V' \subset V$, containing $x(t_1)$, such that, $\forall v \in V'$,

$$\dot{x}(t) \cdot \nabla u_n(v) \geq \frac{3}{4}, \qquad \text{a.e. on } \{t : x(t) \in V'\}. \tag{3.5}$$

Select times $t_1' < t_1 < t_1''$ such that $x(t) \in V'$, $\forall t \in [t_1', t_1'']$ and let

$$\epsilon' := \min\{-u_n(x(t_1')), u_n(x(t_1''))\}. \tag{3.6}$$

Finally, select $\delta' > 0$ such that

$$u_n(v) < -\frac{\epsilon'}{2}, \qquad \forall v \in B(x(t_1'), \delta'), \tag{3.7a}$$

$$u_n(v) > \frac{\epsilon'}{2}, \qquad \forall v \in B(x(t_1''), \delta'), \tag{3.7b}$$

$$B(x(t), \delta') \subset V', \qquad \forall t \in [t_1', t_1''], \tag{3.7c}$$

and choose D to satisfy

$$D\big(e^{K(t_1''-t_0)} + e^{2K(t_1''-t_0)} + 1\big) \leq \delta', \tag{3.8a}$$

$$2DKe^{2K(t_1''-t_0)} \cdot \sup_{v \in V'} |\nabla u_n(v)| \leq \frac{1}{4}. \tag{3.8b}$$

We use the construction in [6]. Let $\{\varphi_t^m(\xi)\}_{m=0}^{\infty}$ denote the sequence of approximate solutions in $C_{ac}([t_0, t_1''])$, with ξ in some neighborhood of $x(t_0)$, converging to $\varphi_t(\xi)$ uniformly in $C_{ac}([t_0, t_1''])$.

Let $\xi_0 := x(t_0)$. We claim that, for all $\xi \in B(\xi_0, \frac{D}{2})$

$$\varphi_t(\xi) \in V', \qquad \forall t \in [t_1', t_1''], \tag{3.9}$$

$$u_n(\varphi_{t_1'}(\xi)) < -\frac{\epsilon'}{2} \quad \text{and} \quad u_n(\varphi_{t_1''}(\xi)) > \frac{\epsilon'}{2}, \tag{3.10}$$

$$\dot\varphi_t(\xi) \cdot \nabla u_n(\varphi_t(\xi)) \ge \frac{1}{2}, \qquad \text{a.e. on } [t_1', t_1'']. \tag{3.11}$$

By (3.4) and (3.8a), $|\varphi_t(\xi) - x(t)| \le \delta'$, which together with (3.7) proves (3.9)–(3.10).

It remains to show (3.11). Using the construction in [6] one can derive the following. Corresponding to each $m \ge 0$ and to each $\xi \in B(\xi_0, \frac{D}{2})$, there is a finite partition $\{I_j(\xi)\}_{j=1}^{n_m}$ of $[t_0, t_1'']$, and a finite set $\Xi_m = \{\xi_j^\ell,\ 0 \le \ell \le m-1,\ 1 \le j \le n_m\}$, $\Xi_m \subset B(\xi_0, \frac{D}{2})$, not depending on ξ, such that, with $\xi_j^m \equiv \xi$, the following estimate holds a.e. on $I_j(\xi)$, for $\ell = 1, \dots, m-1$,

$$|\dot\varphi_t^\ell(\xi_j^\ell) - \dot\varphi_t^{\ell-1}(\xi_j^{\ell-1})| \le DK\Big[\frac{(K(t-t_0))^{\ell-1}}{(\ell-1)!} + \frac{e^{2K(t-t_0)}}{2^\ell}\Big]. \tag{3.12}$$

From (3.12), using a triangle inequality, we obtain

$$\begin{aligned} |\dot\varphi_t^m(\xi) - \dot\varphi_t^0(\xi_j^0)| &\le DK\Big[e^{K(t-t_0)} + e^{2K(t-t_0)}\Big] \\ &\le 2DKe^{2K(t_1''-t_0)}. \end{aligned} \tag{3.13}$$

a.e. on $I_j(\xi)$. Combining (3.5), (3.8b) and (3.13), and using the fact that $\dot\varphi_t^0(\xi_j^0) = \dot x(t)$,

$$\begin{aligned} \dot\varphi_t^m(\xi) \cdot \nabla u_n(\varphi_t^m(\xi)) &\ge \dot x(t) \cdot \nabla u_n(\varphi_t^m(\xi)) - \\ &|\nabla u_n(\varphi_t^m(\xi))| \cdot |\dot\varphi_t^m(\xi) - \dot\varphi_t^0(\xi_j^0)| \ge \tfrac{3}{4} - \tfrac{1}{4} = \tfrac{1}{2}, \end{aligned}$$

a.e. on $I_j(\xi)$, $\forall \xi \in B(\xi_0, \frac{D}{2})$, thus establishing (3.11), by passing to the limit as $m \to \infty$. Parts (a) and (b) of the Lemma follow if we select $W = B(\xi_0, \frac{D}{2})$.

Finally, by (3.11), for each $\xi \in W$ there exists a unique $\tau(\xi) \in (t_1', t_1'')$ and such that $\varphi_{\tau(\xi)}(\xi) \in \partial g_e$ or equivalently $u_n(\varphi_{\tau(\xi)}(\xi)) = 0$. To prove continuity, we argue by contradiction. Suppose $\{\xi_k\} \subset W$ is a sequence converging, as $k \to \infty$, to $\xi^* \in W$, and $\tau(\xi_k) \not\to \tau(\xi^*)$. Then, along some subsequence, denoted also by $\{\xi_k\}$, $\tau(\xi_k) \to \tau^*$, for some $\tau^* \ne \tau(\xi^*)$. It follows that $\varphi_{\tau(\xi_k)}(\xi_k) \to \varphi_{\tau^*}(\xi^*)$, and hence, $u_n(\varphi_{\tau(\xi_k)}(\xi_k)) \to u_n(\varphi_{\tau^*}(\xi^*))$. But $u_n(\varphi_{\tau(\xi_k)}(\xi_k)) = 0$, implying $u_n(\varphi_{\tau^*}(\xi^*)) = 0$, which contradicts the uniqueness of $\tau(\xi^*)$. This proves part (c). □

Theorem 3.2. *Suppose A satisfies Assumption 3.2 and let π_0 be a transversal m-step trajectory of A with initial state $p_0 = (l^0, \xi^0)$. There exists a neighborhood (l^0, U) of p_0, with $U \subset \mathbb{R}^n$ open, and $\Psi(t, \xi)$, a selection of Π_0, such that $\Psi(t, \xi^0) = \pi_0(t)$ and $\Psi(\cdot, \xi)$ is continuous on U.*

Proof: Suppose that π_0 has an m step run $l^0, \dots, l^{m-1}$, each step represented by (2.1), and visits the enabling conditions $g^0, \dots, g^{m-1}$, with $r^0, \dots, r^{m-1}$ denoting the corresponding reset maps. Observe first, that in order to obtain a continuous selection, the selection trajectories must take m steps, and have identical runs $l^0, \dots, l^{m-1}$.

First consider the reset of the kth step. Since r^k is locally selectionable by Michael's Selection Theorem, there exists a continuous selection $\tilde r^k$ of r^k, satisfying

$$\tilde r^k\big(x^k(t^{k+1})\big) = x^{k+1}. \tag{3.14a}$$

Hence, given an open neighborhood W^{k+1} of x^{k+1} there exists an open subset V^k containing $x^k(t^{k+1})$ such that

$$\tilde r^k\big(V^k \cap g^k\big) \subset W^{k+1}. \tag{3.14b}$$

If $x^k(t^{k+1}) \in \partial g^k$ then by Lemma 3.1, given an open set $V^k \ni x^k(t^{k+1})$ there exists a neighborhood W^k of $x^k(t^k)$ and a continuous selection $\psi^k : W^k \to C_{ac}([0, \hat t^{k+1} - t^k])$, for some $\hat t^{k+1} > t^{k+1}$, of solutions of $\dot\psi^k = F_{l^k}(\psi^k)$, along with a continuous $\tau^k : W^k \to [0, \hat t^{k+1} - t^k]$ such that

$$\psi_t^k\big(x^k(t^k)\big) = x^k(t - t^k), \quad t \in [0, t^{k+1} - t^k], \tag{3.15a}$$

$$\tau^k\big(x^k(t^k)\big) = t^{k+1} - t^k, \tag{3.15b}$$

$$\psi_{\tau^k(w)}^k(w) \in V^k \cap g^k, \qquad \forall w \in W^k \tag{3.15c}$$

On the other hand, if $x^k(t^{k+1}) \in \mathrm{int}(g^k)$, then selecting an open neighborhood $V^k \subset g^k$ of $x^k(t^{k+1})$, and defining $\tau^k = t^{k+1} - t^k$, by the results in [6], there is a continuous selection ψ^k defined on some open set $W^k \ni x^k(t^k)$ such that (3.15) holds.

A finite iteration of the arguments in the last two paragraphs, yields collections of open sets $\{W^0, \dots, W^{m-1}\}$ and $\{V^0, \dots, V^{m-1}\}$ along with continuous selections $\{\psi^k\}_{k=0}^{m-1}$ and continuous maps $\{\tilde r^k\}_{k=0}^{m-1}$ and $\{\tau^k\}_{k=0}^{m-1}$, as defined above, such that (3.14) and (3.15) hold.

Define $\tilde\psi^k : W^k \to V^k \cap g^k$ by $\tilde\psi^k(w) = \psi_{\tau^k(w)}^k(w)$.

From the continuity of $w \mapsto \psi_t^k(w)$ and of τ^k along with the absolute continuity of $t \mapsto \psi_t^k(w)$, and the triangle inequality

$$\begin{aligned} |\tilde\psi^k(w) - \tilde\psi^k(w')| &\le |\psi_{\tau^k(w)}^k(w) - \psi_{\tau^k(w')}^k(w)| + \\ &|\psi_{\tau^k(w')}^k(w) - \psi_{\tau^k(w')}^k(w')|, \end{aligned} \tag{3.16}$$

we obtain that $\tilde\psi^k$ is continuous on W^k. Let $U = W^0$ and define for $\xi \in U$ and $k = 1, \dots, m$,

$$\beta^k(\xi) = \tilde r^{k-1} \circ \tilde\psi^{k-1} \circ \dots \circ \tilde r^0 \circ \tilde\psi^0(\xi), \quad \beta^0(\xi) = \xi$$

$$t^k(\xi) = \sum_{\ell=0}^{k-1} \tau^\ell \circ \beta^\ell(\xi),$$

$$\Psi(t, \xi) = \psi_{t - t^k(\xi)}^k \circ \beta^k(\xi), \quad t \in (t^k(\xi), t^{k+1}(\xi)]$$

It follows that $t^k(\cdot)$ and $\Psi(t,\cdot)$, for fixed t, are continuous on U. To show continuity of $\Psi(\cdot,\cdot)$ in the hybrid metric d_s, let $\xi,\xi' \in U$ and define, for $t \in (t^k(\xi), t^{k+1}(\xi)]$,

$$\kappa(t) = \big(t - t^k(\xi)\big)\frac{t^{k+1}(\xi') - t^k(\xi')}{t^{k+1}(\xi) - t^k(\xi)} + t^k(\xi'),$$

It is straightforward to show that $|t - \kappa(t)| \xrightarrow[\xi' \to \xi]{} 0$, uniformly on $[t^0(\xi), t^m(\xi)]$ and using a triangle inequality as in (3.16), we can also show the same holds for $|\Psi(t,\xi) - \Psi(\kappa(t),\xi')|$. The proof is complete. □

4 Homotopic equivalence for hybrid systems with convex inclusions

We are given a hybrid automaton A, whose dynamics follow a constant, convex inclusion $\dot{x} \in F_l$, at each $l \in L$. We also assume that each enabling region g_e is convex and the reset $r_e : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is compact, convex valued.

Definition 4.1. We define the relation $\mathcal{R}$ on Q by: $[(l,x),(l',x')] \in \mathcal{R}$ if for every trajectory starting from (l,x) there is a trajectory starting from (l',x') which takes the same next edge, and vice versa. Let $\mathcal{R}^*$ be the coarsest stable refinement of $\mathcal{R}$.

While we are primarily interested in hybrid systems with differential inclusions, it is worth pointing out the relationship between edge equivalence and the region equivalence of [1] for timed automata, which we call $\mathcal{R}_{AD}$.

Lemma 4.1. *Partition $\mathcal{R}^*$ of timed automaton T is $\mathcal{R}_{AD}$.*

4.1 Homotopic equivalence

Consider two trajectories $\pi_0, \pi_1 \in (\Pi_0, d_s)$. The kth step of π_α, $\alpha \in \{0,1\}$ is represented by

$$(l_\alpha^k, x_\alpha^k) \to (l_\alpha^k, x_\alpha^k(t_\alpha^{k+1})) \to (l_\alpha^{k+1}, x_\alpha^{k+1})$$

where l_α is the location, x_α is the continuous state, and t_α is the time.

Definition 4.2. We say trajectories $\pi_0, \pi_1 \in (\Pi_0, d_s)$ are *homotopically equivalent*, or $\pi_0 \simeq_h \pi_1$ if there is a continuous map $h(\cdot) : [0,1] \to (\Pi_0, d_s)$ such that, for each $\theta \in [0,1]$, $h(\theta)$ is a trajectory of A and $h(0) = \pi_0$, $h(1) = \pi_1$.

We define the *homotopy relation* $\mathcal{H}$ on Q by: $[(l,x),(l',x')] \in \mathcal{H}$ if to every trajectory starting from (l,x) there is a trajectory starting from (l',x') that is homotopically equivalent to it, and vice versa. Clearly $\mathcal{H}$ is an equivalence relation.

Lemma 4.2. $\mathcal{H} \subseteq \mathcal{R}^*$.

Proof: If not, there exist states $(l,x),(l',x')$ and $[(l,x),(l',x')] \in \mathcal{H}$ but $[(l,x),(l',x')] \notin \mathcal{R}$. This means there exists a trajectory π starting at (l,x) and $k \in \mathbb{Z}$, such that location l^k is visited after the kth transition, whereas for any trajectory π' starting from (l',x'), the location visited after the kth transition is not l^k. It is clear from the definition of the hybrid metric d_s that π and π' are not homotopic and consequently $[(l,x),(l',x')] \notin \mathcal{H}$. □

Suppose that π_0 and π_1 have an m step run $l_0^k = l_1^k$, $k = 0,\dots,m-1$, and visit the enabling conditions $\{g^k\}_{k=0}^{m-1}$, with $\{r^k\}_{k=0}^{m-1}$ denoting the corresponding reset maps. We define the homotopy $h(\theta)$ to be a trajectory of A with the kth step denoted by:

$$(l^k, z_\theta^k) \to (l^k, z_\theta^k(u_\theta^{k+1})) \to (l^{k+1}, z_\theta^{k+1}) \tag{4.1}$$

where $z_\theta^k(u)$ is the continuous part of the trajectory over the kth step and u is time. The initial and reset times and states are given by

$$\begin{aligned} u_\theta^k &= (1-\theta)t_0^k + \theta t_1^k, \qquad k = 0,\dots,m, \\ z_\theta^k &= (1-\theta)x_0^k + \theta x_1^k, \qquad k = 0,\dots,m. \end{aligned} \tag{4.2}$$

For $k = 0,\dots,m-1$ and $\alpha = 0,1$, define

$$\begin{aligned} \mu_\alpha^k(\theta) &= \frac{t_\alpha^{k+1} - t_\alpha^k}{u_\theta^{k+1} - u_\theta^k}, \\ \nu_\alpha^k(\theta) &= \frac{u_\theta^{k+1}t_\alpha^k - u_\theta^k t_\alpha^{k+1}}{u_\theta^{k+1} - u_\theta^k}. \end{aligned} \tag{4.3}$$

and, for $u \in (u_\theta^k, u_\theta^{k+1}]$, let

$$z_\theta^k(u) = (1-\theta)x^k\big(\mu_0^k(\theta)u + \nu_0^k(\theta)\big) + \theta y^k\big(\mu_1^k(\theta)u + \nu_1^k(\theta)\big). \tag{4.4}$$

Note that $(1-\theta)\mu_0^k(\theta) + \theta\mu_1^k(\theta) = 1$, hence differentiating (4.4) and using the convexity of the inclusion as well as the convexity of the enabling condition and the reset map, we deduce that $\{l^k, z_\theta^k(u)\}_{k=0}^{m-1}$ is a trajectory of A, for each $\theta \in [0,1]$. To establish continuity we first define, for $\theta,\theta' \in [0,1]$, the map $\kappa_{\theta,\theta'} : [u_\theta^0, u_\theta^m] \to [u_{\theta'}^0, u_{\theta'}^m]$, by

$$\kappa_{\theta,\theta'}(u) = \frac{u_{\theta'}^{k+1} - u_{\theta'}^k}{u_\theta^{k+1} - u_\theta^k}u + \frac{u_\theta^{k+1}u_{\theta'}^k - u_\theta^k u_{\theta'}^{k+1}}{u_\theta^{k+1} - u_\theta^k},$$

for $u \in (u_\theta^k, u_\theta^{k+1}]$. It is straightforward to show that

$$\sup_{u \in [u_\theta^0, u_\theta^m]} \left|\kappa_{\theta,\theta'}(u) - u\right| \xrightarrow[\theta' \to \theta]{} 0$$

and that the same holds for $\left|z_\theta^k(u) - z_{\theta'}^k\big(\kappa_{\theta,\theta'}(u)\big)\right|$.

Theorem 4.3. $\mathcal{R}^* = \mathcal{H}$.

Proof: We need only show that $\mathcal{R}^* \subseteq \mathcal{H}$. Suppose $[(l,x),(l',x')] \in \mathcal{R}^*$. Then, $l = l'$, and for each finite trajectory $\pi_0 = \{l^k, x^k(\cdot)\}_{k=0}^m$ starting from (l,x) there exists, a trajectory $\pi_1 = \{l^k, y^k(\cdot)\}_{k=0}^m$. Construct the homotopy $h(\theta)$ as defined in (4.1)–(4.4). By definition $h(0) = \pi_0$ and $h(1) = \pi_1$, h is continuous in θ and is a trajectory of A. Therefore, $[(l,x),(l',x')] \in \mathcal{H}$. □

5 Homotopic equivalence with Lipschitz inclusions

In this section we consider the class of hybrid automata A satisfying Assumption 3.2. The goal is to characterize the homotopy equivalence relation $\mathcal{H}$ on trajectories of A. In contrast with the constant, convex inclusion case, it is not adequate to consider the partition formed by states whose runs take the same transitions. A finer partition is needed. For each $e \in E$, let $g_e = \cup_{\alpha=1}^{m_e} g_{e,\alpha}$ be the decomposition of g_e into its path connected components.

Definition 5.1. We say $[(l, x), (l', x')] \in \mathcal{R}_p$ iff

(a) $[(l, x), (l', x')] \in \mathcal{R}$.

(b) $\forall e \in E(l)$ and $\forall \alpha$, $x \in g_{e,\alpha}$ iff $y \in g_{e,\alpha}$.

We define *path connected edge equivalence* $\mathcal{R}_p^*$ to be the coarsest stable path connected refinement of the equivalence relation $\mathcal{R}_p$.

It is difficult to show that $\mathcal{R}_p^* \subset \mathcal{H}$, without additional hypotheses. We present here a result on homotopy of solutions to inclusions. This result is a ramification of the results of [6], and essentially guarantees that any two solutions of a Lipschitz inclusion are homotopic. It is stated below without proof, for brevity.

Theorem 5.1. *Let*

$$\dot{x} \in F(x) \tag{5.1}$$

satisfy Assumption 3.1 and $K \subset \mathbb{R}^n$ *be a compact set. Suppose that a set of initial conditions* $\{x_i\}_{i=1}^N \subset K$, *and corresponding solutions* $\{\gamma_i(t)\}_{i=1}^N$ *are given on a time interval* $[0, T]$. *Then there exists a continuous selection of solutions* $\gamma(t, x)$ *of* (5.1), *defined on* $t \in [0, T]$, $x \in K$, *such that* $\gamma(\cdot, x_i) = \gamma_i(\cdot)$, *for* $i = 1, \ldots, N$.

Acknowledgments The author thanks Pravin Varaiya for motivating this work by suggesting homotopic equivalence, and Anuj Puri for his valuable advice.

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