Reachability Analysis of Hybrid Systems with Linear Dynamics

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Abstract

We present results on symbolic reachability analysis for hybrid systems with linear autonomous dynamics in each location.

1 Introduction

A great deal of attention has been being focused on algorithmic approaches to reachability analysis of hybrid systems, as evidenced by the number of papers devoted to this topic in recent workshops [4, 5]. In this paper we present results on symbolic reachability analysis for hybrid systems with linear dynamics. The two step approach involves finding stable partitions for each linear vector field of the hybrid system by finding expressions for the local first integrals in a compact region of the continuous state space, assuming that the dynamics are in Jordan form. Second, the enabling, reset, initial and final conditions are selected or are pre-defined to be compatible with these partitions. This gives an analytical representation of a finite bisimulation of the hybrid system. The analytical representation can be used to obtain the symbolic execution theory of the hybrid automaton [2] which enables a symbolic reachability algorithm to be developed. The practical difficulties in applying this method are in finding the expressions for the first integrals and in obtaining the enabling, and reset conditions that are compatible with the partitions. It is the goal of this paper to address the first difficulty by explicitly carrying out the computation of first integrals for linear systems in Jordan form. These can be hardcoded in a symbolic reachability tool. The latter difficulty, which involves methods of partition refinement, will be addressed in future papers.

2 Bisimulation for hybrid automata

Let $\mathcal{X}(\mathbb{R}^n)$ denote the sets of smooth vector fields on $\mathbb{R}^n$. A hybrid automaton is a tuple $H = (Q, \Sigma, D, E, I, G, R)$ with the following components. $Q = L \times \mathbb{R}^n$ consists of a finite set L of control locations and n continuous variables $x \in \mathbb{R}^n$. Σ is a finite observation alphabet. $D : L \to \mathcal{X}(\mathbb{R}^n)$ is a function assigning a linear autonomous vector field to each location. We use the notation $D(l) = J_l x$, where $J_l \in \mathbb{R}^{n \times n}$. $E \subset L \times \Sigma \times L$ is a set of control switches. $e = (l, \sigma, l')$ is a directed edge between a source location l and a target location l' with event label σ. $I : L \to 2^{\mathbb{R}^n}$ is a mapping assigning a compact invariant condition $I(l) = I^l \subset \mathbb{R}^n$ to each location. $G : E \to \{g_e\}_{e \in E}$ is a function assigning to each edge an enabling (or guard) condition $g_e \subset I^l$, where we use the notation $G(e) = g_e$. $R : E \to \{r_e\}_{e \in E}$ is a function assigning to each edge a reset condition, $r_e : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, where we use the notation $R(e) = r_e$ and $r_e(g_e) \subset I^{l'}$.

Semantics. The state of H is a pair (l, x), where $l \in L$ and $x \in I^l$. Trajectories of H evolve in *steps* of two types. A *σ-step* is a binary relation $\xrightarrow{\sigma} \subset Q \times Q$, and we write $(l, x) \xrightarrow{\sigma} (l', x')$ iff (1)

$e = (l, \sigma, l') \in E$, (2) $x \in g_e$, and (3) $x' = r_e(x)$. A *t-step* is a binary relation $\xrightarrow{t} \subset Q \times Q$, and we write $(l, x) \xrightarrow{t} (l', x')$ iff (1) $l = l'$, and (2) for $t \geq 0$, $x' = \phi_t(x)$, where $\dot{\phi}_t(x) = J_l \phi_t(x)$.

Definition 2.1. *Let $\lambda \in \mathbb{R}$ represent an arbitrary time interval. A* bisimulation *of H is an equivalence relation $\simeq \subset Q \times Q$ such that for all states $p_1, p_2 \in Q$, if $p_1 \simeq p_2$ and $\sigma \in \Sigma \cup \{\lambda\}$, then if $p_1 \xrightarrow{\sigma} p_1'$, there exists p_2' such that $p_2 \xrightarrow{\sigma} p_2'$ and $p_1' \simeq p_2'$.*

Definition 2.2. *For each $l \in L$, let $\simeq^l$ be an equivalence relation on $l \times \mathbb{R}^n$ We say $\simeq^l$ defines a* stable partition with respect to the flow *ϕ^l if $(l, x) \simeq^l (l, x')$ implies that for all $y \in \mathbb{R}^n$ and $t \geq 0$, if $y = \phi^l_t(x)$, then there exists $y' \in \mathbb{R}^n$ and $t' \geq 0$ such that $y' = \phi^l_{t'}(x')$ and $(l, y) \simeq^l (l, y')$.*

Definition 2.3. *Let $e = (l, \sigma, l') \in E$ and let $\{\simeq^l\}_{l \in L}$ define a set of stable partitions. Given $\simeq^l$ at $l \in L$, we say g_e* is compatible with *$\simeq^l$ if $(l, x) \in \{l\} \times g_e$ implies $[(l, x)] \subseteq \{l\} \times g_e$. That is, the enabling condition is a union of cosets of $\simeq^l$. Analogous definitions for compatibility of Q^0, Q^f, and I^l apply. For $e = (l, \sigma, l')$ we say that r_e* is compatible with *$\simeq^{l'}$ if $(l', x') \in \{l'\} \times r_e(x)$ implies $[(l', x')] \subseteq \{l'\} \times r_e(x)$, and $[(l, x)] = [(l, y)]$ implies $r_e(x) = r_e(y)$. Finally, we say A* is compatible with *$\{\simeq^l\}$ if for each $e \in E$, g_e and r_e are compatible with $\simeq^l$, $\simeq^{l'}$, respectively, and for each $l \in L$, I^l is compatible with $\simeq^l$, and Q^0 and Q^f are compatible with $\{\simeq^l\}$.*

Lemma 2.1. *Given H and $\{\simeq^l\}$ defining a set of stable partitions such that H is compatible with $\{\simeq^l\}$, then $\simeq \subset Q \times Q$ defined by: $(l, x) \simeq (l', x')$ iff (1) $l = l'$, and (2) $(l, x) \simeq^l (l', x')$, is a bisimulation for H.*

We build stable partitions using foliations. We know from the Pre-Image theorem [6, p. 31] that the pre-image of a submersion is a foliation with regular leaves. Let $f \in \mathcal{X}(\mathbb{R}^n)$. We require two types of co-dimension one foliations. A *tangential foliation* F of $\mathbb{R}^n$ is a co-dimension one foliation that satisfies $f(x) \in T_x F, \forall x \in \mathbb{R}^n$; that is, f is a cross-section of the tangent bundle of F. A *transversal foliation* $F_\perp$ of $\mathbb{R}^n$ is a co-dimension one foliation that satisfies $f(x) \notin T_x F, \forall x \in \mathbb{R}^n$. A stable partition on I^l is constructed using a set of co-dimension one tangential foliations with submersions $\Psi^l_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n-1$ and a co-dimension one transversal foliation with submersion $\Psi^l_n : \mathbb{R}^n \to \mathbb{R}$, $l \in L$, such that $\Psi^l = (\Psi^l_1, \ldots, \Psi^l_n) : I^l \to [-1, 1]^n$ form coordinates on I^l. Ψ^l_i, $i = 1, \ldots, n-1$ are obtained using local *first integrals* of $J_l x$ on I^l. A first integral of $\dot{x} = f(x)$ is a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ satisfying $L_f \Psi = 0$, where $L_f \Psi$ is the Lie derivative of Ψ along f. We *discretize* the foliations by selecting a finite set of leaves. Fix $k \in \mathbb{Z}^+$ and let $\Delta = \frac{1}{2^k}$. Define $C_k = \{0, \pm\Delta, \pm 2\Delta, \ldots, \pm 1\}$. Each $\Psi^l_i = c$ for $c \in C_k$, $i = 1, \ldots, n$ defines a hyperplane in $\mathbb{R}^n$ denoted $\tilde{W}^l_{i,c}$, and a submanifold $W^l_{i,c} = (\Psi^l)^{-1}(\tilde{W}^l_{i,c})$. The collection of submanifolds is denoted $\mathcal{W}^l = \{\ W^l_{i,c} \mid c \in C_k, i \in \{1, \ldots, n\}\ \}$. Define an equivalence relation $\simeq^l$ on $I^l \times I^l$ by $x \simeq x'$ iff (1) $x \notin I^l$ iff $x' \notin I^l$, and (2) if $x, x' \in I^l$, then for each $i = 1, \ldots, n$, $\Psi_i(x) \in (c, c + \Delta)$ iff $\Psi_i(x') \in (c, c + \Delta)$, and $\Psi_i(x) = c$ iff $\Psi_i(x') = c$, for all $c \in C_k$. It was shown in [1] that $\simeq^l$ defines a stable partition.

3 Jordan form

We derive expressions for Ψ_i, $i = 1, \ldots, n-1$ when the linear dynamics are in Jordan form. The procedure consists of the following steps: (1) for each type of elementary Jordan block derive expressions for the local first integrals, and (2) for each pair of Jordan blocks derive an expression for the coupling first integral, defining another tangential foliation.

Consider the linear system $\dot{x} = Jx$ where $J \in \mathbb{R}^{n\times n}$ is of the form $J = diag(J^r, \cdots, J^r, J^c, \cdots, J^c)$. J^r and J^c are elementary Jordan blocks corresponding to the real (repeated) eigenvalues and complex (repeated) eigenvalues of J, respectively. The expression $F(t, x, c) = x - \phi_t(c)$ vanishes on solutions of the linear system. For values of t, x, and c where F is non-singular the implicit function theorem can be applied to obtain $c_i = g_i(x, t)$, $i = 1, \ldots, n$ and $t = g_n(x, c)$. $g_1, \ldots, g_{n-1}$ are time-varying first integrals of the linear system. To obtain time-invariant first integrals we substitute t in $F(t, x, c)$ to obtain $\overline{F}(x, c)$. Using $\overline{F}$ we seek functions $\Psi_i(\cdot) : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, n-1$ such that $\overline{F}_i(x, c) = 0 = \Psi_i(x) - \Psi_i(c)$. $\Psi_i(x)$ are time-invariant first integrals.

3.0.1 Real Eigenvalues

Consider the elementary Jordan block $J^r \in \mathbb{R}^{m\times m}$ given by

$$J^r = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda \end{bmatrix} \tag{3.1}$$

where $\lambda \in \mathbb{R}$. The solution of $\dot{x} = J^r x$ with initial condition $c \in \mathbb{R}^m$ is

$$x(t) = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \ldots & \frac{t^{m-1}}{(m-1)!} \\ & 1 & t & \ldots & \\ & & \ddots & \ddots & \vdots \\ & & & 1 & t \\ & & & & 1 \end{bmatrix} c. \tag{3.2}$$

We obtain $m-1$ first integrals $\Psi_1^r, \ldots, \Psi_m^r$ as follows. From the solution of x_m we find $e^{\lambda t} = \frac{x_m}{c_m}$. The solution of x_{m-1} gives $t = \frac{x_{m-1}}{x_m} - \frac{c_{m-1}}{c_m}$. Substituting t in $e^{\lambda t}$ we obtain the first integral

$$\Psi_{m-1}^r := x_m \exp\left(-\lambda \frac{x_{m-1}}{x_m}\right) = d_{m-1} \tag{3.3}$$

where $d_{m-1} \in \mathbb{R}$. The remaining $m-2$ first integrals are found by substituting $e^{\lambda t}$ and t in the solutions for x_1 through x_{m-2}. Carrying out this operation recursively, we obtain the first integrals

$$\Psi_{m-2}^r := \frac{x_{m-2}}{x_m} - \frac{x_{m-1}^2}{2x_m^2} = d_{m-2} \tag{3.4}$$

$$\Psi_{m-3}^r := \frac{x_{m-3}}{x_m} - \frac{x_{m-2}x_{m-1}}{x_m^2} - \frac{x_{m-1}^3}{3x_m^3} = d_{m-3} \tag{3.5}$$

$$\vdots$$

$$\Psi_{m-k}^r := \frac{x_{m-k}}{x_m} - \sum_{j=1}^{k-2} \frac{1}{j!} \frac{x_{m-1}^j}{x_m^j} \Psi_{m-(k-j)}^r - \frac{1}{k!} \frac{x_{m-1}^k}{x_m^k} = d_{m-k} \tag{3.6}$$

where $d_j \in \mathbb{R}$. We show these are first integrals by an inductive argument. First, $D\Psi_{m-2} \cdot J^r x = 0$. Suppose $D\Psi^r_{m-j} \cdot J^r x = 0$ for $j = 2, \dots, k-1$. Then we obtain

$$D\Psi_{m-k} \cdot J^r x = \frac{x_{m-k+1}}{x_m} - \frac{x_{m-1}^{k-1}}{(k-1)! x_m^{k-1}} - \sum_{j=1}^{k-2} \frac{x_{m-1}^{j-1}}{(j-1)! x_m^{j-1}} \Psi^r_{m-k+j} = 0.$$

3.0.2 Complex Eigenvalues

Consider the elementary Jordan block $J^c \in \mathbb{R}^{m \times m}$ given by

$$J^c = \begin{bmatrix} D & I_2 & & \\ & \ddots & \ddots & \\ & & & I_2 \\ & & & D \end{bmatrix} \tag{3.7}$$

where

$$D = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad ; \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The solution of $\dot{x} = J^c x$ is found by converting to the complex domain. Let $z : \mathbb{R} \to \mathbb{C}^{\frac{m}{2}}$, $i \cdot i = -1$, and consider $\dot{z} = Bz$, where

$$B = \begin{bmatrix} \mu & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \mu \end{bmatrix} \quad ; \qquad \mu = a + ib. \tag{3.8}$$

We identify $\mathbb{C}^{\frac{m}{2}}$ with $\mathbb{R}^m$ by the correspondence $(z_1, \dots, z_{\frac{m}{2}}) = (x_1 + ix_2, \dots, x_{m-1} + ix_m)$. The solution of $\dot{z} = Bz$ is

$$z_k(t) = e^{\mu t} \sum_{j=k}^{\frac{m}{2}} \frac{t^{j-k}}{(j-k)!} c_j.$$

We obtain $m-1$ first integrals $\Psi^c_1, \dots, \Psi^c_m$ as follows. First, from the solutions of x_{m-1} and x_m we derive the useful expressions:

$$e^{at} = \left(\frac{x_{m-1}^2 + x_m^2}{c_{m-1}^2 + c_m^2} \right)^{\frac{1}{2}} \tag{3.9}$$

$$e^{at} \cos bt = \frac{c_{m-1} x_{m-1} + c_m x_m}{c_{m-1}^2 + c_m^2} \tag{3.10}$$

$$e^{at} \sin bt = \frac{c_{m-1} x_m - c_m x_{m-1}}{c_{m-1}^2 + c_m^2}. \tag{3.11}$$

Let

$$X_{k+} = \frac{x_{m-k} x_{m-1} + x_{m-k+1} x_m}{x_{m-1}^2 + x_m^2}$$

$$X_{k-} = \frac{x_{m-k} x_m - x_{m-k+1} x_{m-1}}{x_{m-1}^2 + x_m^2}.$$

Evaluating X_{3+} gives

$$t = \frac{x_{m-3}x_{m-1} + x_{m-2}x_m}{x_{m-1}^2 + x_m^2} - \frac{c_{m-3}c_{m-1} + c_{m-2}c_m}{c_{m-1}^2 + c_m^2}. \tag{3.12}$$

Equipped with (3.9) - (3.12) we can find $m-1$ first integrals. Considering the last two equations of $\dot{x} = J^c x$ and using polar coordinates, we obtain a first integral

$$\Psi_{m-1}^c := \sqrt{x_m^2 + x_{m-1}^2}\exp\left(-aX_{3+}\right) = d_{m-1} \tag{3.13}$$

where $d_{m-1} \in \mathbb{R}$. The remaining $m-2$ first integrals are found by evaluating X_{k+} and X_{k-} for $k = 3, 5, 7, \dots, m-1$ and substituting (3.9) - (3.12) in the solutions for x_m to x_1. Considering the evaluation of X_{k-} we obtain the first integrals

$$\begin{aligned}
\Psi_{m-2}^c &:= X_{3-} = d_{m-2} \\
&\vdots \\
\Psi_{m-k+1}^c &:= X_{k-} - \sum_{j=1}^{\frac{k-3}{2}} \frac{1}{j!} X_{3+}^j \Psi_{m-k+1+2j}^c = d_{m-k+1}.
\end{aligned}$$

Considering the evaluation of X_{k+}, we first obtain the first integral

$$\Psi_{m-3}^c := \frac{x_{m-3}^2 + x_{m-2}^2}{x_{m-1}^2 + x_m^2} - X_{3+}^2 = d_{m-3}.$$

The first integrals for $k = 5, 7, \dots$ are

$$\begin{aligned}
\Psi_{m-5}^c &:= X_{5+} - \frac{1}{2}X_{3+}^2 = d_{m-5} \\
&\vdots \\
\Psi_{m-k}^c &:= X_{k+} - \sum_{j=1}^{\frac{k-5}{2}} \frac{1}{j!} X_{3+}^j \Psi_{m-k+2j}^c - \frac{1}{p!}X_{3+}^p = d_{m-k}
\end{aligned}$$

where $p = \frac{k-1}{2}$. We can verify by a recursive argument as in the real repeated case that these are first integrals.

3.0.3 Coupling integrals

It remains to find the first integrals describing the coupling between elementary Jordan blocks. We consider the pairs (J^r, J^r), (J^r, J^c), and (J^c, J^c).

For the coupling between a J^r and a J^c block, it suffices to find a coupling first integral for the system

$$\dot{x} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix} x. \tag{3.14}$$

Using polar coordinates $x_2 = r\cos\theta$, $x_3 = r\sin\theta$, we have $\dot{r} = ar$, from which it is seen that $x_1^a(x_2^2 + x_3^2)^{-\frac{\lambda}{2}} = d$ where $d \in \mathbb{R}$. For the coupling between two J^r blocks it suffices to find a first integral for the system

$$\dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x \tag{3.15}$$

which corresponds to the last row of each J^r block. We obtain $\lambda_2 x_1 - \lambda_1 x_2 = d$. For the coupling between two J^c blocks it suffices to consider the system

$$\dot{x} = \begin{bmatrix} \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} & \\ & \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} \end{bmatrix} x. \tag{3.16}$$

Converting to polar coordinates, we have $\dot{\theta}_1 = b_1$ and $\dot{\theta}_2 = b_2$, so

$$b_2 \arctan(\frac{x_2}{x_1}) - b_1 \arctan(\frac{x_4}{x_3}) = d.$$

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