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Continuous Interpolation of Solutions of Lipschitz Inclusions

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We show that given any finite set of trajectories of a Lipschitz differential inclusion there exists a continuous selection from the set of its solutions that interpolates the given trajectories. In addition, we present a result on lipschitzian selections.

Key Words: differential inclusions; continuous selections.

1. INTRODUCTION

In this paper we consider differential inclusions of the form

$$\dot{x}(t) \in F(t, x), \qquad x(0) = \xi, \tag{1}$$

where F is a set valued map defined on $[0, T] \times \mathbb{R}^n$ and taking values on closed, nonempty subsets of $\mathbb{R}^n$ and is lipschitzian with respect to x. There is a fair amount of results on the existence of a continuous selection of solutions of (1) (see [2, 4, 5, 7]). In the present paper we investigate a related

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problem: Given a finite set of trajectories of (1) starting from distinct initial points, does there exist a continuous selection from the set of solutions of (1) that interpolates these trajectories? We use the methodology in [7] to show that a continuous interpolating selection exists. In particular, any two solutions of (1) are homotopic along any continuous path that connects their initial points. This problem is motivated by current research in hybrid automata [3].

A second result concerns lipschitzian selections. It is well known that if F has compact, convex values, a lipschitzian vector-field f can be obtained which is a selection from F. As a result, the solutions to f are a selection from the set of trajectories of F that is Lipschitz continuous with respect to the initial condition. However, if a certain trajectory of (1) is specified it is not clear that there exists a selection from the set of trajectories of (1) which is Lipschitz continuous with respect to the initial data and also agrees with the given trajectory. We show that if the map F is convex valued and γ is a one dimensional submanifold then there exists a selection of solutions of (1) that agrees with any given trajectory and is Lipschitz continuous on γ.

2. PRELIMINARIES

We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$ and by $d(x, B)$ the distance from a point $x \in \mathbb{R}^n$ to a set $B \subset \mathbb{R}^n$. Also, $B(x, r)$ denotes the open ball of radius r around a point $x \in \mathbb{R}^n$. The Hausdorff distance between two sets $A, B \subset \mathbb{R}^n$ is denoted by $d_H(A, B)$. For an interval $I = [0, T]$ let $\mathscr{C}(I)$ and $\mathscr{C}_{ac}(I)$ denote the spaces of continuous and absolutely continuous functions $f : I \to \mathbb{R}^n$, endowed with the sup norm $\|f\|_\infty$ and the norm $\|f\|_{ac} = |f(0)| + \int_0^T |\dot{f}(s)|ds$, respectively. We denote by $\mathscr{L}^1(I)$ the Lebesgue integrable functions on I. Finally, χ_E stands for the characteristic function of a set E.

The basic assumptions concerning the map F in (1) are as follows:

Assumption A. The set-valued map $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ satisfies:

(a) The values of F are closed, nonempty subsets of $\mathbb{R}^n$.

(b) For each $x \in \mathbb{R}^n$, $t \mapsto F(t, x)$ is measurable.

(c) There exists $k \in \mathscr{L}^1(I)$ such that, for all $x, x' \in \mathbb{R}^n$,

$$d_H(F(t, x), F(t, x')) \leq k(t)|x - x'|, \qquad \text{a.e. on } I.$$

3. MAIN RESULTS

THEOREM 3.1. *Let F satisfy Assumption A and $X \subset \mathbb{R}^n$ be a compact set with diameter D. Suppose that a set of distinct initial conditions $\{\xi_i\}_{i=1}^N \subset X$*

and corresponding solutions $\{y_i(t)\}_{i=1}^N$ *of* (1) *are given on a time interval* $[0, T]$. *Then there exists a continuous selection of solutions* $\varphi_t(\xi)$ *of* (1), *defined on* $t \in [0, T]$, $\xi \in X$, *such that* $\varphi_t(\xi_i) = y_i(t)$, *for* $i = 1, \dots, N$.

Proof. For $\xi \in X$ define

$$\delta(\xi) = \begin{cases} \frac{1}{2} \min_{1 \le j \le N} |\xi - \xi_j|, & \xi \ne \xi_i, i = 1, \dots, N \\ \frac{1}{2} \min_{i,j} |\xi_i - \xi_j|, & \text{otherwise.} \end{cases}$$

Let $\{B(\eta_j, \delta(\eta_j))\}_{j=1}^{q_0}$ be a finite subcover of the cover $\{B(\xi, \delta(\xi))\}_{\xi \in X}$ of X, and let $\{\psi_j\}_{j=1}^{q_0}$ be a partition of unity subordinate to it. Note that each ξ_i belongs to exactly one member of this subcover. For each $\xi \in X$, let

$$I_j(\xi) = \left[T \sum_{i=1}^{j-1} \psi_i(\xi), T \sum_{i=1}^{j} \psi_i(\xi) \right], \qquad 1 \le j \le q_0.$$

We define a partition $\{J_1, \dots, J_N\}$ of $\{1, \dots, q_0\}$, by

$$J_k = \{j : |\eta_j - \xi_k| < |\eta_j - \xi_\ell|, \ell > k\} \bigcap \{j : |\eta_j - \xi_k| \le |\eta_j - \xi_\ell|, \ell \le k\},$$

for $1 \le k < N$, and

$$J_N = \{1, \dots, q_0\} \setminus \bigcup_{k=1}^{N-1} J_k.$$

Let

$$\alpha_k(\xi, t) = \sum_{j \in J_k} \chi_{I_j(\xi)}(t), \qquad k = 1, \dots, N.$$

It follows from the previous definitions that $J_k \ne \emptyset$ and $\alpha_k(\xi_k, t) = 1$, for all $k \in \{J_1, \dots, J_N\}$. In addition,

$$\sum_{k=1}^{N} \alpha_k(\xi, t) = 1, \qquad \forall \xi \in X, \quad \forall t \in [0, T].$$

Consider the family of functions $\{\varphi_t^0(\xi)\} \subset \mathscr{C}_{ac}([0, T])$ given by

$$\varphi_t^0(\xi) = \xi + \int_0^t \sum_{1 \le k \le N} \alpha_k(\xi, t) \dot{y}_k(t). \tag{2}$$

Note that $\varphi_t^0(\xi_k) = y_k(t)$. We obtain the estimate

$$\begin{aligned} d\big(\dot{\varphi}_t^0(\xi), F\big(t, \varphi_t^0(\xi)\big)\big) &\le \max_{1 \le k \le N} d_H(F(t, y_k(t)), F(t, \varphi_t^0(\xi))) \\ &\le k(t) \max_{1 \le k \le N} \Bigg[|\xi_k - \xi| + \int_0^t |\dot{y}_k(s) \\ &\qquad - \sum_{1 \le \ell \le N} \alpha_\ell(\xi, s) \dot{y}_\ell(s) \mid ds \Bigg] \\ &\le L k(t), \end{aligned} \tag{3}$$

where

$$L = D + \min_{i,j} \int_0^T |\dot{y}_i(s) - \dot{y}_j(s)| ds.$$

Note that, for $\xi, \xi' \in X$,

$$\left|\varphi_t^0(\xi) - \varphi_t^0(\xi')\right| \le L. \tag{4}$$

Choose $v_t^0(\xi)$ to be a measurable selection from $F(t, \varphi_t^0(\xi))$ such that

$$\left|\dot{\varphi}_t^0(\xi) - v_t^0(\xi)\right| = d\big(\dot{\varphi}_t^0(\xi), F\big(t, \varphi_t^0(\xi)\big)\big) \tag{5}$$

and set

$$\varphi_t^1(\xi) = \xi + \int_0^t \sum_{j=0}^{q_0} \chi_{I_j(\xi)}(s) v_s^0(\eta_j)\, ds. \tag{6}$$

It follows from (2), (3), and (6) that

$$\begin{aligned}
\left|\dot{\varphi}_t^1(\xi) - \dot{\varphi}_t^0(\xi)\right| &= \left|\sum_{j=0}^{q_0} \chi_{I_j(\xi)}(s) v_t^0(\eta_j) - \sum_{1\le k\le N} \alpha_k(\xi, t)\dot{y}_k(t)\right| \\
&\le \max_{j,k} d_H\big(F(t, \varphi_t^0(\eta_j)), F\big(t, y_k(t)\big)\big) \\
&= \max_{j,k} d_H\big(F\big(t, \varphi_t^0(\eta_j)\big), F\big(t, \varphi_t^0(\xi_k)\big)\big) \\
&\le Lk(t).
\end{aligned} \tag{7}$$

By (4),

$$\begin{aligned}
d\big(\dot{\varphi}_t^1(\xi), F\big(t, \varphi_t^0(\xi)\big)\big) &\le \max_{1\le j\le q_0} d\big(v_t^0(\eta_j), F\big(t, \varphi_t^0(\xi)\big)\big) \\
&\le \max_{1\le j\le q_0} d_H\big(F\big(t, \varphi_t^0(\eta_j)\big), F\big(t, \varphi_t^0(\xi)\big)\big) \\
&\le Lk(t).
\end{aligned} \tag{8}$$

Therefore, by (6) and (7),

$$\begin{aligned}
d\big(\dot{\varphi}_t^1(\xi), F\big(t, \varphi_t^1(\xi)\big)\big) &\le d\big(\dot{\varphi}_t^1(\xi), F\big(t, \varphi_t^0(\xi)\big)\big) \\
&\quad + d_H\big(F\big(t, \varphi_t^0(\xi)\big), F\big(t, \varphi_t^1(\xi)\big)\big) \\
&\le Lk(t) + Lk(t)g(t),
\end{aligned}$$

where

$$g(t) = \int_0^t k(s)\, ds.$$

Using the method in [7], we can construct a sequence $\varphi^n : X \mapsto \mathscr{C}_{ac}([0, T])$ of continuous maps, satisfying $\varphi_0^n(\xi) = \xi$, $\varphi_t^n(\xi_i) = y_i(t)$, for all $i = 1, \dots, N$, and

$$\int_0^t |\dot{\varphi}_s^n(\xi) - \dot{\varphi}_s^{n-1}(\xi)|\, ds \le L\left[\frac{g^n(t)}{n!} + \frac{4}{2^n}\sum_{i=1}^{n}\frac{(2g(t))^i}{i!} + \frac{1}{2^n}\right], \tag{9}$$

$$d\big(\dot{\varphi}_t^n(\xi), F\big(\varphi_t^n(\xi)\big)\big) \le Lk(t)\left[\frac{g^n(t)}{n!} + \frac{4}{2^n}\sum_{i=0}^{n}\frac{\big(2g(t)\big)^i}{i!}\right]. \tag{10}$$

The construction is accomplished by induction, as follows: For $n \ge 2$, using the Proposition in [7], select $\delta_n > 0$ such that $|\xi - \xi'| < \delta_n$ implies

$$\int_0^T |\dot{\phi}_t^{n-1}(\xi) - \dot{\phi}_t^{n-1}(\xi')|\, dt \le \frac{L}{2^n}.$$

Define

$$\delta_n(\xi) = \begin{cases} \min\left\{\frac{L}{2^n}, \delta_n, \frac{1}{2}\min_{1\le j\le N}|\xi - \xi_j|\right\}, & \xi \ne \xi_i, i = 1, \dots, N \\ \min\left\{\frac{L}{2^n}, \delta_n, \frac{1}{2}\min_{i,j}|\xi_i - \xi_j|\right\}, & \text{otherwise.} \end{cases}$$

Let $\{B(\eta_j^n, \delta_n(\eta_j^n))\}_{j=1}^{q_n}$ be a finite subcover of the cover $\{B(\xi, \delta_n(\xi))\}_{\xi\in X}$ of X, and let $\{\psi_j^n\}_{j=1}^{q_n}$ be a partition of unity subordinate to it. Note that each ξ_i belongs to exactly one member of this subcover. For each $\xi \in X$, let

$$I_j^n(\xi) = \left[T\sum_{i=1}^{j-1}\psi_i^n(\xi), T\sum_{i=1}^{j}\psi_i^n(\xi)\right], \qquad 1 \le j \le q_n.$$

In analogy to (5)–(6), choose $v_t^{n-1}(\xi)$ to be a measurable selection from $F(t, \varphi_t^{n-1}(\xi))$ such that

$$|\dot{\varphi}_t^{n-1}(\xi) - v_t^{n-1}(\xi)| = d\big(\dot{\varphi}_t^{n-1}(\xi), F\big(t, \varphi_t^{n-1}(\xi)\big)\big)$$

and set

$$\varphi_t^n(\xi) = \xi + \int_0^t \sum_{j=0}^{q_n}\chi_{I_j^n(\xi)}(s)v_s^{n-1}(\eta_j^n)\, ds.$$

The estimates in (9)–(10) can be easily proved by induction (see [7]).

It follows by (9) that the sequence $\{\varphi^n(\xi)\}$ is uniformly Cauchy in $\mathscr{C}_{ac}([0, T])$ and thus it converges uniformly to a continuous map $\varphi : X \mapsto \mathscr{C}_{ac}([0, T])$. In turn, (10) implies that $\varphi_t(\xi)$ is a solution of (1). ∎

Let $\xi_0 \in X$ and $x(\cdot)$ be a solution of (1) such that $x(0) = \xi_0$. As shown in [7], there exists a sequence of approximate trajectories, $\{\varphi_t^m(\xi)\}_{m=0}^{\infty}$ which forms a Cauchy sequence in the normed space $\mathscr{C}_{ac}([0, T])$ and converges to the continuous selection $\varphi_t(\xi)$. In particular, this sequence can be chosen to satisfy

$$\|\varphi^m(\xi) - \varphi^{m-1}(\xi)\|_{ac} \leq D\left(\frac{g^m(T)}{m!} + \frac{e^{2g(T)}}{2^{m+1}}\right),$$

where D is the diameter of the compact set X. Thus,

$$\|\varphi(\xi) - \varphi^0(\xi)\|_{ac} \leq D\left(e^{g(T)} + e^{2g(T)}\right),$$

where

$$\varphi_t^0(\xi) = \xi + \int_0^t \dot{\varphi}_s(\xi_0)ds.$$

Hence, we have the estimate

$$\|\varphi(\xi) - \varphi(\xi_0)\|_{ac} \leq D\left(e^{g(T)} + e^{2g(T)} + 1\right) \leq 3De^{2g(T)}. \tag{11}$$

We conclude with a result providing a lipschitzian selection along a path in $\mathbb{R}^n$ of solutions of a Lipschitz inclusion having compact, convex values.

THEOREM 3.2. *Suppose that F satisfies Assumption A and, in addition, suppose that it has compact, convex values, and it is upper semicontinuous in (t, x). Let $\gamma : [-1, 1] \mapsto \mathbb{R}^n$ be a continuous, injective path parameterized by its arc-length; i.e., $|\theta' - \theta|$ is the arc-length of the segment from $\gamma(\theta)$ to $\gamma(\theta')$. Let $\xi_0 := \gamma(0)$, and $x(t)$, $t \in [0, T]$, be a solution of the inclusion $\dot{x} = F(t, x)$, with $x(0) = \xi_0$. Then there is a selection $\varphi_t(\xi)$, $\xi \in \gamma([-1, 1])$, of solutions of the inclusion such that $\varphi_t(\xi_0) = x(t)$ and*

$$\|\varphi(\gamma(\theta)) - \varphi(\gamma(\theta'))\|_\infty \leq 3|\theta - \theta'|e^{2g(T)}, \qquad \text{for all } \theta, \theta' \in [-1, 1].$$

Proof. Let J_n be the set of the binary rationals of the form $k/2^n$, where $-2^n \leq k \leq 2^n$, and set $\xi_k^n := \gamma(k/2^n)$. For each $n \in \mathbb{N}$ we construct a continuous selection φ^n of solutions of the inclusion as follows. First, obtain a continuous selection interpolating $x(t)$ on a ball of radius $1/2^n$, centered at ξ_0, and denote its restriction on $\gamma([-1/2^n, 1/2^n])$ by $\varphi^{n,0}$. Proceed iteratively to define, in the same manner, for each $k = 1, \ldots, 2^n$, a selection $\varphi^{n,k}$ interpolating $\varphi_t^{n,k-1}(\xi_k^n)$ on $[\xi_k^n, \xi_{k+1}^n]$. Repeat the analogous procedure for negative k. Let φ^n be the selection which agrees with each $\varphi^{n,k}$ on its domain of definition. If $\xi \in \gamma([k/2^n, (k+1)/2^n])$, for some $k \geq 0$, noting that ξ_k^n is centered in the ball on which $\varphi^{n,k}$ is defined, the estimate (10) yields

$$\begin{aligned}\|\varphi^n(\xi) - \varphi^n(\xi_k^n)\|_\infty &= \|\varphi^{n,k}(\xi) - \varphi^{n,k}(\xi_k^n)\|_\infty \\ &\leq 3|\xi - \xi_k^n|e^{2g(T)} \leq \frac{3}{2^n}e^{2g(T)},\end{aligned} \tag{12}$$

and

$$\|\varphi^n(\xi) - \varphi^n(\xi^n_{k+1})\|_\infty \le \|\varphi^n(\xi) - \varphi^n(\xi^n_k)\|_\infty + \|\varphi^n(\xi^n_k) - \varphi^n(\xi^n_{k+1})\|_\infty$$
$$\le \frac{6}{2^n} e^{2g(T)}. \tag{13}$$

For $k < 0$, the estimates in (11)–(12) are interchanged. Suppose $\xi = \gamma(\theta)$, $\xi' = \gamma(\theta')$, and $k < k'$ are such that $k/2^n \le \theta \le (k+1)/2^n$ and $(k'-1)/2^n \le \theta' \le k'/2^n$. Using a triangle inequality, we obtain from (11)–(12),

$$\|\varphi^n(\xi) - \varphi^n(\xi')\|_\infty \le (k'-k+1)\frac{3}{2^n} e^{2g(T)} \le 3|\theta' - \theta| e^{2g(T)} + \frac{9}{2^n} e^{2g(T)}. \tag{14}$$

Since F is compact valued and upper semicontinuous, the family of solutions $\{\varphi^n(\xi)\}_{n,\xi}$ is equicontinuous [6]. Therefore, for each ξ, there exists a subsequence which converges uniformly. Using Cantor's diagonal principle we can extract a subsequence, also denoted by $\{\varphi^n\}$, which converges at every point of the countable set $J = \bigcup\{\xi^k_m : k, m \in \mathbb{N}\}$. Let $\xi = \gamma(\theta)$ be arbitrary. Given $\varepsilon > 0$, using (13), select a dyadic rational $\tilde{\theta}$ and $N_0 > 0$ large enough such that, with $\tilde{\xi} := \gamma(\tilde{\theta})$, on the one hand

$$\|\varphi^n(\xi) - \varphi^n(\tilde{\xi})\|_\infty \le \frac{\varepsilon}{4}, \qquad \text{for all } n \ge N_0, \tag{15}$$

while at the same time, since $\{\varphi^n(\tilde{\xi})\}$ is Cauchy,

$$\|\varphi^n(\tilde{\xi}) - \varphi^m(\tilde{\xi})\|_\infty \le \frac{\varepsilon}{2}, \qquad \text{for all } n, m \ge N_0. \tag{16}$$

Hence, by (15)–(16),

$$\|\varphi^n(\xi) - \varphi^m(\xi)\|_\infty \le \|\varphi^n(\xi) - \varphi^n(\tilde{\xi})\|_\infty + \|\varphi^n(\tilde{\xi}) - \varphi^m(\tilde{\xi})\|_\infty$$
$$+ \|\varphi^m(\xi) - \varphi^m(\tilde{\xi})\|_\infty$$
$$\le \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon, \qquad \text{for all } n, m \ge N_0,$$

showing that $\{\varphi^n(\xi)\}$ is Cauchy for all ξ. Let φ be the limit of $\{\varphi^n\}$. Since F is convex and continuous, φ, being the uniform limit of solutions, is itself a solution of the inclusion. Taking limits as $n \to \infty$ in (13), we obtain the desired result. ∎

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