EFFICIENT SOLUTION OF OPTIMAL CONTROL PROBLEMS USING HYBRID SYSTEMS

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Abstract. We consider the synthesis of optimal controls for continuous feedback systems by recasting the problem to a hybrid optimal control problem: *synthesize optimal enabling conditions for switching between locations in which the control is constant.* An algorithmic solution is obtained by translating the hybrid automaton to a finite automaton using a bisimulation and formulating a dynamic programming problem with extra conditions to ensure non-Zenoness of trajectories. We show that the discrete value function converges to the viscosity solution of the Hamilton-Jacobi-Bellman equation as a discretization parameter tends to zero.

Key words. optimal control, hybrid systems, bisimulation

1. Introduction. The goal of this paper is the development of a computationally appealing technique for synthesizing optimal controls for continuous feedback systems $\dot{x} = f(x, u)$, by recasting the problem as a hybrid optimal control problem. The hybrid problem is obtained by approximating the control set $U \subset \mathbb{R}^m$ by a finite set $\Sigma \subset U$ and defining vector fields for the locations of the hybrid system of the form $f(x, \sigma)$, $\sigma \in \Sigma$; that is, the control is constant in each location. The hybrid control problem is to synthesize enabling conditions such that a target set $\Omega_f \subset \Omega$ is reached while a hybrid cost function is minimized, for each initial condition in a specified set $\Omega \subset \mathbb{R}^n$.

Casting the problem as a hybrid control problem is not necessarily a simplification because, while algorithmic approaches for solving the controller synthesis problem for specific classes of hybrid systems have appeared [33, 52], no general, efficient algorithm is available. To be able to solve the (nonlinear) hybrid optimal control problem, we must exploit some additional property. We have a feasible and appealing approach if we can translate the problem to an equivalent discrete problem, which abstracts completely the continuous behavior. This translation is possible if we can construct a finite *bisimulation* defined on the hybrid state space; that is, an equivalence relation that induces a partition in each hybrid automaton location that is consistent with the continuous dynamics of that location. A finite bisimulation can be constructed using the geometric approach reported in [10], based on the following key assumption: $n-1$ *local (on Ω) first integrals can be expressed analytically for each vector field $f(x, \sigma)$, $\sigma \in \Sigma$.* This assumption is imposed in the transient phase of a feedback system's response, when the vector field is non-vanishing and local first integrals always exist, though finding closed form expressions for them is not always easy or possible. Also, the assumption that the partition be a bisimulation is sufficient but not necessary for the overall approach.

If the assumption is met, then we can transform the hybrid system to a quotient system associated with the finite bisimulation, which is a finite automaton. The

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control problem posed on the finite automaton is to synthesize a discrete supervisor, providing a switching rule between automaton locations, that minimizes a discrete cost function approximating the original cost function, for each initial discrete state. We provide a dynamic programming solution to this problem, with extra constraints to ensure non-Zenoness of the closed-loop trajectories. By imposing non-Zeno conditions on the synthesis we obtain piecewise constant controls with a finite number of discontinuities in bounded time.

The discrete value function depends on the discretizations of U and of Ω using the bisimulation. We quantify these discretizations by parameters δ and δ_Q. The main theoretical contribution is to show that as $\delta, \delta_Q \to 0$, the discrete value function converges to the unique viscosity solution of the Hamilton-Jacobi-Bellman (HJB) Equation.

There is a similarity between our approach to optimal control and *regular synthesis*, introduced in [8], in the sense that both restrict the class of controls to a set that has some desired property and both use a finite partition to define switching behavior. For linear systems, the results on regular synthesis are centered on the Bang-Bang principle [38], stating that a sufficient class of optimal controls is piecewise constant. If U is a convex polyhedron, then the number of discontinuities of the control is bounded. There is no hope that general Bang-Bang results are available due to Fuller's example [20, 28]. Nevertheless, in many applications the optimal control is a piecewise continuous function, and methods of regular synthesis of such controls are worth investigating. Our paper focuses on piecewise constant controls and provides a constructive approach to obtain a cell decomposition, in the spirit of regular synthesis, by using a finite bisimulation, which further allows us to formulate the synthesis problem on its quotient system - a finite automaton.

The idea of using a time abstract model formed by partitioning the continuous state space has been pursued in a number of papers recently. Lemmon, Antsaklis, Stiver and coworkers [48], [53] use a partition of the state space to convert a hybrid model to a discrete event system (DES). This enables them to apply controller synthesis for DES's to synthesize a supervisor. While our approach is related to this methodology, it differs in that we provide conditions for obtaining the partition. In [41] hybrid systems consisting of a linear time-invariant system and a discrete controller that has access to a quantized version of the linear system's output is considered. The quantization results in a rectangular partition of the state space. This approach suffers from spurious solutions that must be trimmed from the automaton behavior. Hybrid optimal control problems have been studied in papers by Witsenhausen [51], Branicky et.al. [9], and Bensoussan and Menaldi [6]. The first two concentrate on problems of well-posedness, necessary conditions, and existence of optimal solutions but do not provide algorithmic solutions. Bensoussan and Menaldi consider a more general model than ours that includes continuous dynamics with a measurable control input and a discrete part with impulsive control. Control switches can be autonomous or controlled and may have time delays. They characterize the viscosity solution of a dynamic programming problem on their model. They construct open-loop controls whereas we obtain feedback controls, and they do not consider the numerical implementation.

There has been recently significant progress in developing numerical methods that incorporate geometric invariants of a dynamical or control system. In particular, in the area of geometric mechanics, numerical integrators have been developed that preserve the Hamiltonian, Lie group symmetries, and other integrals of motion [18, 26, 24, 27, 40]. See the reference [36] for an overview of problems where geometric structure is exploited in numerical methods. Our work represents the first general methodology in which geometric invariants are explicitly considered in the numerical solution of optimal control problems. The geometric structure that is present in the optimal control problem is encoded in the bisimulation partition. In effect, the two step procedure of a time discretization followed by the state discretization via finite element methods that together lead to a fixed point formulation of the approximate solution of a continuous time optimal control problem is circumvented. Instead an exact representation of the time evolution of the system is encoded in the finite element partition, enabling a simplified and more efficient formulation.

The paper is organized as follows. In section 2 we state the optimal control problem, while in section 3 the associated hybrid system is given. In Section 4 we review how the bisimulation is constructed. Section 5 formulates the proposed solution using bisimulation and dynamic programming. In section 6 we prove the main theoretical result. In section 7 we present an algorithmic solution of the dynamic programming problem including a formal justification of the algorithm's optimality. In Section 8 we give two simple examples. Section 9 summarizes our findings.

2. Optimal control problem. **Notation.** $\mathbf{1}(\cdot)$ is the indicator function. $cl(A)$ denotes the closure of set A. $\|\cdot\|$ denotes the Euclidean norm. Let $C^1(\mathbb{R}^n)$ and $\mathcal{X}(\mathbb{R}^n)$ denote the sets of continuously differentiable real-valued functions and smooth vector fields on $\mathbb{R}^n$, respectively. $\phi_t(x_0, \mu)$ denotes the trajectory of $\dot{x} = f(x, \mu)$ starting from x_0 and using control $\mu(\cdot)$.

Let U be a compact subset of $\mathbb{R}^m$, Ω an open, bounded, connected subset of $\mathbb{R}^n$, and Ω_f a compact subset of Ω. Define $\mathcal{U}_m$ to be the set of measurable functions mapping $\mathbb{R}^+$ to U. We define the minimum hitting time $T : \mathbb{R}^n \times \mathcal{U}_m \to \mathbb{R}^+$ by

$$T(x,\mu) := \begin{cases} \infty & \text{if } \{t \mid \phi_t(x,\mu) \in \Omega_f \} = \emptyset \\ \min\{t \mid \phi_t(x,\mu) \in \Omega_f\} & \text{otherwise.} \end{cases} \tag{2.1}$$

A control $\mu \in \mathcal{U}_m$ specified on $[0, T]$ is *admissible* for $x \in \Omega$ if $\phi_t(x, \mu) \in \Omega$ for all $t \in [0, T]$. The set of admissible controls for x is denoted $\mathcal{U}_x$. Let

$$\mathcal{R} := \{ \ x \in \Omega \mid \exists \mu \in \mathcal{U}_x .\ T(x,\mu) < \infty \ \}.$$

We consider the following stationary optimal control problem. Given $y \in \Omega$,

$$\text{minimize} \qquad J(y,\mu) = \int_0^{T(y,\mu)} L(x(t),\mu(t))dt + h(x(T(y,\mu))) \tag{2.2}$$

$$\text{subject to} \qquad \dot{x} = f(x,\mu), \qquad a.e.\ t \in [0, T(y,\mu)] \tag{2.3}$$

$$x(0) = y \tag{2.4}$$

among all admissible controls $\mu \in \mathcal{U}_y$. $J : \mathbb{R}^n \times \mathcal{U}_m \to \mathbb{R}$ is the *cost-to-go* function, $h : \mathbb{R}^n \to \mathbb{R}$ is the *terminal cost*, and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the *instantaneous cost*. At $T(y, \mu)$ the terminal cost $h(x(T(y, \mu)))$ is incurred and the dynamics are stopped. The control objective is to reach Ω_f from $y \in \Omega$ with minimum cost.

Assumption 2.1.
(1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies $\|f(x', u') - f(x, u)\| \le L_f \left[\|x' - x\| + \|u' - u\|\right]$ for some $L_f > 0$. Let M_f be the upper bound of $\|f(x, u)\|$ on $\Omega \times U$.
(2) $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies $|L(x', u') - L(x, u)| \le L_L \left[\|x' - x\| + \|u' - u\|\right]$ and $1 \le L(x, u) \le M_L$, $x \in \Omega$, $u \in U$, for some $L_L, M_L > 0$.
(3) $h : \mathbb{R}^n \to \mathbb{R}$ satisfies $|h(x') - h(x)| \le L_h\|x' - x\|$ for some $L_h > 0$, and $h(x) \ge 0$ for all $x \in \Omega$. Let M_h be the upper bound of $|h(x)|$ on Ω.

Remark 2.1. These assumptions ensure existence of solutions to (2.3) and uniqueness of the trajectories $\phi_t(x,\mu)$. Weaker assumptions are possible; see [4].

The *value function* or optimal cost-to-go function $V : \mathbb{R}^n \to \mathbb{R}$ is given by

$$V(y) = \inf_{\mu \in \mathcal{U}_y} J(y,\mu)$$

for $y \in \Omega \setminus \Omega_f$, and by $V(y) = h(y)$ for $y \in \Omega_f$. A control μ is called ϵ-*optimal* for x if $J(x,\mu) \le V(x) + \epsilon$.

It is well-known [19] that V satisfies the *Hamilton-Jacobi-Bellman* (HJB) equation

$$-\inf_{u \in U} \left\{ L(x,u) + \frac{\partial V}{\partial x} f(x,u) \right\} = 0 \tag{2.5}$$

at each point of $\mathcal{R}$ at which it is differentiable. The HJB equation is an infinitesimal version of the equivalent *Dynamic Programming Principle* (DPP) which says that

$$\begin{array}{rcll} V(x) & = & \inf_{\mu \in \mathcal{U}_x} \left\{ \int_0^t L(\phi_s(x,\mu),\mu(s))ds \ + \ V(\phi_t(x,\mu)) \right\}, & x \in \Omega \setminus \Omega_f \\ V(x) & = & h(x) & x \in \Omega_f. \end{array}$$

The subject of assiduous effort has been that the HJB equation may not have a C^1 solution. This gap in the theory was closed by the introduction of the concept of viscosity solution [31, 13], which can be shown to provide the unique solution of (2.5) without any differentiability assumption. In particular, a bounded uniformly continuous function V is called a *viscosity solution* of HJB provided, for each $\psi \in C^1(\mathbb{R}^n)$, the following hold:

(i) if $V - \psi$ attains a local maximum at $x_0 \in \mathbb{R}^n$, then

$$-\inf_{u \in U} \left\{ L(x_0,u) + \frac{\partial \psi}{\partial x}(x_0) f(x_0,u) \right\} \le 0,$$

(ii) if $V - \psi$ attains a local minimum at $x_1 \in \mathbb{R}^n$, then

$$-\inf_{u \in U} \left\{ L(x_1,u) + \frac{\partial \psi}{\partial x}(x_1) f(x_1,u) \right\} \ge 0.$$

Assumption 2.2. For every $\epsilon > 0$ and $x \in \mathcal{R}$, there exists $N_\epsilon > 0$ and an admissible piecewise constant ϵ-optimal control μ having at most N_ϵ discontinuities and such that $\phi_t(x,\mu)$ is transverse to $\partial\Omega_f$.

The transversality assumption implies that the viscosity solution is continuous at the boundary of the target set, a result needed in proving uniform continuity of V over a finite horizon. The assumption can be replaced by a small-time controllability condition. For a treatment of small time controllability and compatibility of the terminal cost with respect to continuity of the value function, see [4]. The finite switching assumption holds under mild assumptions such as Lipschitz continuity of the vector field and cost functions, and is based on approximating measurable functions by piecewise constant functions.

3. Hybrid system.

The approach we propose for solving the continuous optimal control problem first requires a mapping to a hybrid system and, second, employs a bisimulation of the hybrid system to formulate a dynamic programming problem on

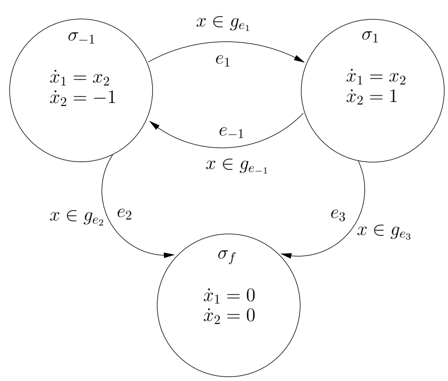


FIG. 3.1. *Hybrid automaton for time optimal control of a double integrator system*

the quotient system. In this section we define the hybrid system. First, we discretize U by defining a finite set $\Sigma_\delta \subset U$ which has a mesh size

$$\delta := \sup_{u \in U} \min_{\sigma \in \Sigma_\delta} \|u - \sigma\|.$$

We define the hybrid automaton $H := (\Sigma \times \mathbb{R}^n, \Sigma_\delta, D, E_h, G)$ with the following components:

State set $\Sigma \times \mathbb{R}^n$ consists of the finite set $\Sigma = \Sigma_\delta \cup \{\sigma_f\}$ of control locations and n continuous variables $x \in \mathbb{R}^n$. σ_f is a terminal location when the optimal control problem is "stopped" and the target set is reached. The controller for σ_f may, for instance, be a linear feedback designed using the linearization of the system.

Events Σ_δ is a finite set of control events.

Vector fields $D : \Sigma \to \mathcal{X}(\mathbb{R}^n)$ is a function assigning an autonomous vector field to each location. We use the notation $D(\sigma) = f_\sigma$.

Control switches $E_h \subset \Sigma \times \Sigma$ is a set of control switches. $e = (\sigma, \sigma')$ is a directed edge between a source location σ and a target location σ'. If $E_h(\sigma)$ denotes the set of edges that can be enabled at $\sigma \in \Sigma$, then $E_h(\sigma) := \{(\sigma, \sigma') \mid \sigma' \in \Sigma \setminus \sigma\}$ for $\sigma \in \Sigma_\delta$ and $E_h(\sigma_f) = \emptyset$. Thus, from a source location not equal to σ_f, there is an edge to every other location (but not itself), while location σ_f has no outgoing edges.

Enabling conditions $G : E_h \to \{g_e\}_{e \in E_h}$ is a function assigning to each edge an enabling (or guard) condition $g \subset \mathbb{R}^n$. We use the notation $G(e) = g_e$. The optimal enabling conditions are unknown and must be synthesized.

3.1. Semantics. A state is a pair (σ, x), $\sigma \in \Sigma$ and $x \in \mathbb{R}^n$. In location $\sigma \in \Sigma_\delta$ the continuous state evolves according to the vector field $f(x, \sigma)$. In location σ_f, the vector field is $\dot{x} = f(x, \mu_f)$ where μ_f is the (not necessarily constant) control of the terminal location. Trajectories of H evolve in *steps* of two types. A σ-*step* is a binary relation $\xrightarrow{\sigma} \subset (\Sigma \times \mathbb{R}^n) \times (\Sigma \times \mathbb{R}^n)$, and we write $(\sigma, x) \xrightarrow{\sigma'} (\sigma', x')$ iff (1) $e = (\sigma, \sigma') \in E_h$, (2) $x \in g_e$, and (3) $x = x'$. A t-*step* is a binary relation $\xrightarrow{t} \subset (\Sigma \times \mathbb{R}^n) \times (\Sigma \times \mathbb{R}^n)$, and we write $(\sigma, x) \xrightarrow{t} (\sigma', x')$ iff (1) $\sigma = \sigma'$, and (2) for some $t \geq 0$, $x' = \phi_t(x, \sigma)$, where $\dot{\phi}_t(x) = f(\phi_t(x, \sigma), \sigma)$. Enabling conditions are *forced* in that an edge is taken instantaneously and as soon as it is enabled.

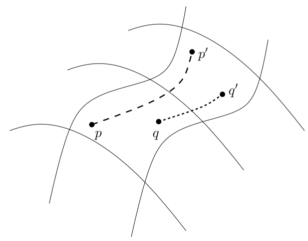


FIG. 4.1. *Illustration of the definition of bisimulation.*

Example 3.1 Consider the time optimal control problem for the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u. \tag{3.1}$$

Suppose $\Omega = (-1,1) \times (-1,1)$ and $\Omega_f = \overline{B}_\epsilon(0)$, the closed epsilon ball centered at 0. The cost-to-go function is $J(x,\mu) = \int_0^{T(x,\mu)} dt$ and $U = \{u \;:\; |u| \leq 1\}$. We select $\Sigma_\delta = \{-1, 1\}$, so that $\delta = 1$. The hybrid system is show in Figure 3.1. The state set is $\{\sigma_{-1} = -1, \sigma_1 = 1, \sigma_f\} \times \mathbb{R}^2$. $g_{e_{-1}}$ and g_{e_1} are unknown and must be synthesized, while $g_{e_2} = g_{e_3} = \Omega_f$.

4. Bisimulation. Let λ represent an arbitrary time interval. A *bisimulation* of H is an equivalence relation $\simeq \subset (\Sigma_\delta \times \mathbb{R}^n) \times (\Sigma_\delta \times \mathbb{R}^n)$ such that for all states $p_1, p_2 \in \Sigma_\delta \times \mathbb{R}^n$, if $p_1 \simeq p_2$ and $\beta \in \Sigma_\delta \cup \{\lambda\}$, then if $p_1 \xrightarrow{\beta} p_1'$, there exists p_2' such that $p_2 \xrightarrow{\beta} p_2'$ and $p_1' \simeq p_2'$. See Figure 4.1. Intuitively, a bisimulation is an equivalence relation defining a partition on the hybrid state space that preserves reachability over σ-steps and time steps. However, the definition leaves ambiguity as to how the partition should be obtained. Alur and Dill [1] gave a construction for timed automata that was based on the first integrals of the continuous dynamics and on the syntax of the enabling and reset conditions. Their approach was first generalized in [10]. Time evolution of the original system is modelled as untimed transitions from equivalence class to equivalence class in the quotient system associated with the bisimulation. Transitions between locations of the hybrid automaton appear also as transitions in the quotient system. Thus if there are a finite number of equivalence classes of $\simeq$, then a finite transition system or finite automaton is obtained which gives a time abstract model of the original system, with reachability properties exactly preserved. For a more thorough discussion of results on bisimulations for hybrid systems, see [25, 2].

We declare the set of "interesting" equivalence classes of $\simeq$, which is assumed to be finite and is denoted Q, to be those that intersect $\Sigma_\delta \times cl(\Omega)$. For each $q \in Q$ we define a distinguished point $(\sigma, \xi) \in q$, and we use the notation $q = [(\sigma, \xi)]$. We define a mesh size on Q by

$$\delta_Q = \max_{q \in Q} \sup_{(\sigma,x),(\sigma,y) \in q} \|x - y\|.$$

For each $q = [(\sigma, \xi)] \in Q$ we associate the duration τ_q, the maximum time to traverse q using constant control σ. That is,

$$\tau_q = \sup_{(\sigma,x),(\sigma,y) \in q} \{\; t \mid y = \phi_t(x, \sigma) \;\}.$$

This extra data assocated with the bisimulation is required in our problem to obtained approximations of the cost functions L and h.

4.1. Geometric construction. We review our method for obtaining finite bisimulations [10] which relies on the following assumptions on the vector fields on Ω.

Assumption 4.1.
(1) For each $\sigma \in \Sigma_\delta$, there exist $n-1$ C^1 functions $\gamma_i^\sigma : \Omega \to \mathbb{R}$, $i = 1, \ldots, n-1$, whose time derivative along solutions of $\dot{x} = f(x, \sigma)$ in Ω is zero.
(2) There exists $m_f > 0$ such that $\|f(x, u)\| \geq m_f$ for all $x \in cl(\Omega)$, $u \in U$.

Remark 4.1. There is an uncontested view promulgated by Poincare that differential equations possessing a complete set of first integrals, i.e. completely integrable systems, are the exception rather than the norm. This has lead to some confusion as to when one can or cannot find first integrals for non-Hamiltonian systems. The primary source of confusion comes from the multiple meanings of the term "integrability". It appears as a Liouville integrability (the version alluded to by Poincare and further developed by Arnold [3]), local integrability, algebraic integrability, etc. A type of integrability suitable for non-Hamiltonian systems was proposed by Llibre [32] with the terminology *weak integrability* (to contrast with "strong integrability" in the sense of Liouville). Weak integrability is meant to capture that many systems do not exhibit complex behavior such as chaos, even if they are not Hamiltonian.

Let $\dot{x} = f(x)$ be a differential equation with domain of definition $\mathcal{D} \subset \mathbb{R}^n$, and let $\mathcal{O}$ be a set of orbits of the system such that $\mathcal{D} \setminus \mathcal{O}$ is open. Following [32], we say a C^1 function $\gamma : \mathcal{D} \setminus \mathcal{O} \to \mathbb{R}$ is a *weak first integral* of the system $\dot{x} = f(x)$ if γ is constant on each solution of the system contained in $\mathcal{D} \setminus \mathcal{O}$, and γ is non-constant on any open subset of $\mathcal{D} \setminus \mathcal{O}$. A system is said to be *weakly integrable* if it has $n-1$ functionally independent (on $\mathcal{D} \setminus \mathcal{O}$) weak first integrals.

The relaxation of the requirement that the first integral be a differentiable function on the entire domain of the differential equation means that, for instance, all linear systems are weakly integrable [11], whereas only the Hamiltonian linear systems (centers and saddles in the case of second order linear systems) are integrable in the strong sense. The assumption 4.1(1) is a weak integrability assumption.

There are many methods for finding first integrals including Lie group symmetry analysis [7, 37], Lax pairs, Painleve analysis, and Frobenius theorem, among others [15, 21, 46, 49]. A general reference and overview of the methods can be found in [23]. The best known result for symbolic computation of first integrals is the Prelle-Singer procedure [39]. Reduce and Macsyma implementations of the Prelle-Singer procedure are described in [34, 46], while an implementation in higher dimensions is described in [35]. Algorithms for finding polynomial first integrals are described in [43, 44].

A bisimulation of $\Sigma_\delta \times \mathbb{R}^n$ is found by first constructing partitions for each location of H such that reachability properties are preserved over time steps. In Section 5 we describe how to accomodate σ-steps in the quotient system. To obtain a partition consistent with the dynamics of location $\sigma \in \Sigma_\delta$ we use the level sets of the $n-1$ first integrals $\gamma_i^\sigma(x) = y_i^\sigma$, $i = 1, \ldots, n-1$ to bound the flow in $n-1$ independent directions, thus obtaining "tubes" of trajectories with a rectangular cross section. Next, the level sets of a submersion $\gamma_n^\sigma = y_n^\sigma$ that is transverse to the flow of $\dot{x} = f(x, \sigma)$ is used to divide the tube of trajectories into boxes, so that $(y_1^\sigma, \ldots, y_n^\sigma)$ form a set of Euclidean coordinates $\gamma^\sigma : \Omega \to [-1, 1]^n$ on Ω. That is, we assume that the level sets of γ_i^σ *foliate* the set Ω (see [29] for background on foliations) and, by appropriate scaling, their level values lie between -1 and 1 on Ω. We *discretize* the foliations associated with each γ_i^σ by selecting a finite number of level values. More precisely, fix $k \in Z^+$

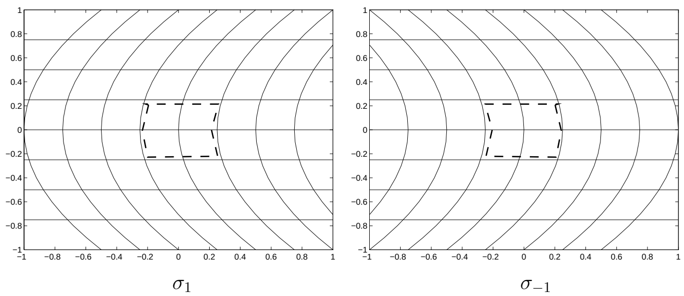


FIG. 4.2. *Partitions for states σ_1 and σ_{-1} of the hybrid automaton of Figure 3.1*

and let $\Delta = \frac{1}{2^k}$. Define

$$C_k = \{0, \pm\Delta, \pm 2\Delta, \ldots, \pm 1\}. \tag{4.1}$$

Each $y_i^\sigma = c$ for $c \in C_k$, $i = 1, \ldots, n$ defines a hyperplane in $\mathbb{R}^n$ denoted $\tilde{W}_{i,c}^\sigma$, and a submanifold $W_{i,c}^\sigma = (\gamma^\sigma)^{-1}(\tilde{W}_{i,c}^\sigma)$. The collection of submanifolds for $\sigma \in \Sigma_\delta$ is

$$\mathcal{W}_k^\sigma = \{\ W_{i,c}^\sigma \mid c \in C_k, i \in \{1, \ldots, n\}\ \}. \tag{4.2}$$

$\Omega \backslash \mathcal{W}_k^\sigma$ is the union of $2^{n(k+1)}$ disjoint open sets $\mathcal{V}_k^\sigma = \{V_j^\sigma\}$. We define an equivalence relation $\simeq^e$ on $\mathbb{R}^n$ as follows. $y \simeq^e y'$ iff
(1) $y \notin [-1,1]^n$ iff $y' \notin [-1,1]^n$, and
(2) if $y, y' \in [-1,1]^n$, then for each $i = 1, \ldots, n$, $y_i \in (c, c+\Delta)$ iff $y_i' \in (c, c+\Delta)$, and $y_i = c$ iff $y_i' = c$, for all $c \in C_k$.
We define the equivalence relation $\simeq$ on $\Sigma_\delta \times \mathbb{R}^n$ as follows. $(\sigma, x) \simeq (\sigma', x')$ iff (1) $\sigma = \sigma'$, and (2) $\gamma^\sigma(x) \simeq^e \gamma^\sigma(x')$.

Remark 4.2. A consequence of this construction is that if any trajectory of H passing through $q \in Q$ spends zero time in it, then $\tau_q = 0$.

Example 4.1 Continuing example 3.1, a first integral for vector field $\dot{x}_1 = x_2$, $\dot{x}_2 = 1$ is $x_1 - \frac{1}{2}x_2^2 = c_1$, $c_1 \in \mathbb{R}$. For $\dot{x}_1 = x_2$, $\dot{x}_2 = -1$ a first integral is $x_1 + \frac{1}{2}x_2^2 = c_2$, $c_2 \in \mathbb{R}$. We select a transverse foliation for each vector field, given by $x_2 = c_3$. Partitions for locations σ_1 and σ_{-1} and $\Omega = (-1,1) \times (-1,1)$ are shown in Figure 4.2. The equivalence classes of $\simeq$ are pairs consisting of a control event in Σ_δ and of the interiors of regions, open line segments and curves forming the boundaries of two regions, and the points at the corners of regions. $\tau = 0$ for the segments transverse to the flow and the corner points. $\tau = \Delta$ for the interiors of regions and segments tangential to the flow, where $\Delta = .25$ in Figure 4.2.

5. Discrete problem. In this section we transform the hybrid optimal synthesis problem to a dynamic programming problem on a non-deterministic finite automaton. Consider the class of non-deterministic finite automata with cost structure represented by the tuple

$$A = (Q, \Sigma_\delta, E, \hat{L}, \hat{h}).$$

Q is the finite state set, as defined above, and Σ_δ is the set of control events as before. $E \subseteq Q \times Q$ is the transition relation encoding t-steps and σ-steps of H. $(q, q') \in E$, where $q = [(\sigma, \xi)]$ and $q' = [(\sigma', \xi')]$ if either (a) $\sigma = \sigma'$, there exists $x \in \Omega$ such that $(\sigma, x) \in q$, and there exists $\tau > 0$ such that $\forall t \in [0, \tau]$, $(\sigma, \phi_t(x, \sigma)) \in q$ and

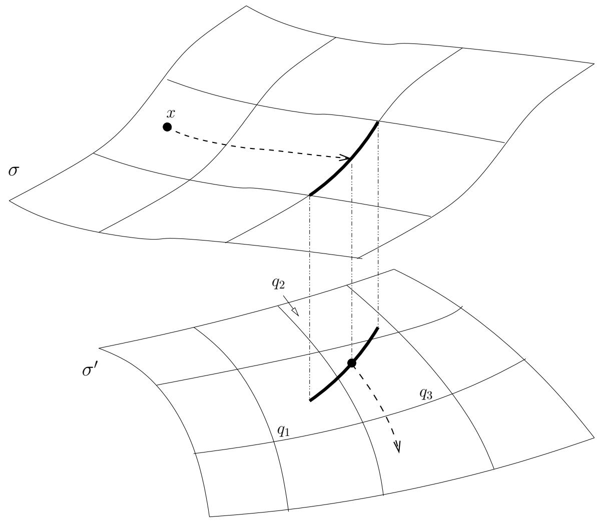


FIG. 5.1. *Partitions for states σ and σ' of a hybrid automaton, and the resulting non-determinism in A.*

$(\sigma, \phi_{\tau+\epsilon}(x, \sigma)) \in q'$ for arbitrarily small $\epsilon > 0$, or (b) $\sigma = \sigma'$, there exists $x \in \Omega$ such that $(\sigma, x) \in q$, and there exists $\tau > 0$ such that $\forall t \in [0, \tau)$, $(\sigma, \phi_t(x, \sigma)) \in q$ and $(\sigma, \phi_\tau(x, \sigma)) \in q'$, or (c) $\sigma \neq \sigma'$ and there exists $x \in \Omega$ such that $(\sigma, x) \in q$ and $(\sigma', x) \in q'$. Cases (a) and (b) say that from a point in q, q' is the first state (different from q) reached after following the flow of $f(x, \sigma)$ for some time. Case (c) says that an edge exists between q and q' if their projections to $\mathbb{R}^n$ have non-empty intersection.

Remark 5.1. The requirement that there be an edge from q to q' if their projections to $\mathbb{R}^n$ have non-empty intersection is illustrated in Figure 5.1. We have partitions for controls σ and σ', respectively. In the partition for σ suppose a trajectory starting at x flows in time using control σ until state q of A is reached, at which time the control is set to σ'. The possible states of A that can be reached from q are q_1, q_2, q_3, and the one-dimensional equivalence classes between them. Hence, edges corresponding to these possible futures for the trajectory must be included in the definition of A. A consequence is that multiple trajectories of A can be defined starting from an initial state. One can think of this construction as over-approximating the identity map in terms of the equivalence classes of $\simeq$. This is the source of non-determinacy of A.

Let $e = (q, q')$ with $q = [(\sigma, \xi)]$ and $q' = [(\sigma', \xi')]$. $\hat{L} : E \to \mathbb{R}$ is the *discrete instantaneous cost* given by

$$\hat{L}(e) := \begin{cases} \tau_q L(\xi, \sigma) & \text{if } \sigma = \sigma' \\ 0 & \text{if } \sigma \neq \sigma'. \end{cases} \tag{5.1}$$

$\hat{h} : Q \to \mathbb{R}$ is the *discrete terminal cost* given by

$$\hat{h}(q) := h(\xi).$$

The domain of $\hat{h}$ can be extended to Ω, with a slight abuse of notation, by

$$\hat{h}(x) := \hat{h}(q) \tag{5.2}$$

where $q = \arg\min_{q'} \{\|x - \xi'\| \mid q' = [(\sigma', \xi')]\}$. Finally, Q_f is the target set given by the over-approximation of Ω_f,

$$Q_f = \{q \in Q \mid \exists x \in \Omega_f \ s.t. \ (\sigma, x) \in q \ \}. \tag{5.3}$$

5.1. Semantics. A transition or *step* of A from $q = [(\sigma, \xi)] \in Q$ to $q' = [(\sigma', \xi')] \in Q$ is denoted $q \stackrel{\sigma'}{\to} q'$. If $\sigma \neq \sigma'$ the transition is referred to as a *control switch*; otherwise, it is referred to as a *time step*. If $E(q)$ is the set of edges that can be enabled from $q \in Q$, then for $\sigma \in \Sigma_\delta$,

$$E_\sigma(q) = \{e \in E(q) \mid e = (q, q'), q = [(\sigma, \xi)], q' = [(\sigma', \xi')]\}.$$

If $|E_\sigma(q)| > 1$, then we say that $e \in E_\sigma(q)$ is *unobservable* in the sense that when control event σ is issued, it is unknown which edge among $E_\sigma(q)$ is taken. If $\sigma = \sigma'$, then $|E_\sigma(q)| = 1$, by the uniqueness of solutions of ODE's and by the definition of bisimulation.

A *control policy* $c : Q \to \Sigma_\delta$ is a map assigning a control event to each state; $c(q) = \sigma$ is the control event issued when the state is at q. A *trajectory* π of A over c is a sequence $\pi = q_0 \stackrel{\sigma_1}{\to} q_1 \stackrel{\sigma_2}{\to} q_2 \stackrel{\sigma_3}{\to} \ldots$, $q_i \in Q$. A trajectory is *non-Zeno* if between any two non-zero duration time steps there are a finite number of control switches. Note that this definition is slightly different from the traditional definition of non-Zeno trajectories of H [25] in which it is assumed that time steps always have a non-zero duration. Here zero-duration time steps can occur. Let $\Pi_c(q)$ be the set of trajectories starting at q and applying control policy c, and let $\tilde{\Pi}_c(q)$ be the set of trajectories starting at q, applying control policy c, and eventually reaching Q_f. If for every $q \in Q$, $\pi \in \Pi_c(q)$ is non-Zeno then we say c is an *admissible control policy.* The set of all admissible control policies for A is denoted $\mathcal{C}$.

A control policy c is said to have a *loop* if A has a trajectory $q_0 \stackrel{c(q_0)}{\to} q_1 \stackrel{c(q_1)}{\to} \ldots \stackrel{c(q_{m-1})}{\to} q_m = q_0$, $q_i \in Q$. A control policy has a *Zeno loop* if it has a loop made up of control switches and/or zero duration time steps (i.e. $\tau_q = 0$) only.

LEMMA 5.1. *A control policy c is admissible if and only if it has no Zeno loops.*

Proof. First we show that a non-deterministic automaton with non-Zeno trajectories has a control policy without Zeno loops. For suppose not. Then a trajectory starting on a state belonging to the loop can take infinitely many steps around the loop before taking a non-zero duration time step. Such a trajectory must necessarily include a control switch (since a zero duration time step is always followed either by a non-zero duration time step or a control switch). Since this control switch occurs infinitely often in a finite time interval, the trajectory is Zeno, a contradiction.

Second, we show that a control policy without Zeno loops implies non-Zeno trajectories. Suppose not. Consider a Zeno trajectory that takes an infinite number of control switches in some finite time interval. Because there are a finite number of states in Q, by the Dirichlet Principle [30], one of the states must be repeated in the sequence of states visited during the infinite number of control switches. Note this sequence can include zero duration time steps. This implies the existence of a loop in the control policy. Now we argue this loop is Zeno.

First, by Remark 4.2, if $\tau_q = 0$, then all trajectories spend zero time in q. Second, if $\tau_q > 0$, then there exists $\overline{\tau}_q > 0$ such that all trajectories spend at least $\overline{\tau}_q$ time in q. This follows from the boundedness of f and the bisimulation construction (trajectories cannot move between two level sets of γ_n^σ in arbitrarily small time). Since there are a finite number of states in Q, there exists $\overline{\tau} > 0$, the minimum time spent by any trajectory in a state $q \in Q$ with $\tau_q > 0$. The result is that Zeno trajectories only arise

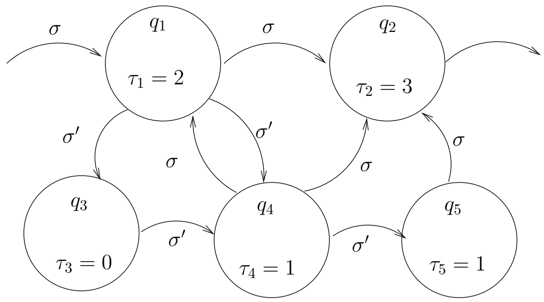


FIG. 5.2. *Fragment of automaton with a zero duration time step.*

by an infinite number of control switches in a zero duration time interval. Hence, we have shown that the loop consists of control switches and/or zero duration time steps only, i.e. it is a Zeno loop. □

Example 5.1 Consider the automaton in Figure 5.2. Suppose that we define a control policy $c(q_1) = \sigma'$, $c(q_3) = \sigma'$, $c(q_4) = \sigma$, and $c(q_5) = \sigma$. Starting at q_1 two possible trajectories are $q_1 \overset{\sigma'}{\to} q_3 \overset{\sigma'}{\to} q_4 \overset{\sigma}{\to} q_2$, or $q_1 \overset{\sigma'}{\to} q_3 \overset{\sigma'}{\to} q_4 \overset{\sigma}{\to} q_1$. The first trajectory has a zero duration time step. The control is inadmissible since the second trajectory has a Zeno loop.

5.2. Dynamic programming. We formulate the dynamic programming problem on A. This involves defining a cost-to-go function and a value function that minimizes it over control policies suitable for non-deterministic automata.

Let $\pi = q_0 \overset{\sigma_1}{\to} q_1 \to \ldots \to q_{N-1} \overset{\sigma_N}{\to} q_N$, where $q_i = [(\sigma_i, \xi_i)]$ and π takes the sequence of edges $e_1 e_2 \ldots e_N$. We define a *discrete cost-to-go* $\hat{J} : Q \times \mathcal{C} \to \mathbb{R}$ by

$$\hat{J}(q, c) = \begin{cases} \max_{\pi \in \tilde{\Pi}_c(q)} \left\{ \sum_{j=1}^{N_\pi} \hat{L}(e_j) + \hat{h}(q_{N_\pi}) \right\} & \text{if } \Pi_c(q) = \tilde{\Pi}_c(q) \\ \infty & \text{otherwise} \end{cases}$$

where $N_\pi = \min\{j \geq 0 \mid q_j \in Q_f\}$. We take the maximum over $\tilde{\Pi}_c(q)$ because of the non-determinacy of A: it is uncertain which among the (multiple) trajectories allowed by c will be taken so we must assume the worst-case situation. The *discrete value function* $\hat{V} : Q \to \mathbb{R}$ is

$$\hat{V}(q) = \min_{c \in \mathcal{C}} \hat{J}(q, c)$$

for $q \in Q \backslash Q_f$ and $\hat{V}(q) = \hat{h}(q)$ for $q \in Q_f$. We show in Proposition 5.2 that $\hat{V}$ satisfies a DPP that takes into account the non-determinacy of A and ensures that optimal control policies are admissible. Let $\mathcal{A}_q$ be the set of control assignments $c(q) \in \Sigma_\delta$ at q such that c is admissible.

PROPOSITION 5.2. *$\hat{V}$ satisfies*

$$\hat{V}(q) = \min_{c(q) \in \mathcal{A}_q} \left\{ \max_{e=(q,q') \in E_{c(q)}(q)} \left\{ \hat{L}(e) + \hat{V}(q') \right\} \right\}, \quad q \in Q \setminus Q_f \tag{5.4}$$

$$\hat{V}(q) = \hat{h}(q), \qquad q \in Q_f. \tag{5.5}$$

Proof. Fix $q \in Q$. By definition of $\hat{J}$

$$\hat{J}(q, c) = \max_{e=(q,q')\in E_{c(q)}(q)} \{\hat{L}(e) + \hat{J}(q', c)\}. \tag{5.6}$$

By definition of $\hat{V}$

$$\hat{J}(q, c) \geq \max_{e=(q,q')\in E_{c(q)}(q)} \{\hat{L}(e) + \hat{V}(q')\}.$$

Since $c(q) \in \mathcal{A}_q$ is arbitrary

$$\hat{V}(q) \geq \min_{c(q)\in\mathcal{A}_q} \big\{ \max_{e=(q,q')\in E_{c(q)}(q)} \{\hat{L}(e) + \hat{V}(q')\}\big\}.$$

To prove the reverse inequality suppose, by way of contradiction, there exists $\sigma' \in \Sigma_\delta$ such that

$$\hat{V}(q) > \max_{e=(q,q')\in E_{\sigma'}(q)} \{\hat{L}(e) + \hat{V}(q')\} := \hat{L}(e) + \hat{V}(\overline{q}). \tag{5.7}$$

Suppose an optimal admissible policy for $\overline{q}$ is $\overline{c}$. Define $c = \overline{c}$ on $Q \setminus \{q\}$ and $c(q) = \sigma'$. Then $\hat{J}(q, c) = \hat{L}(e) + \hat{V}(\overline{q}) < \hat{V}(q)$. This gives rise to a contradiction if we can show c is admissible. Suppose not. Then there exists a loop of control switches and zero duration time steps containing q. Either the loop includes $\overline{q}$, implying $\hat{V}(\overline{q}) = \hat{V}(q)$, which contradicts hypothesis (5.7). Alternatively, the loop includes some other q' such that $(q, q') \in E_{\sigma'}(q)$, implying $\hat{V}(q') = \hat{V}(q)$. But $\hat{V}(\overline{q}) \geq \hat{V}(q')$ since $\overline{q}$ gives the worst-case cost over edges with label σ'. This again contradicts hypothesis (5.7). $\square$

5.3. Synthesis of g_e. The synthesis of enabling conditions or *hybrid controller synthesis* is typically a post-processing step of a backward reachability analysis (see, for example, [52]). This situation prevails here as well: equations (5.4)-(5.5) describe a backward analysis to construct an optimal policy $c \in \mathcal{C}$. Once c is known the enabling conditions of H are extracted as follows.

Consider each $e = (\sigma, \sigma') \in E$ of H with $\sigma \neq \sigma'$. There are two cases. If $\sigma' \neq \sigma_f$ then $g_e = \big\{x \mid (\sigma, x) \in q, q \in Q, c(q) = \sigma'\big\}$. That is, if the control policy designates switching from $q \in Q$ with label σ to $q' \in Q$ with label σ', then the corresponding enabling condition in H includes the projection to $\mathbb{R}^n$ of q. The second case when $\sigma' = \sigma_f$ is for edges going to the terminal location of H. Then $g_e = \big\{x \mid (\sigma, x) \in q, q \in Q_f\big\}$.

6. Main Result. We will prove that $\hat{V}$ converges to V, the viscosity solution of the HJB equation, as $\delta_Q, \delta \to 0$. We make use of a filtration of control sets $\Sigma_k \equiv \Sigma_{\delta_k}$ corresponding to a sequence $\delta_k \to 0$ as $k \to \infty$, in such a manner that $\Sigma_k \subset \Sigma_{k+1}$. Considering (4.2), we define a filtration of families of submanifolds such that $\mathcal{W}_k^\sigma \subset \mathcal{W}_{k+1}^\sigma$, for each $\sigma \in \Sigma_k$.

The proof proceeds in three steps. In the first step we restrict the class of controls to piecewise constant functions whose constant intervals are a function of the state. In particular, the control is constant on equivalence classes of $\simeq$. As δ_k tends to zero this class of piecewise constant controls well approximates ϵ-optimal controls. Arzela-Ascoli theorem is invoked to show that the limit of a sequence of approximations V_k of the value function using the aforementioned controls is a continuous function V_*. Using techniques of [4], V_* is shown to be the unique viscosity solution of HJB. In the second step we introduce the discrete approximations of L and h. The discrete approximation of h is a one time error while the error between L and $\hat{L}$ is shown to be $O(\delta_k^2)$ per interval τ. Since the number of intervals is $O(1/\delta_k)$, the error is $O(\delta_k)$. In the last step we introduce the discrete states Q. The error introduced at each control switch by the non-determinacy of A is $O(\delta_k)$ and since there are a fixed number of control switches as $\delta_k \to 0$, this error can be made arbitrarily small.

Step 1: piecewise constant controls.

In the first step we define a class of piecewise constant functions that depend on the state and show that the value function which minimizes the cost-to-go over this class converges to the viscosity solution of HJB as $\delta_k \to 0$. The techniques of this step are based on those in Bardi and Capuzzo-Dolcetta [4] and are related to those in [12].

We consider the optimal control problem (2.2)-(2.4) when the set of admissible controls is $\mathcal{U}_k^1$, piecewise constant functions consisting of finite sequences of control events $\sigma \in \Sigma_k$ where each σ is applied for a time $\tau(\sigma, x)$ and the trajectory remains in Ω. Let $(\sigma, x) \in q$ for $q \in Q$ and define $\tau(\sigma, x)$ to be the minimum of the time it takes the trajectory starting at x and using control $\sigma \in \Sigma_k$ to reach (ta) $\partial\Omega_f$, or (tb) some x' such that $(\sigma, x') \notin q$. If a trajectory is at x_i at the start of the $(i+1)$th step, then the control σ_{i+1} is applied for time $\tau_{i+1} := \tau(\sigma_{i+1}, x_i)$ and $x_{i+1} = \phi_{\tau_{i+1}}(x_i, \sigma_{i+1})$. Thus U_k^1 is a class of piecewise constant controls whose constant intervals are based on the state partition induced by $\simeq$ (in contrast with a partition of the time interval): the control can only change values on the boundary of equivalence classes.

Let

$$\mathcal{R}_k^1 := \{\ x \in \Omega \mid \exists \mu \in \mathcal{U}_k^1\ .\ T(x, \mu) < \infty\ \}.$$

We define the cost-to-go function $J_k^1 : \Omega \times \mathcal{U}_k^1 \to \mathbb{R}$ as follows. For $x \in \Omega$ and $\mu = \sigma_1 \sigma_2 \ldots \in \mathcal{U}_k^1$, if $T(x, \mu) < \infty$ then

$$J_k^1(x, \mu) = \sum_{j=1}^{N} \int_0^{\tau(\sigma_j, x_{j-1})} L(\phi_s(x_{j-1}, \sigma_j), \sigma_j) ds + h(x_N)$$

where $N = \min\{j \geq 0 \mid x_j \in \Omega_f\}$. $J_k^1(x, \mu) = \infty$, otherwise. We define the value function $V_k^1 : \mathbb{R}^n \to \mathbb{R}$ as follows. For $x \in \Omega \setminus \Omega_f$,

$$V_k^1(x) = \inf_{\mu \in \mathcal{U}_k^1} J_k^1(x, \mu) \tag{6.1}$$

and for $x \in \Omega_f$, $V_k^1(x) = h(x)$. The following result is proved using standard arguments from dynamic programming [19].

Proposition 6.1. *V_k^1 satisfies, for all $x \in \mathcal{R}_k^1$,*

$$V_k^1(x) = \min_{\sigma \in \Sigma_k} \left\{ \int_0^{\tau(\sigma, x)} L(\phi_s(x, \sigma), \sigma) ds + V_k^1(\phi_{\tau(\sigma, x)}(x, \sigma)) \right\}. \tag{6.2}$$

We would like to show that V_k^1 is uniformly bounded and locally uniformly continuous. Considering uniform continuity of V_k^1, let C_k be as in (4.1) and γ_n^σ the submersion whose level sets are transverse to the flow of $\dot{x} = f(x, \sigma)$. Referring to Figure 6.1, for each $\sigma \in \Sigma_k$ and for each fixed $c \in C_k$ we define the regions in $\mathbb{R}^n$

$$M_c^\sigma := \{\ x \mid \gamma_n^\sigma(x) = c\ \}$$
$$M_{c-}^\sigma := \{\ x \mid \gamma_n^\sigma(x) \in (-1, c)\ \},$$

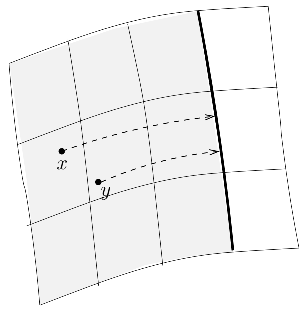


FIG. 6.1. *The shaded region is M^σ_{c-} while the bold curve on its boundary is M^σ_c.*

that is, M_{c-} is the strip of points belonging to a level set of γ^σ_n whose level value is between -1 and c.

Remark 6.1.

(a) If $x, y \in M^\sigma_{c-}$ for some $c \in C_k$ and $\tau(\sigma, x)$ and $\tau(\sigma, y)$ are defined using (tb) then $|\tau(\sigma, x) - \tau(\sigma, y)| \to 0$ and $\|\phi_{\tau(\sigma,x)}(x, \sigma) - \phi_{\tau(\sigma,y)}(y, \sigma)\| \to 0$ as $\|x - y\| \to 0$ in M^σ_{c-}, since M^σ_c is a smooth submanifold. See Figure 6.1. For the details, see Theorem 6.1, p. 91-94, [19]. If instead $\tau(\sigma, x)$ and $\tau(\sigma, y)$ are defined using (ta) and σ is an ϵ-optimal control for x, then by Assumption 2.2 the same results hold.

(b) For each $x \in \cup_k \mathcal{R}^1_k$ and $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ and $\mu \in \mathcal{U}^1_m$ such that μ is an ϵ-optimal control for x w.r.t. V^1_k with at most N_ϵ discontinuities and such that $\phi_t(x, \mu)$ is transverse to $\partial\Omega_f$. This follows from Assumption 2.2, $V^1_k(x) \geq V(x)$, and the fact that we can well-approximate an ϵ-optimal control for V by a control in $\mathcal{U}^1_m$, for large enough m.

The following lemma shows that V^1_k is locally uniformly continuous.

LEMMA 6.2. *For each $y \in \cup_k \mathcal{R}^1_k$ and $\epsilon > 0$, there exists $m_\epsilon \in \mathbb{Z}^+$ and $\eta_\epsilon > 0$ such that $|V^1_k(x) - V^1_k(y)| < 2\epsilon$ for all $|x - y| < \eta_\epsilon$ with $x \in \mathcal{R}^1_k$ and for all $k > m_\epsilon$.*

Proof. Fix $y \in \cup_k \mathcal{R}^1_k$. By Remark 6.1(b) there exists $m_1 > 0$ and $\mu \in \mathcal{U}^1_{m_1}$ such that μ is an ϵ-optimal control for y satisfying Assumption 2.2. Let $x \in \mathcal{R}^1_{m_1}$. Then $V^1_k(x) - V^1_k(y) \leq J^1_k(x, \mu_x) - J^1_k(y, \mu) + \epsilon$ for any $\mu_x \in \mathcal{U}^1_{m_1}$ and $k > m_1$. If we can show that for fixed y and μ there exists $\mu_x \in \mathcal{U}^1_{m_1}$ such that

$$J^1_k(x, \mu_x) - J^1_k(y, \mu) < \epsilon \tag{6.3}$$

for all $x \in \mathcal{R}^1_{m_1}$ sufficiently close to y, then $V^1_k(x) - V^1_k(y) \leq 2\epsilon$ for all $k \geq m_1$.

Conversely, by Remark 6.1(b) there exists $m_2 > 0$ and $\mu_x \in \mathcal{U}^1_{m_2}$ such that μ_x is an ϵ-optimal control for x satisfying Assumptions 2.2. Then $V^1_k(y) - V^1_k(x) \leq J^1_k(y, \mu) - J^1_k(x, \mu_x) + \epsilon$ for any $\mu \in \mathcal{U}^1_{m_2}$ and $k > m_2$. If we can show that for fixed y there exists $\mu \in \mathcal{U}^1_{m_2}$ such that

$$J^1_k(y, \mu) - J^1_k(x, \mu_x) < \epsilon \tag{6.4}$$

for all $x \in \mathcal{R}^1_{m_2}$ sufficiently close to y, then $V^1_k(x) - V^1_k(y) \geq -2\epsilon$ for all $k \geq m_2$. The result follows by letting $m_\epsilon = \min\{m_1, m_2\}$. Thus, we must show (6.3) and (6.4).

Consider first (6.3). Let $\mu = \sigma_1\sigma_2 \ldots \in \mathcal{U}^1_k$ be an ϵ-optimal control for y such that $y_N \in \partial\Omega_f$. By redefining indices, we can associate with μ the open-loop control $\tilde{\mu} = (\sigma_1, \tau_1)(\sigma_2, \tau_2) \ldots$, where τ_i is the time σ_i is applied. We claim there exists

$\tilde{\mu}^x = (\sigma_1, \tau_1^x)(\sigma_2, \tau_2^x)\ldots$ such that as $x \to y$, (a) $x_j \to y_j$, (b) $\tau_j^x \to \tau_j$, and (c) $x_N \in \partial\Omega_f$. Let $T_k = \max_i \tau_i$. Then we have

$$\begin{aligned} J_k^1(x, \tilde{\mu}^x) - J_k^1(y, \tilde{\mu}) \leq & \sum_{j=1}^{N} \int_0^{\tau_j} |L(\phi_s(x_{j-1}, \sigma_j), \sigma_j) - L(\phi_s(y_{j-1}, \sigma_j), \sigma_j)| ds \\ & + \sum_{j=1}^{N} \Big| \int_{\tau_j}^{\tau_j^x} L(\phi_s(x_{j-1}, \sigma_j), \sigma_j) ds \Big| + |h(y_N) - h(x_N)| \\ \leq & L_L T_k \exp(L_f T_k) \sum_{j=1}^{N} \|x_{j-1} - y_{j-1}\| \\ & + M_L \sum_{j=1}^{N} |\tau_j^x - \tau_j| + L_h |x_N - y_N|. \end{aligned}$$

By the previous claim the r.h.s. can be made less than ϵ. Thus, we need only show there exists $\tilde{\mu}^x = (\sigma_1, \tau_1^x)(\sigma_2, \tau_2^x)\ldots$ which satisfies the claim and $\mu^x \in \mathcal{U}_k^1$ can be reconstructed from it, based on the discrete states in Q visited by $\phi_t(x, \tilde{\mu}^x)$.

We argue by induction. Suppose (a)-(c) hold at $j-1$. We show they hold at j. We need only consider the case when $y_{j-1} \in M_{c-}^{\sigma_j}$ and $y_j \in M_c^{\sigma_j}$ for some $c \in C_k$; that is, y_{j-1} lies "upstream" of y_j (trajectories flow in the increasing $\gamma_n^{\sigma_j}$ direction), while y_j lies on the boundary of an equivalence class where the control is allowed to switch values. For x_{j-1} sufficiently close to y_{j-1}, $x_{j-1} \in M_{c-}^{\sigma_j}$. By Remark 6.1(a) there exists τ_j^x such that $x_j = \phi_{\tau_j^x}(x_{j-1}, \sigma_j) \in M_c^{\sigma_j}$ and $\tau_j^x \to \tau_j$ and $x_j \to y_j$ as $x_{j-1} \to y_{j-1}$. The case $y_{j-1} \in M_{c-}^{\sigma_j}$ and $y_j \in \partial\Omega_f$ follows in the same way from Assumption 2.2. Proving (6.4) follows along the same lines as the proof for (6.3). ∎

To show boundedness of V_k^1, let $T(x) := \inf_{\mu \in \mathcal{U}_k^1} T(x, \mu)$. In light of Assumption 2.1(2), we have that for all $x \in \mathbb{R}^n$, $|V_k^1(x)| \leq T(x) \cdot M_L + M_h$. Consider the set $K_a := \{x \in \mathcal{R}_k^1 \mid T(x) < a\}$. Then $|V_k^1(x)| \leq a \cdot M_L + M_h$, $\forall x \in K_a$.

We have shown that on each $K_a \subseteq \mathbb{R}^n$, $\{V_k^1\}$ forms a family of equibounded, locally equicontinuous functions. It follows by Arzela-Ascoli Theorem [42] that along some subsequence k_n, $V_{k_n}^1$ converges to a continuous function V_*. The proof of the following result closely follows [4].

Proposition 6.3. *V_* is the unique viscosity solution of HJB.*

Proof. We show that V_* solves HJB in the viscosity sense. Let $\psi \in C^1(\mathbb{R}^n)$ and suppose $x_0 \in \Omega$ is a strict local maximum for $V_* - \psi$. There exists a closed ball B centered at x_0 such that $(V_* - \psi)(x_0) > (V_* - \psi)(x)$, for all $x \in B$. Let $x_{0\delta_k}$ be a maximum point for $V_k^1 - \psi$ over B. Since $V_k^1 \to V_*$ locally uniformly it follows that $x_{0\delta_k} \to x_0$ as $\delta_k \to 0$. Then, for any $\sigma \in \Sigma_k$, the point $\phi_\tau(x_{0\delta_k}, \sigma)$ is in B (using boundedness of f), for sufficiently small δ_k and $0 \leq \tau \leq \tau(x_{0\delta_k}, \sigma)$, since $\tau(x_{0\delta_k}, \sigma) \to 0$ as $\delta_k \to 0$. Therefore,

$$V_k^1(x_{0\delta_k}) - \psi(x_{0\delta_k}) \geq V_k^1(\phi_\tau(x_{0\delta_k}, \sigma)) - \psi(\phi_\tau(x_{0\delta_k}, \sigma)).$$

Considering Equation 6.2, we have

$$\begin{aligned} 0 &= -\min_{\sigma \in \Sigma_k} \Big\{ V_k^1(\phi_\tau(x_{0\delta_k}, \sigma)) - V_k^1(x_{0\delta_k}) + \int_0^\tau L(\phi_s(x_{0\delta_k}, \sigma), \sigma) ds \Big\} \\ &\geq -\min_{\sigma \in \Sigma_k} \Big\{ \psi(\phi_\tau(x_{0\delta_k}, \sigma)) - \psi(x_{0\delta_k}) + \int_0^\tau L(\phi_s(x_{0\delta_k}, \sigma), \sigma) ds \Big\}. \end{aligned}$$

Since $\psi \in C^1(\mathbb{R}^n)$, we have by the Mean Value Theorem,

$$0 \geq -\min_{\sigma \in \Sigma_k} \left\{ \frac{\partial \psi}{\partial x}(y) \cdot \int_0^\tau f(\phi_s(x_{0\delta_k}, \sigma), \sigma) ds + \int_0^\tau L(\phi_s(x_{0\delta_k}, \sigma), \sigma) ds \right\}$$

where $y = \alpha x_{0\delta_k} + (1-\alpha)\phi_\tau(x_{0\delta_k}, \sigma)$ for some $\alpha \in (0,1)$. Dividing by $\tau > 0$ on each side and taking the limit as $\delta_k \to 0$, we have $V_k^1 \to V_*$, $x_{0\delta_k} \to x_0$, $\tau \to 0$, and $y \to x_{0\delta_k}$. By the Fundamental Theorem of Calculus, the continuity of f and L, and the uniform continuity in u of the expression in brackets, we obtain

$$0 \geq -\inf_{u \in U} \left\{ \frac{\partial \psi}{\partial x}(x_0) \cdot f(x_0, u) + L(x_0, u) \right\}.$$

This confirms part (i) of the viscosity solution definition. Part (ii) is proved in an analogous manner. ◻

Step 2: approximate cost functions and over-approximation of Ω_f by Q_f.

In this step we define a class of piecewise constant controls, denoted $\mathcal{U}_k^2$, nearly the same as $\mathcal{U}_k^1$, to accommodate that trajectories terminate at Q_f not Ω_f, and we replace the cost functions L and h by approximations L^2 and $\hat{h}$, respectively. We define $\mathcal{U}_k^2 \subset \mathcal{U}_k^1$ to be the class of piecewise continuous controls whose constant time intervals $\tau(\sigma, x)$ are determined by the equivalence classes of $\simeq$, but not $\partial\Omega_f$. That is, the case (ta) in the definition of $\tau(\sigma, x)$ in Step 1 is omitted. Next we define an approximate instantaneous cost $L^2 : \Omega \times \Sigma_k \to \mathbb{R}$ by

$$L^2(x, \sigma) := \hat{L}(e) \tag{6.5}$$

where $(\sigma, x) \in q$ and $e = (q, q')$ represents the time step. For $x \in \Omega$ and $\mu = \sigma_1\sigma_2 \ldots \in \mathcal{U}_k^2$, if $T(x, \mu) < \infty$, the cost-to-go function $J_k^2 : \Omega \times \mathcal{U}_k^2 \to \mathbb{R}$ is

$$J_k^2(x, \mu) = \sum_{j=1}^{N} L^2(x_{j-1}, \sigma_j) + \hat{h}(x_N)$$

where $N = \min\{j \geq 0 \mid x_j \in Q_f\}$. In other words, J_k^2 is a worst-case cost over a set of trajectories starting at x that visit the same sequence of equivalence classes of $\simeq$; and it is a worst-case cost w.r.t. J_k^1 because $\int_0^{\tau(\sigma,x)} L(\phi_s(x,\sigma),\sigma) ds \leq L^2(x, \sigma)$.

We define a value function $V_k^2 : \mathbb{R}^n \to \mathbb{R}$ as follows. For $x \in \Omega \setminus Q_f$,

$$V_k^2(x) = \inf_{\mu \in \mathcal{U}_k^1} J_k^2(x, \mu) \tag{6.6}$$

and for $x \in Q_f$, $V_k^2(x) = \hat{h}(x)$. For $x \in \Omega$ such that $V_k^2(x) < \infty$, V_k^2 satisfies the DPP

$$V_k^2(x) = \min_{\sigma \in \Sigma_k} \left\{ L^2(x, \sigma) + V_k^2(\phi_{\tau(\sigma,x)}(x, \sigma)) \right\}.$$

The proof is along the same lines as that of Proposition 5.2.

The following facts are useful for the subsequent result. The first lemma says that τ_q is order δ_k. The second lemma says that given two times τ and τ' that two trajectories spend, respectively, in the same equivalence class, $|\tau - \tau'|$ is order δ_k^2.

LEMMA 6.4. *If $\delta_k < \frac{m_f}{L_f}$, then for all $q \in Q$,*

$$\tau_q \leq \frac{\delta_k}{m_f - L_f \delta_k}. \tag{6.7}$$

Proof. Let $q \in Q$. Fix $x \in \Omega$ and $\sigma \in \Sigma_k$ such that $(\sigma, x) \in q$. We know $\|\phi_{\tau(\sigma,x)}(x,\sigma) - x\| \leq \delta_k$. We have

$$\begin{aligned}\delta_k \geq \|\phi_{\tau(\sigma,x)} - x\| &= \Big\|\int_0^{\tau(\sigma,x)} f(\phi_s(x,\sigma),\sigma)ds\Big\| \\ &\geq \Big\|\int_0^{\tau(\sigma,x)} f(x,\sigma)ds\Big\| - \Big\|\int_0^{\tau(\sigma,x)} [f(\phi_s(x,\sigma),\sigma) - f(x,\sigma)]ds\Big\| \\ &\geq \tau(\sigma,x)\|f(x,\sigma)\| - \tau(\sigma,x)L_f\delta_k,\end{aligned}$$

where in the last step we use the fact that $\|\phi_s(x,\sigma) - x\| \leq \delta_k$. Therefore,

$$\tau(\sigma,x) \leq \frac{\delta_k}{\|f(x,\sigma)\| - L_f\delta_k}.$$

Using Assumption 4.1(2) and taking the sup over all $\tau(\sigma,x)$ for q, the result follows. ∎

LEMMA 6.5. *Let $x, x' \in M_c^\sigma$ for some $c \in C_k$ and $\sigma \in \Sigma_k$ such that $\|x - x'\| \leq \delta_k$. Let τ, τ' be times such that $\phi_\tau(x,\sigma), \phi_{\tau'}(x',\sigma) \in M_{c+\Delta}^\sigma$. Then $|\tau - \tau'| \leq c_\gamma \tau\delta_k$ for some $c_\gamma > 0$.*

Proof. We have

$$\int_0^\tau \frac{d}{ds}(\gamma_n^\sigma(\phi_s(x,\sigma)))ds = \int_0^{\tau'} \frac{d}{ds}(\gamma_n^\sigma(\phi_s(x',\sigma)))ds.$$

Let $f = f(\phi_s(x,\sigma),\sigma)$, $f' = f(\phi_s(x',\sigma),\sigma)$, $d\gamma = \frac{d\gamma_n^\sigma(z)}{dz}|_{z=\phi_s(x,\sigma)}$ and $d\gamma' = \frac{d\gamma_n^\sigma(z)}{dz}|_{z=\phi_s(x',\sigma)}$. Then rearranging terms

$$\int_0^\tau (f' \cdot d\gamma')ds - \int_0^\tau (f \cdot d\gamma)ds = \int_{\tau'}^\tau (f' \cdot d\gamma')ds.$$

Let L_1 be the Lipschitz constant of $f \cdot d\gamma$ (using the fact that γ_n^σ is smooth). Then

$$\int_{\tau'}^\tau f' \cdot d\gamma' \leq L_1\tau\|x - x'\| \leq L_1\tau\delta_k.$$

Since γ_n^σ defines a transversal foliation to vector field $f(\cdot,\sigma)$, $f \cdot d\gamma > 0$. Let $c = \min_{s\in[\tau,\tau']}\{f' \cdot d\gamma'\} > 0$. Letting $c_\gamma = \frac{L_1}{c}$ we obtain the result. ∎

Remark 6.2. If $\mu \in \mathcal{U}_k^1$ is an ϵ-optimal control for x and the first time the trajectory $\phi_t(x,\mu)$ reaches Ω_f (Q_f) is T (T^2), then $T - T^2 \to 0$ and $|\phi_T(x,\mu) - \phi_{T^2}(x,\mu)| \to 0$ as $k \to \infty$. This follows from the fact that the distance between Ω_f and Q_f tends to zero as $k \to \infty$.

We denote by $\mu^2 \in \mathcal{U}_k^2$ the restriction of μ to $[0, T^2]$. Note that if the length of μ is $|\mu| = N$ then $|\mu_2| := N^2 \leq N$. Then we have the following result.

PROPOSITION 6.6. *Let $k_0 \in \mathbb{Z}^+$ be arbitrary, $x \in \mathcal{R}_{k_0}^1$, and $\mu \in \mathcal{U}_{k_0}^1$ be an ϵ-optimal control for x. Then $|J_k^1(x,\mu) - J_k^2(x,\mu^2)| \to 0$ as $k \to \infty$.*

Proof. Suppose $\mu = (\sigma_1,\tau_1)\dots(\sigma_N,\tau_N)$ and $\mu^2 = (\sigma_1,\tau_1)\dots(\sigma_{N^2},\tau_{N^2})$ where $N^2 \leq N$. Thus, $N - N^2$ additional steps are required to reach $\partial\Omega_f$ after reaching Q_f. Then we have

$$\big|J_k^1(x,\mu) - J_k^2(x,\mu^2)\big| \leq \Big|\sum_{j=1}^N \Big[\int_0^{\tau(\sigma_j,x_{j-1})} L(\phi_s(x_{j-1},\sigma_j),\sigma_j)ds\Big] + h(x_N)$$

$$-\sum_{j=1}^{N^2}\left[\tau_{q_{j-1}}L(\xi_{j-1},\sigma_j)\right]-\hat{h}(x_{N^2})\Big|$$

where $(\sigma_j, x_{j-1}) \in q_{j-1}$ and $q_{j-1} = [(\sigma_j, \xi_{j-1})]$. There exists ξ_{N^2} such that $\hat{h}(x_{N^2}) = h(\xi_{N^2})$ and $\|x_{N^2} - \xi_{N^2}\| \le \delta_k$. Also, using the Mean Value Theorem, there exists $\tilde{t}_{j-1}$ with $\tilde{x}_{j-1} = \phi_{\tilde{t}_{j-1}}(x_{j-1}, \sigma_j)$ and $\|\tilde{x}_{j-1} - \xi_{j-1}\| \le \delta_k$ such that

$$\begin{aligned}
\left|J_k^1(x,\mu) - J_k^2(x,\mu^2)\right| &\le \sum_{j=1}^{N^2}\left|\tau(\sigma_j,x_{j-1})L(\tilde{x}_{j-1},\sigma_j) - \tau_{q_{j-1}}L(\xi_{j-1},\sigma_j)\right| \\
&+\left|\sum_{j=N^2+1}^{N}\left[\int_0^{\tau(\sigma_j,x_{j-1})} L(\phi_s(x_{j-1},\sigma_j),\sigma_j)ds\right]\right| + \left|h(x_N) - \hat{h}(x_{N^2})\right| \\
&\le \sum_{j=1}^{N^2}\tau_{q_{j-1}}L_L\delta_k + \sum_{j=1}^{N^2}[\tau_{q_{j-1}} - \tau(\sigma_j,x_{j-1})]L(\tilde{x}_{j-1},\sigma_j) \\
&+(T-T^2)M_L + L_h\|x_N - x_{N^2}\| + L_h\delta_k.
\end{aligned}$$

The last three terms on the r.h.s. go to zero as $k \to \infty$ because of Remark 6.2 and since $\delta_k \to 0$. Using Lemma 6.4 the first summation decreases linearly as δ_k. Call the second summation on the r.h.s. "B". Splitting B into sums over control switches and time steps, we have

$$\begin{aligned}
B &\le M_L\sum_{j=1}^{N^2}[\tau_{q_{j-1}} - \tau(\sigma_j,x_{j-1})]\mathbf{1}(\sigma_j = \sigma_{j-1}) + M_L\sum_{j=1}^{N^2}[\tau_{q_{j-1}} - \tau(\sigma_j,x_{j-1})]\mathbf{1}(\sigma_j \ne \sigma_{j-1}) \\
&\le M_L\sum_{j=1}^{N^2}c_{j-1}\tau_{q_{j-1}}\delta_k + M_L\sum_{j=1}^{N^2}\tau_{q_{j-1}}\mathbf{1}(\sigma_j \ne \sigma_{j-1})
\end{aligned}$$

for some $c_{j-1} \in \mathbb{R}$. In the second line we used Lemma 6.5 and the fact that $\tau_{q_{j-1}} \ge \tau(\sigma_j, x_{j-1})$. Using Lemma 6.4 the first summation on the r.h.s. decreases linearly as δ_k. The second term on the r.h.s. goes to zero since, by Assumption 2.2, μ has a fixed number of control switches for all $k \ge k_0$. $\square$

Step 3: discrete states and non-determinacy.

In the last step we compare the value function $V_k^2(x)$ with the discrete value function $\hat{V}$ defined on A. The difference between the two is that trajectories defined over $\mathcal{U}_k^2$ do not include jumps while trajectories whose time abstract versions are accepted by A can have jumps due to the non-determinacy of A. Nevertheless, as $k \to \infty$ this decrepancy can be made negligible and we show that the difference between V_k^2 and $\hat{V}$ can be made arbitrarily small.

First we extend the domain of $\hat{V}(q)$, with an abuse of notation, by defining

$$\hat{V}_k(x) := \min_{\sigma\in\Sigma_k}\{\ \hat{V}(q)\mid(\sigma,x)\in q\ \}.$$

Also let $\hat{\mathcal{R}}_k = \{x \in \Omega \mid \hat{V}_k(x) < \infty\}$ and $\hat{\mathcal{R}} = \cup_k\hat{\mathcal{R}}_k$.

Remark 6.3.

(a) For each $x \in \cup_k\mathcal{R}_k^1$ and $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ and $\mu \in \mathcal{U}_m^2$ such that μ is an ϵ-optimal control for x w.r.t. V_k^2 with at most N_ϵ discontinuities. This follows from Remark 6.1(b) and the fact that trajectories in $\mathcal{U}_m^2$ are merely truncations of trajectories in $\mathcal{U}_m^1$.

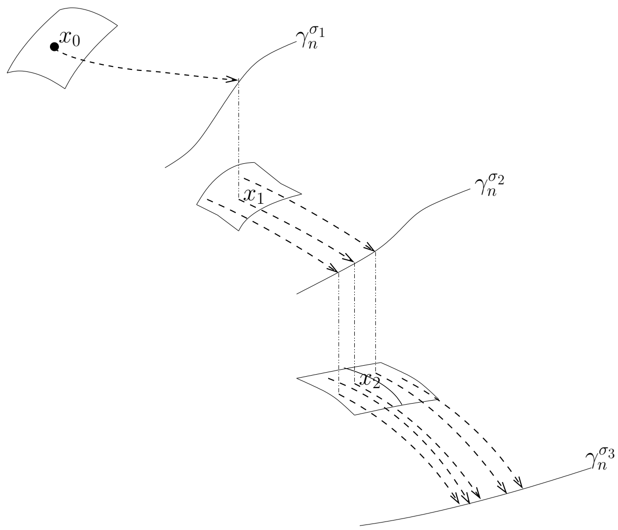


FIG. 6.2. *The family of trajectories Ψ_k in the proof of Proposition 6.7.*

(b) $\hat{\mathcal{R}} \subset \cup_k \mathcal{R}_k^1$, but the converse is not true, in general.
(c) If μ is an ϵ-optimal control for x w.r.t. V_k^2, then we can assume $\phi_t(x, \mu)$ does not self-intersect, for if it did we could find $\tilde{\mu}$, also ϵ-optimal, which eliminates loops in $\phi_t(x, \mu)$.
(d) $\|x - x'\| \to 0$ as $k \to \infty$ for all x,x', σ, $\sigma' \neq \sigma$ such that $([(\sigma, x)], [(\sigma', x')]) \in E$.
(e) For all $x \in \hat{R}$, $\hat{V}_k(x) \geq V_k^2(x)$. This follows because the argument of the max in the definition of $\hat{J}(q, c)$ is equal to $J_k^2(x, \mu)$, where c is a control policy as defined in Section 5.1 with $c(q) = \sigma_1$, $(\sigma_1, x) \in q$, and $\mu = (\sigma_1, \tau_1) \ldots$ is the piecewise continuous control that corresponds to following policy c starting at x. Thus, $J_k^2(x, \mu)$ is the cost for a particular trajectory in $\tilde{\Pi}_c(q)$ which has no jumps at the control switches. Then we have $\hat{J}(q, c) \geq J_k^2(x, \mu)$, since $\hat{J}(q, c)$ maximizes over all trajectories in $\tilde{\Pi}_c(q)$.

PROPOSITION 6.7. *For all $x \in \hat{\mathcal{R}}$, $|\hat{V}_k(x) - V_k^2(x)| \to 0$ as $k \to \infty$.*

Proof.

Fix $\epsilon > 0$ and $x \in \hat{\mathcal{R}}$. By Remark 6.3(a) there exists $m > 0$ and an ϵ-optimal control $\mu \in \mathcal{U}_m^2$ for x w.r.t. V_m^2. Denote $\mu = ((\sigma_1, \tau_1) \ldots (\sigma_N, \tau_N))$, where τ_i is the time σ_i is applied. Let c be any control policy on Q that is generated using δ_k and C_k, for $k \geq m$. Then, using Remark 6.3(e),

$$0 \leq \hat{V}_k(x) - V_k^2(x) \leq \hat{J}_k(q, c) - J_k^2(x, \mu) + \epsilon,$$

where $q = [(\sigma_1, x)]$. If we can show there exists $\overline{k} \geq m$ such that for $k > \overline{k}$, there exists a policy $\overline{c}$ such that

$$\hat{J}_k(q, \overline{c}) - J_k^2(x, \mu) < \epsilon,$$

then the result follows.

By Remark 6.3(d) and the transversality of $\phi_t(x, \mu)$ with the level sets of γ_n, we can find $\overline{k} \geq m$ such that for $k > \overline{k}$, there exists a family of (both continuous and discontinuous) trajectories Ψ_k starting at x with the following properties:

1. $\phi_t(x, \mu) \in \Psi_k$.

2. $\phi \in \Psi_k$ is defined over a control $\tilde{\mu} = ((\sigma_1, \tilde{\tau}_1), \ldots, (\sigma_N, \tilde{\tau}_N)) \in \mathcal{U}_k^2$ with the same sequence of control values as μ.
3. $\phi \in \Psi_k$ switches controls on the same (transversal) submanifolds as $\phi_t(x, \mu)$ and reaches Q_f.
4. If $x_j^- = \phi_{\tau_j}(x_{j-1}, \sigma_j)$, then x_j, the initial condition of the next step, satisfies $([(\sigma_j, x_j^-)], (\sigma_{j+1}, x_j)]) \in E$. Thus, the trajectories of Ψ_k include jumps at the control switches modeling the non-determinacy of A.
5. If $\phi \in \Psi_k$ intersects $q \in Q$ in $\mathbb{R}^n$ at the jth step, then that is the only step where it intersects q. Also, all other $\phi' \in \Psi_k$ that intersect q do so at the jth step only. This requirement can be met, for sufficiently large k by the fact that ϕ' has no self-intersections, by the fact that there are a finite number of steps, and by Remark 6.3(d). For if ϕ' has a self-intersection, then since ϕ' approaches $\phi_t(x, \mu)$ as $k \to \infty$, this would imply $\phi_t(x, \mu)$ has a self-intersection, contradicting Remark 6.3(c).

The family Ψ_k includes all trajectories starting at x, using the same sequence of control values as μ, and switching on the same equivalence class boundaries $\phi_t(x, \mu)$. Moreover, the initial condition at the start of each step can be any point in an equivalence class that has a non-empty intersection in $\mathbb{R}^n$ with the equivalence class reached at the end of the previous step. One visualizes a tube of trajectories that fans out with each successive control switch, as depicted in Figure 6.2. By choosing k sufficiently large and by transversality, all these trajectories reach Q_f.

Let $W_k(\phi) = \sum_{j=1}^{N} L^2(x_{j-1}, \sigma_j) + \hat{h}(x_N)$. Observe that for $\phi, \phi' \in \Psi_k$, $|W_k(\phi) - W_k(\phi')| \to 0$ as $k \to \infty$, using Lipschitz continuity of L and h, and Remark 6.3(d). We can define a control policy $\overline{c}$ in which $q \in Q$ is assigned a time step if q is not visited by any trajectory in Ψ_k. If $q \in Q$ is visited by some $\phi \in \Psi_k$ in its jth step, then we assign $\overline{c}(q) = \sigma_j$. This gives a well-defined value for c because of Property 4. By construction A accepts the time abstract trajectory starting at q corresponding to each trajectory of Ψ_k. $\overline{c}$ is admissible because otherwise some time abstract trajectory of A would have a Zeno loop. But a time abstract trajectory of A with a Zeno loop has a corresponding timed trajectory in Ψ_k that violates Property 4 of Ψ_k.

Now we observe that

$$\hat{J}(q, \overline{c}) = \max_{\phi \in \Psi_k} W_k(\phi) := W_k(\overline{\phi}).$$

Thus, $\hat{J}_k(q, \overline{c}) - J_k^2(x, \mu) \leq |W_k(\overline{\phi}) - W_k(\phi(x, \mu))| \to 0$ as $k \to \infty$. ∎

Combining Propositions 6.3, 6.6, and 6.7, we have

THEOREM 6.8. *For all $x \in \hat{\mathcal{R}}$, $\hat{V}_k(x) \to V(x)$ as $k \to \infty$.*

7. Implementation. So far we have developed a discrete method for solving an optimal control problem based on hybrid systems and bisimulation. Now we focus on the pragmatic question of how the discretized problem can be efficiently solved. In this section we propose a modification of the Dijkstra algorithm suitable for nondeterministic automata and prove that it is optimal and does not synthesize Zeno loops.

7.1. Motivation. Capuzzo-Dolcetta [12] introduced a method for obtaining approximations of viscosity solutions based on time discretization of the HJB equation. The approximations of the value function correspond to a discrete time optimal control problem, for which an optimal control can be synthesized which is piecewise constant. Finite difference approximations were also introduced in [14] and [47]. In general, the time discretized approximation of the HJB equation is solved by finite element methods. Gonzales and Rofman [22] introduced a discrete approximation by triangulating the domain of the problem, while the admissible control set is approximated by a finite set. Gonzales and Rofman's approach is adapted in several papers, including [17]. The approach of [50] uses the special structure of an optimal control problem to obtain a single-pass algorithm to solve the discrete problem, thus bypassing the expensive iterations of a finite element method. See [45] for a recent adaptation of Tsitsiklis' approach. The essential property needed to find a single pass algorithm is to obtain a partition of the domain so that the cost-to-go value from any equivalence class of the partition is determined from knowledge of the cost-to-go from those equivalence classes with strictly smaller cost-to-go values. We obtain a partition of the domain provided by a bisimulation partition. *The combination of the structure of the bisimulation partition and the requirement of non-Zeno trajectories enables us reproduce the essential property of [50], so that we obtain a Dijkstra-like algorithmic solution.* Our approach has complexity $O(N \log N)$ if suitable data structures are used, where N is the number of locations of the finite automaton. The number N is, of course, exponential in n, the dimension of the continuous state space.

7.2. Non-deterministic Dijkstra algorithm. The dynamic programming solution (5.4)-(5.5) can be viewed as a shortest path problem on a non-deterministic finite graph subject to all optimal paths satisfying a non-Zeno condition. We propose an algorithm which is a modification of the Dijkstra algorithm for deterministic graphs [16]. First we define the notation. F_n is the set of states that have been assigned a control and are deemed "finished" at iteration n, while U_n are the unfinished states. At each n, $Q = U_n \cup F_n$. $\Sigma_n(q) \subseteq \Sigma_\delta$ is the set of control events at iteration n that take state q to finished states exclusively. $\tilde{U}_n$ is the set of states for which there exists a control event that can take them to finished states exclusively. $\tilde{V}_n(q)$ is a tentative cost-to-go value at iteration n. B_n is the set of "best" states among $\tilde{U}_n$.

The non-deterministic Dijkstra (NDD) algorithm first determines $\tilde{U}_n$ by checking if any q in U_n can take a step to states belonging exclusively to F_n. For states belonging to $\tilde{U}_n$, an estimate of the value function $\tilde{V}$ following the prescription of (5.4) is obtained: among the set of control events constituting a step into states in F_n, select the event with the lowest worst-case cost. Next, the algorithm determines B_n, the states with the lowest $\tilde{V}$ among $\tilde{U}_n$, and these are added to F_{n+1}. The iteration counter is incremented until it reaches $N = |Q|$. It is assumed in the following description that initially $\hat{V}(q) = \infty$ and $c(q) = \emptyset$ for all $q \in Q$.

```
Procedure NDD:

F_1 = Q_f; U_1 = Q − Q_f;
for each q ∈ Q_f, V̂(q) = ĥ(q);

for n = 1 to N, do
    for each q ∈ U_n,
            Σ_n(q) = {σ' ∈ Σ_δ | if q --σ'--> q', then q' ∈ F_n};
    Ũ_n = {q ∈ U_n | Σ_n(q) ≠ ∅};
    for each q ∈ Ũ_n,
            Ṽ_n(q) = min_{σ'∈Σ_n(q)} {max_{e=(q,q')∈E_σ'(q)} {L̂(e) + V̂(q')}};
    B_n = argmin_{q∈Ũ_n} {Ṽ_n(q)};
    for each q ∈ B_n,
            V̂(q) = Ṽ_n(q);
            c(q) = argmin_{σ'∈Σ_n(q)} {max_{e=(q,q')∈E_σ'(q)} {L̂(e) + V̂(q')}};
    endfor
    F_{n+1} = F_n ∪ B_n; U_{n+1} = Q − F_{n+1};
endfor
```

7.3. Justification. In this section we prove that the algorithm is *optimal*; that is, it synthesizes a control policy so that each $q \in Q$ reaches Q_f with the best worst-case cost. We observe a few properties of the algorithm. First, if all states of Q can reach Q_f in a non-deterministic sense, then $Q - Q_f = \cup_n B_n$. By non-deterministic sense we mean that for each $q \in Q$, there exists a control policy c such that $\Pi_c(q) = \tilde{\Pi}_c(q)$. Note that if this condition is not met then it can happen that at some iteration of NDD, $\tilde{U}_n = \emptyset$ but $U_n \neq \emptyset$. Second, as in the deterministic case, the algorithm computes $\hat{V}$ in order of level sets of $\hat{V}$. In particular, $\hat{V}(B_n) \leq \hat{V}(B_{n+1})$. Finally, we need the following property.

Lemma 7.1. *For all $q \in Q$ that can reach Q_f in a non-deterministic sense and for all $\sigma' \in \Sigma_\delta$,*

$$\hat{V}(q) \leq \max_{e=(q,q')\in E_{\sigma'}(q)} \{\hat{L}(e) + \hat{V}(q')\}.$$

Proof. Fix $q \in Q$ and $\sigma' \in \Sigma_\delta$. There are two cases.
Case 1.

$$\hat{V}(q) \leq \max_{e=(q,q')\in E_{\sigma'}(q)} \{\hat{V}(q')\}.$$

In this case the result is obvious.
Case 2.

$$\hat{V}(q) > \max_{e=(q,q')\in E_{\sigma'}(q)} \{\hat{V}(q')\}. \tag{7.1}$$

By assumption, q belongs to some B_n. Suppose w.l.o.g. that $q \in B_j$. Together with (7.1) this implies $q' \in F_j$ for all q' such that $q \xrightarrow{\sigma'} q'$. This, in turn, means that $\sigma' \in \Sigma_j(q)$ and according to the algorithm

$$\hat{V}(q) = \tilde{V}_j(q) \leq \max_{e=(q,q')\in E_{\sigma'}(q)} \{\hat{L}(e) + \hat{V}(q')\}$$

which proves the result. ∎

THEOREM 7.2. *Algorithm NDD is optimal and synthesizes a control policy with no Zeno loops.*

Proof. First we prove optimality. Let $V(q)$ be the optimal (best worst-case) cost-to-go for $q \in Q$ and $\overline{Q} = \{q \in Q \mid V(q) < \hat{V}(q)\}$. Let $l(\pi_q)$ be the number of edges taken by the shortest optimal (best worst-case) trajectory π_q from q. Define $\overline{q} = \arg\min_{q\in\overline{Q}}\{l(\pi_q)\}$. Suppose that the best worst-case trajectory starting at $\overline{q}$ is $\pi_{\overline{q}} = \overline{q} \xrightarrow{\sigma'} \overline{\overline{q}} \to \ldots$. We showed in the previous lemma that

$$\hat{V}(\overline{q}) \le \max_{e=(\overline{q},q')\in E_{\sigma'}(\overline{q})}\{\hat{L}(e) + \hat{V}(q')\} = \hat{L}(e) + \hat{V}(\overline{\overline{q}}).$$

Since $\pi_{\overline{q}}$ is the best worst-case trajectory from $\overline{q}$ and by the optimality of $V(\overline{q})$

$$V(\overline{q}) = \max_{e=(\overline{q},q')\in E_{\sigma'}(\overline{q})}\{\hat{L}(e) + V(q')\} = \hat{L}(e) + V(\overline{\overline{q}}).$$

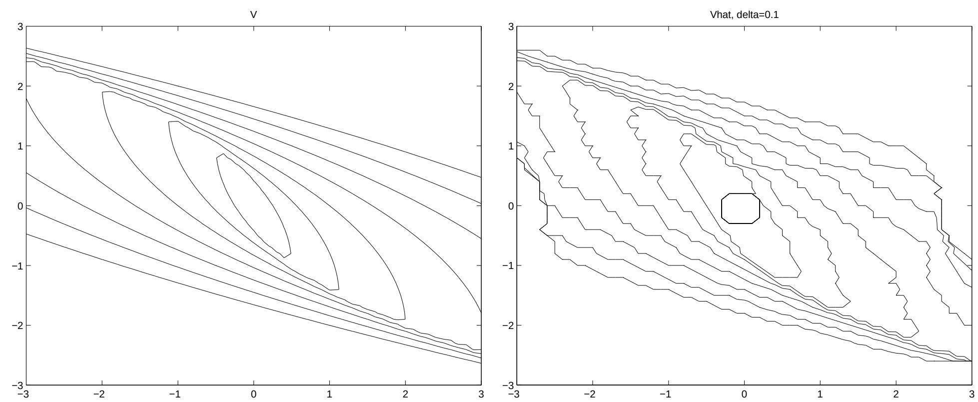
Since $\pi_{\overline{q}}$ is the shortest best worst-case trajectory, we know that $\overline{\overline{q}} \notin \overline{Q}$, so $V(\overline{\overline{q}}) = \hat{V}(\overline{\overline{q}})$. This implies $\hat{V}(\overline{q}) \le \hat{L}(e) + V(\overline{\overline{q}}) = V(\overline{q})$, a contradiction.

To prove that the algorithm synthesizes a policy with no Zeno loops we argue by induction. The claim is obviously true for F_1. Suppose that the states of F_n have been assigned controls forming no Zeno loops. Consider F_{n+1}. Each state of B_n takes either a time step or a control switch to F_n so there cannot be a Zeno loop in B_n. The only possibility is for some $q \in B_n$ to close a Zeno loop with states in F_n. This implies there exists a control assignment that allows an edge from F_n to q to be taken; but this is not allowed by NDD. Thus, F_{n+1} has no Zeno loops. ∎

8. Examples. We consider two simple examples where the solution of the optimal control problem is known in order to illustrate the correctness of the method. The software that generates the optimal enabling conditions is broken into two programs, one that generates the automaton given the information about the bisimulation and the second that runs the algorithm NDD. The first program takes as input the control values Σ_δ and the level values of γ_i^σ, $i = 1, \ldots, n$, $\sigma \in \Sigma_\delta$ defining the bisimulation. The functions γ_i^σ, $\hat{L}$ and $\hat{h}$ are compiled with the executable. A data structure that associates to each location of the finite automaton the lower and upper level values of each γ_i^σ allows time steps to be encoded symbolically; namely by sorting nodes with equal upper and lower first integral level values in ascending order of γ_n^σ level values. The edges of the finite automaton that correspond to σ-steps are generated numerically by evaluating γ_i^σ for $i = 1, \ldots, n$ and each $\sigma \in \Sigma_\delta$ and thereby determining which equivalence classes overlap for each pair (l, l') of locations. In our implementation the grid of sample points is $\{x \in \Omega, \gamma_i \in C^k\}$, in order to correlate with the meshsize of the bisimulation partition. This numerical step can also be performed symbolically if the functions γ_i^σ are polynomials using a quantifier elimination algorithm [5]. However, the quantifier elimination step is expensive and for approximate solutions it suffices to use a numerical approach.

First we apply our method to Example 3.1 and 4.1. The bang-bang solution obtained using Pontryagin's maximum principle is well known to involve a single switching curve. The continuous value function V is shown in Figure 8.1(a).

The results of algorithm NDD are shown in Figure 8.1(b) and Figure 8.2. In Figure 8.2 the dashed line is the smooth switching curve for the continuous problem. The black dots identify equivalence classes where NDD assigns a control switch. Considering $g_{e_{-1}}$ we see that the boundary of the enabling condition in the upper left



(a) V　　　(b) $\hat{V}$ for $\Delta = 0.1$.

FIG. 8.1. *Continuous and discrete value functions for double integrator*

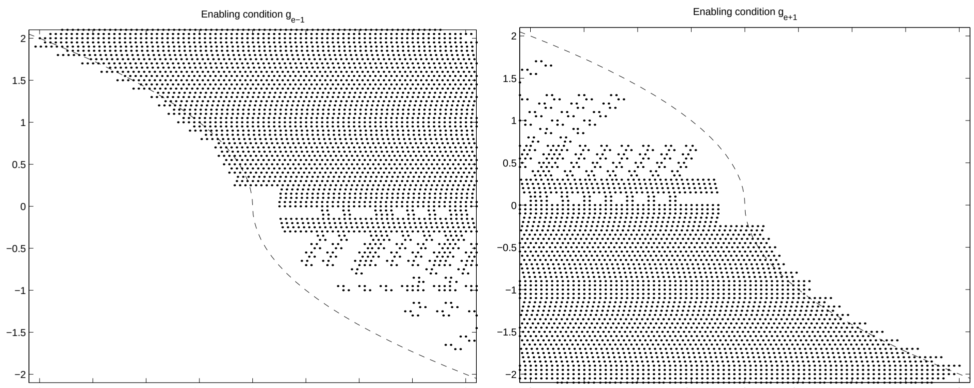
corner is a jagged approximation using equivalence classes of the smooth switching curve. Initial conditions in the upper left corner just inside the enabling condition must switch to a control of $u = -1$, otherwise the trajectory will increase in the x_2 direction and not reach the target. Initial conditions in the upper left corner just outside the enabling condition must allow time to pass until they reach the enabling condition, for if they switched to $u = -1$ they would be unable to reach the target. Hence the upper left boundary of the enabling condition is crisp. The lower right side of the enabling condition which has islands of time steps shows the effect of the non-determinacy of automaton A. These additional time steps occur because it can be less expensive to take a time step than to incur the cost of the *worst case* control switch. Indeed consider an initial condition in Figure 8.2(a) which lies in an equivalence class that takes a time step but should take a control switch according to the continuous optimal control. Such a point will move up and to the left before it takes a control switch. By moving slightly closer to the target, the worst-case cost-to-go incurred in a control switch is reduced. Notice that all such initial conditions eventually take a control switch. This phenomenon of extra time steps is a function of the mesh size δ: as δ decreases there are fewer extra time steps. Finally we note that the two enabling conditions have an empty intersection, as expected in order to ensure non-Zeno trajectories.

Figure 8.3 shows trajectories of the closed-loop system using the controller synthesized by NDD. The central shaded region is an enlarged target set.

Next we consider the time optimal control problem for the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u. \end{aligned} \tag{8.1}$$

Suppose $\Omega = (-1,1) \times (-1,1)$ and $\Omega_f = \overline{B}_\epsilon(0)$, the closed epsilon ball centered at 0. The cost-to-go function is $J(x,\mu) = \int_0^{T(x,\mu)} dt$ and $U = \{u \ : \ |u| \leq 1\}$. We select $\Sigma_\delta = \{-1, 1\}$, so that $\delta = 1$. The hybrid system is show in Figure 8.4. The state set is $\{\sigma_{-1} = -1, \sigma_1 = 1, \sigma_f\} \times \mathbb{R}^2$. $g_{e_{-1}}$ and g_{e_1} are unknown and must be synthesized,



(a) $g_{e_{-1}}$ (b) g_{e_1}

Fig. 8.2. *Enabling conditions*

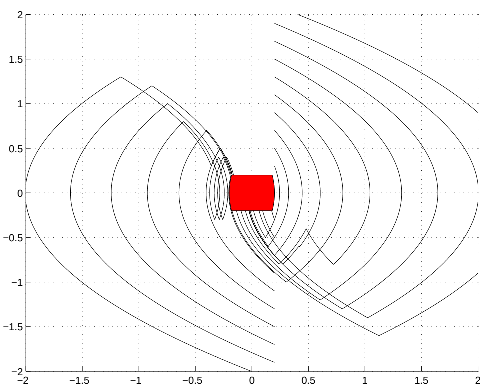


Fig. 8.3. *Trajectories of the closed-loop system*

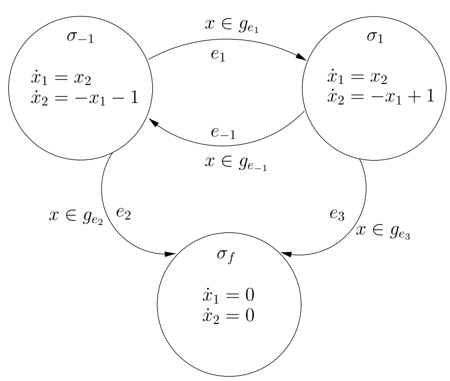
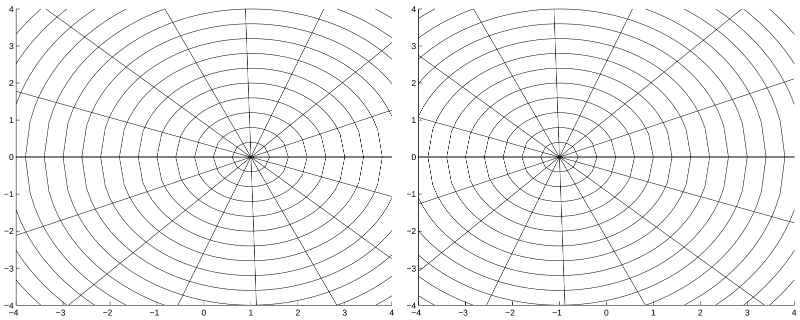
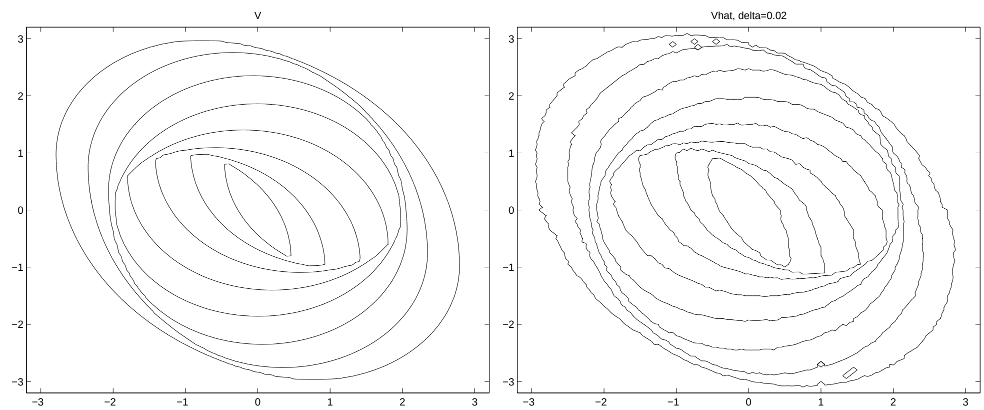


Fig. 8.4. *Hybrid automaton for Example 2.*



(a) σ_1 (b) σ_{-1}

FIG. 8.5. *Partitions for states σ_1 and σ_{-1} of the hybrid automaton of Figure 8.5*

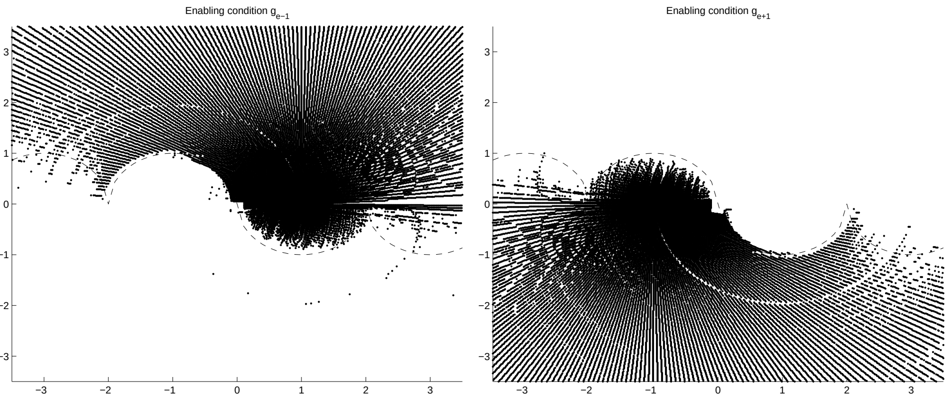


(a) V (b) $\hat{V}$ for $\Delta = 0.02$.

FIG. 8.6. *Continuous and discrete value functions for Example 2.*

while $g_{e_2} = g_{e_3} = \Omega_f$. A first integral for Equation 8.1 is $\sqrt{(x_1 - u)^2 + x_2^2} = c_1$ where $u = \pm 1$. The transverse foliation is chosen to be defined by the function $\arctan(\frac{x_2}{x_1 - u}) = c_2$. Partitions for locations σ_1 and σ_{-1} and are shown in Figure 8.5. The results of algorithm NDD are shown in Figure 8.6(b) and Figure 8.7. In Figure 8.7 the dashed line is the switching curve for the continuous problem. As in the previous example the black dots identify equivalence classes where NDD assigns a control switch. Figure 8.8 shows trajectories of the closed-loop system using the controller synthesized by NDD. An enlarged target set is at the origin.

Remark 8.1. From these examples we observe that our method is best suited to problems when there are relatively few control switches, as each control switch incurs an error of order δ. Also the method is suited to problems where bang-bang controls are used. The method has advantages in situations where a fine time discretization of the vector field is needed for standard finite element methods. We do not require time discretization because of the particular choice of grid which captures time evolution



(a) $g_{e_{-1}}$ (b) g_{e_1}

Fig. 8.7. *Enabling conditions for Example 2.*

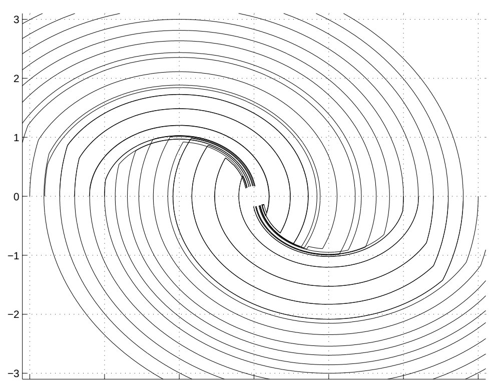


Fig. 8.8. *Trajectories of the closed-loop system for Example 2.*

exactly. Finally, because the method requires computation of weak first integrals, only systems for which first integrals are computable in closed form are considered.

The following table shows the computation times for the two examples as a function of δ. The automaton size and the time in seconds to generate it appear in the second and third columns. We report the time for file I/O which otherwise would dominate the computation times. The time in seconds to run NDD and the size of the set F_n of finished states appear in the last two columns. Note that not all nodes are finished in the first example because the regions of $\mathbb{R}^2$ that are partitioned in the two locations do not overlap perfectly, resulting in the non-existence of a trajectory that can reach the origin starting in a subset of the non-overlapping areas.

Example	δ	N	Automaton	I/O	NDD	Finished
1	.2	7200	.11	4.57	.02	3274
1	.1	28800	.46	5.69	.08	12835
1	.05	115200	1.97	9.09	.38	51490
1	.025	460800	7.83	22.91	1.77	205624
2	.2	1920	.03	4.42	.01	1920
2	.1	7560	.09	4.78	.07	7560
2	.05	30240	.39	6.13	.28	30240
2	.025	120960	1.55	11.04	1.3	189000
2	.0125	482880	7.99	35.29	7.12	482880

9. Conclusion. In this paper we have developed a methodology for the synthesis of optimal controls based on hybrid systems and bisimulations. The idea is to translate the optimal control problem to a switching problem on a hybrid system whose locations describe the dynamics when the control is constant. When the vector fields for each location of the hybrid automaton have local first integrals which can be expressed analytically we are able to define a finite bisimulation using the approach of [10]. From the finite bisimulation we obtain a (time abstract) finite automaton upon which a dynamic programming problem can be formulated that can be solved efficiently. We proposed an efficient single-pass algorithm to solve this dynamic programming problem and demonstrated its correctness on two simple examples.

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